CHARLES PUGH AND MICHAEL SHUB*

Stable ergodicity and partial hyperbolicity

Dedicated to the memory of Ricardo Mañé.

1. Introduction

By the results of Anosov in 1967 volume preserving uniformly hyperbolic systems are ergodic and are open. Thus they exhibit robust statistical behavior. Averages are the same for almost all points, not only for the system in question but also for any small perturbation which preserves the same volume. If the perturbation only preserves a close by volume, then the averages of continuous functions are still close by. On the other hand, in 1954, Kolmogorov announced that there are no ergodic Hamiltonian systems in a neighborhood of a completely integrable one. Completely integrable systems have no hyperbolic behavior at all.

In this paper we will review the results of [Grayson, Pugh and Shub, 1994], [Pugh and Shub, 1996], [Pugh, Shub and Wilkinson, 1996], and [Brezin and Shub, 1995] which study the mixed situation in which the system is only partially hyperbolic.

Our themes are:

1) A little hyperbolicity goes a long way toward guaranteeing ergodic behavior.
2) Stably ergodic systems are considerably more general than one might have feared from Kolmogorov's theorem.
3) Some hyperbolicity may be necessary for stable ergodicity.

We consider C^2 diffeomorphisms f of closed manifolds M which preserve a fixed smooth volume on M. We say that f is stably ergodic if there is a neighborhood U of f in the C^2 volume preserving diffeomorphisms of M such that every $g \in U$ is ergodic.

* Partially supported by an NSF grant.

Each of our main themes is developed in a section below. Finally in section 5, we ggest some generalizations to dissipative systems.

2. Partial Hyperbolicity and Ergodicity

The Main Theorem of this section gives sufficient conditions for a diffeomorphism to e ergodic. We find stably ergodic diffeomorphisms by finding open sets of diffeo-rphisms satisfying these conditions. The theorem may be interpreted to say that en for systems which are not uniformly hyperbolic, the same phenomenon which oduces chaotic behavior i.e. some hyperbolicity may also guarantee ergodicity.

Main Theorem: [Pugh and Shub, 1996] Let $f : M \to M$ be partially hyperbolic and dynamically coherent. Suppose that the stable and unstable bundles have the accessibility property and that the invariant bundles are sufficiently Hölder. Then f is ergodic.

The accessibility property is a concept from control theory which we apply to the rong unstable and stable foliations of a partially hyperbolic diffeomorphism. We ll soon explain these concepts. Partially hyperbolic diffeomorphisms which are ynamically coherent have some more properties which we will eventually come to. All three distributions E^s, E^c and E^u, the strong stable, center and strong unstable sub-bundles of the tangent bundle, of C^2 diffeomorphisms are Hölder. That they are sufficiently Hölder is expressed in terms of relationships of the Hölder exponents and timately in terms of the contraction and expansion constants of the various natural ariant bundles for the derivative. We leave these details to be consulted in [Pugh and Shub, 1996] and [Pugh, Shub and Wilkinson, 1996], but note that foliations with C^1 tangent bundles are sufficiently Hölder. Partially hyperbolic systems and the accessibility property were to our knowledge first considered in [Brin and Pesin, 1974].

We say that a C^r diffeomorphism $f : M \to M$ is *partially hyperbolic* iff $r \geq 1$ and there is a continuous Tf-invariant direct sum decomposition

$$TM = E^s \oplus E^c \oplus E^u$$

where E^s and E^u are non-trivial, some Finsler $\| \;\|$ on TM and some real constants $0 < a < b < c < 1 < d < e < g$ such that

$$\begin{aligned} a\|v\| &< \|Tf(v)\| < b\|v\| \text{ for } v \in E^s - \{0\} \\ c\|v\| &< \|Tf(v)\| < d\|v\| \text{ for } v \in E^c - \{0\} \\ e\|v\| &< \|Tf(v)\| < g\|v\| \text{ for } v \in E^u - \{0\}. \end{aligned}$$

Since $Tf : E^c \to E^c$ may have some contraction and expansion E^s and E^u are strong contracting and strong expanding Tf invariant subbundles. Tangent to E^s and E^u are the strong contracting and strong expanding f invariant foliations which we will denote by W^s and W^u.

Given continuous sub-bundles $F, H \subset TM$ and points $m_0, m_1 \in M$ we say that m_1 is *accessible* from m_0 iff there is a continuous piecewise C^1 path $\phi[0, 1]$ joining