
Combining results of Bowen, Smale, Shub and Nitecki one can find an open dense set in $\text{Diff}(M)$ with the $\mathcal{C}^0$ topology in which each diffeomorphism $f$ satisfies the following lower bound on its topological entropy

$$h(f) \geq \log s(\lambda)$$

where $s(\lambda)$ is the spectral radius or largest eigenvalue of $f : H(M) \to H(M)$.

It was conjectured that (*) holds for all $\Omega$-stable diffeomorphisms and in fact for all diffeomorphisms and even all smooth maps. Even though there is less evidence for these last two there is still no counter-example known.

Bowen has proved that if $f$ satisfies Axiom A then $h(f) = \limsup (1/n) \log \# \text{Fix}(f^n)$ so using the Lefschetz formula we get $h(f) = \limsup (1/n) \log |\Sigma(-1)|^i \text{trace } f^n_i$.

On the other hand $\log s(\lambda) = \limsup (1/n) \log |\Sigma \text{ trace } f^n_i|$. Thus (*) gives a significantly sharper asymptotic estimate on the growth rate of the number of periodic points of an Axiom A no cycle diffeomorphism than the Lefschetz number does.

A simplest diffeomorphism in an isotopy class is a structurally stable diffeomorphism with entropy minimal among stable diffeomorphisms in the class. There is not always a simplest diffeomorphism satisfying Axiom A, see the work on Morse-Smale diffeomorphisms in [1]. In this case we can ask for a sequence $f_i$ of diffeomorphisms in the isotopy class $s.t. \ h(f_i) \to \log s(\lambda)$. If there is no such sequence there must be a better lower bound than (*). Several of us at this symposium have just found a homeomorphism of an 8-manifold with $\Omega(h) = 4$ points and $\log s(\lambda) \neq 0$. This cannot be smoothed because of a $C^1$ Lefschetz index argument.

Proposition. Almost every $C^\infty$ degree 2 map of $S^2$ has $h(f) \geq \log s(\lambda)$.

Consider those maps with only folds or cusps. By a local degree argument $\exists \delta$ s.t. almost every point on $S^2$ has two inverse images $\delta$ apart. 

Hence the intersected sets $h(f) \geq \log 2$. 


2 but zero entropy.

We shall outline a proof of the following new result obtained with R. Williams.

**Theorem.** If \( f \) satisfies Axiom A and the no cycle condition then \( h(f) \geq \log s(f) \).

**Proof.** For simplicity we work here with \( f \colon M \rightarrow M \) Anosov with \( \Omega(f) = M \) and \( E^s \) and \( E^u \) orientable. By taking powers we can assume \( f \) has a fixed point, \( p \) say. Suppose \( r \) is a real eigenvalue of \( f^u \) and let \( \sigma = \Sigma_i \sigma_i \) be a cycle representing a corresponding eigenvector in \( H_1(M; \mathbb{R}) \). Take a closed form \( \eta \) dual to \( \sigma \) so that \( \int \eta = 1 \). We can assume that each \( \sigma_i \) is a smooth simplex transverse to \( E^s \). \( \forall \epsilon, \delta \geq n \) s.t. \( V(\epsilon^r W_\delta^u(p)) = M \) where \( V \) means an \( \epsilon \)-neighbourhood. Chop up \( \sigma_i \) into pieces \( \sigma_{ij} \) in an \( \epsilon \)-neighbourhood of a small part of \( W^u(p) \). For example \( \sigma_{i1} \subset V(\epsilon W_\delta^u(p)) \). \( f^k \sigma_{ij} \) approaches \( W^u(p) \) in the \( C^1 \) sense and \( f^k \sigma_{i1} \subset V(\epsilon^k W_\delta^u(p)) \). Project \( f^k \sigma_{i1} \) down to \( f^k \sigma_{i1} \) by a map \( \pi \). Then \( \int f^k \sigma_{i1} \eta \) is close to \( \int f^k \sigma_{i1} \eta \) and this is bounded by a constant multiple of \( \text{Vol}(\sigma_{i1}) \leq \text{const.} \text{Vol}(f^k W_\delta^u(p)) \). By counting how many boxes \( f^k W_\delta^u(p) \) crosses in a Markov partition for \( f \) we find that

\[
\text{Vol}(f^k W_\delta^u(p)) = \lambda = \exp h(f) \text{ as } k \rightarrow \infty.
\]

Thus \( r^k = \left| \int f^k \sigma_{i1} \eta \right| < \text{const.} \lambda^k \) which gives the result when \( \eta \) has the same dimension as \( f^k \sigma_{i1} \). For lower dimensions take a cycle, fatten it with homologous cycles and do the same. For higher dimensions work with \( f^{-1} \). In the case of Axiom A and no cycles use the relative homology theory for a filtration.

More care is needed with \( W^u(p) \).

**Reference.**


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