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21. Topological entropy and stability.

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Combining results of Bowen, Smale, Shub and Nitecki one can find an open dense set in Diff(M) with the C^0 topology in which each diffeomorphism f satisfies the following lower bound on its topological entropy

$$h(f) \geqslant \log s(f_*) \qquad (*)$$

where $s(f_*)$ is the spectral radius or largest eigenvalue of $f_*: H_*(M) \to H_*(M)$. It was conjectured that (*) holds for all Ω-stable diffeomorphisms and in fact for all diffeomorphisms and even all smooth maps. Even though there is less evidence for these last two there is still no counter-example known.

Bowen has proved that if f satisfies Axiom A then $h(f) = \lim\sup(1/n)\log \# \mathrm{Fix}(f^n)$ so using the Lefschetz formula we get $h(f) \geqslant \lim\sup(1/n)\log|\Sigma(-1)^i \,\mathrm{trace}\, f_{*i}^n|$. On the other hand $\log s(f_*) = \lim\sup(1/n)\log|\Sigma\, \mathrm{trace}\, f_{*i}^n|$. Thus (*) gives a significantly sharper asymptotic estimate on the growth rate of the number of periodic points of an Axiom A no cycle diffeomorphism than the Lefschetz number does.

A simplest diffeomorphism in an isotopy class is a structurally stable diffeomorphism with entropy minimal among stable diffeomorphisms in the class. There is not always a simplest diffeomorphism satisfying Axiom A, see the work on Morse-Smale diffeomorphisms in [1]. In this case we can ask for a sequence f_i of diffeomorphisms in the isotopy class s.t. $h(f_i) \to \log s(f_{1*})$. If there is no such sequence there must be a better lower bound than (*). Several of us at this symposium have just found a homeomorphism of an 8-manifold with $\Omega(h) = 4$ points and $\log s(f_*) \neq 0$. This cannot be smoothed because of a C^1 Lefschetz index argument.

Proposition. Almost every C^∞ degree 2 map of S^2 has $h(f) \geqslant \log s(f_*)$.

Consider those maps with only folds or cusps. By a local degree argument $\exists\delta$ s.t. almost every point on S^2 has two inverse images δ apart. [illegible] separated sets $h(f) \geqslant \log 2$

2 but zero entropy.

We shall outline a proof of the following new result obtained with R. Williams.

Theorem. If f satisfies Axiom A and the no cycle condition then $h(f) \geq \log s(f_*)$.

Proof. For simplicity we work here with $f:M \to M$ Anosov with $\Omega(f) = M$ and E^s and E^u orientable. By taking powers we can assume f has a fixed point, p say. Suppose r is a real eigenvalue of f_{*u} and let $\sigma = \Sigma r_i \sigma_i$ be a cycle representing a corresponding eigenvector in $H_u(M;R)$. Take a closed form η dual to σ so that $\int_\sigma \eta = 1$. We can assume that each σ_i is a smooth simplex transverse to E^s. $\forall \epsilon, \delta \; \exists$ n s.t. $V_\epsilon(f^n W_\delta{}^u(p)) = M$ where V_ϵ means an ϵ-neighbourhood. Chop up σ_i into pieces σ_{ij} in an ϵ-neighbourhood of a small part of $W^u(p)$. For example $\sigma_{i1} \subset V_\epsilon(W_\delta{}^u(p))$. $f^k \sigma_{ij}$ approaches $W^u(p)$ in the C^1 sense and $f^k \sigma_{i1} \subset V_\epsilon(f^{k+1} W_\delta{}^u(p))$. Project $f^k \sigma_{i1}$ down to $f^{k+1} W_\delta{}^u(p)$ by a map π. Then $\int_{f^k \sigma_{ij}} \eta$ is close to $\int_{\pi f^k \sigma_{i1}} \eta$ and this is bounded by a constant multiple of $\mathrm{Vol}(\pi f^k \sigma_{i1}) \leq$ const. $\mathrm{Vol}(f^{k+1} W_\delta{}^u(p))$. By counting how many boxes $f^k W_\delta{}^u(p)$ crosses in a Markov partition for f we find that

$$\mathrm{Vol}(f^k W_\delta{}^u(p))/\mathrm{Vol}(f^{k-1} W_\delta{}^u(p)) \to \lambda = \exp h(f) \text{ as } k \to \infty.$$

Thus $r^k = \left| \int_{f^k \sigma} \eta \right| <$ const. λ^k which gives the result when η has the same dimension as W^u. For lower dimensions take a cycle, fatten it with homologous cycles and do the same. For higher dimensions work with f^{-1}. In the case of Axiom A and no cycles use the relative homology theory for a filtration. More care is needed with $W^u(p)$.

Reference.

1. M. Shub, Dynamical systems, filtrations and entropy, Bull. Amer. Math. Soc., 80 (1974) 27–41.

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