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STABILITY IN DYNAMICAL SYSTEMS

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RESUME

Les utilisateurs des systèmes dynamiques sont le plus souvent intéressés par la connaissance des attracteurs structurellement stables d'un système donné. Leur attention va cependant rarement au-delà d'attracteurs qui sont des points fixes ou des solutions périodiques. Le papier présente un exemple simple d'attracteur structurellement stable qui n'entre pas dans ces catégories et donne un théorème général de caractéristation des attracteurs structurellement stables.

There are numerous notions of stability in dynamical systems. Among these are asymptotic stability and structural stability. We say that a set of orbits is asymptotically stable or an attractor if nearby orbits tend to the set as time increases, that is, even if the initial conditions are perturbed to be slightly off the set the limiting positions are still in the set. A system is structurally stable if perturbing the system a little does not change the orbit structure up to a continuous change of variable. Systems which are structurally stable in a region around an attractor are particularly fruitful for study. The behavior of such systems is neither sensitive to perturbation of the system nor to perturbations of the initial point.

The study of differential equations has classically known two cases of compact attractors which are structurally stable in a region around the attractors. The most prevalent that arise are the hyperbolic stable equilibria of ordinary differential equations, that is, the zeros of equations of the form :

$$\frac{dx}{dt} = A(x(t)) + 0(x(t))$$

where A is linear with the real part of all eigenvalues of A negative and 0 is a perturbation term. When stable equilibria cannot be found the hyperbolic stable periodic solutions are generally the next examples to come to mind ; for example, Van der Pohl's equations :

$$\dot{x} = y - x^3 + x$$

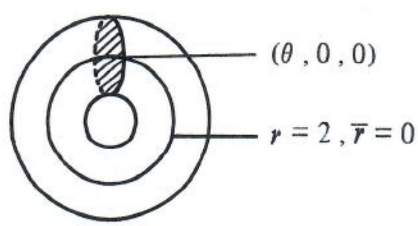
$$\dot{y} = -x \, .$$

One finds these attractors in the literature frequently. Since the attractors consist of a single orbit there is non question of the structural stability of the equation on the orbit. In fact, it is not much more difficult to believe that these systems are structurally stable in a region around the orbit. One may draw some pictures as follows :

Since 1960 we have learned about more structurally stable attractors, manifolds and not.

I will begin by describing a structurally stable attractor which was described by Smale in 1966. Smale began with the structurally stable expanding map $\theta \to 2\theta$ of the circle which was studied in my thesis and so this example is particularly dear to me. The presentation here not only incorporates Smale's but goes on to study the example from the point of view of the work of Bowen, Ruelle, Sinai, Williams and others.

Start with a solid ring, R, in three dimensional space E^3.



we take $(\theta, \bar{r}, s)$ coordinates on R where θ is the angle on the central circle and r, s are coordinates in the plane orthogonal to the circle. This amounts to $(\theta, \bar{r})$ coordinates on a annular region in the plane with $\bar{r} = r - 2$, $-1 \leqslant \bar{r} \leqslant 1$, and a third coordinate s perpendicular to the plane. Now define f : R $\to$ R by

$$f(\theta, \bar{r}, s) = (2\theta, \epsilon_1 \cos\theta + \epsilon_2 \bar{r}, \epsilon_1 \sin\theta + \epsilon_2 s) \quad \text{where} \quad \epsilon_1, \epsilon_2 > 0$$

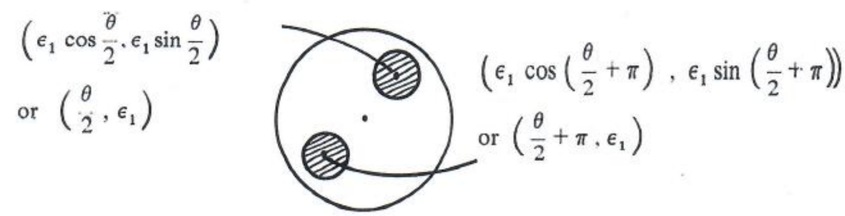
are small. The image $f(\mathrm{R})$ is inside R and the perpendicular disc to the circle at the point θ is mapped into the perpendicular disc to the circle at the point $2\,\theta$.

It is easy to see that if $(\bar{r}, s)$ are restricted to

$$r^2 + s^2 \leqslant \quad \text{and} \quad \epsilon_2 < \epsilon_1 < \frac{1}{2}$$

that this map is an embedding. That is, it is a 1-1 differentiable mapping and it takes R into R.

If we denote a disc cross section through the point θ on the circle by $\mathrm{D}(\theta)$ we see that $f\left(\mathrm{D}\left(\frac{\theta}{2}\right)\right)$ and $f\left(\mathrm{D}\left(\frac{\theta}{2}+\pi\right)\right) \subset \mathrm{D}(\theta)$



and they are contracted by a factor of ϵ_2. The centers are at

$$\left(\epsilon_1 \cos\frac{\theta}{2}, \epsilon_1 \sin\frac{\theta}{2}\right)$$

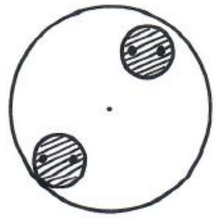
and

$$\left(\epsilon_1 \cos\left(\frac{\theta}{2}+\pi\right), \epsilon_1 \sin\left(\frac{\theta}{2}+\pi\right)\right) = \left(-\epsilon_1 \cos\frac{\theta}{2}, -\epsilon_1 \sin\frac{\theta}{2}\right)$$

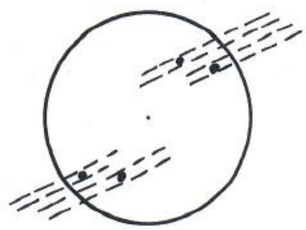
$$= -\left(\epsilon_1 \cos\frac{\theta}{2}, \epsilon_1 \sin\frac{\theta}{2}\right)$$

so the two centers are $2\,\epsilon_1$ apart and the image discs have radius ϵ_2.

The image of the second iterate of f, in the Disc $D(\theta)$ then looks like :



In the third dimension there are lines coming out and twisting around two times. Near $D(\theta)$ the picture lookds like :



and the four strands braid around to make one tube.

Returning to the disc $D(\theta)$, we may write polar coordinates in these discs as well. $D(\theta) = \{(\theta, \bar{r}, s) \mid \bar{r}^2 + s^2 + 1\}$. The first image of f in $D(\theta)$ is the two discs centered at $\left(\frac{\theta}{2}, \epsilon_1\right)$, $\left(\frac{\theta}{2} + \pi, \epsilon_1\right)$.

For the second iterate of f, f^2, the image of f^2 in $D(\theta)$ can be found first by taking the image of f in $D\left(\frac{\theta}{2}\right)$ and $D\left(\frac{\theta}{2} + \pi\right)$, i.e. the ϵ_2 discs centered at $\left(\frac{\theta}{4}, \epsilon_1\right)$ and $\left(\frac{\theta}{4} + \pi, \epsilon_1\right)$ in $D\left(\frac{\theta}{2}\right)$ and the ϵ_2 discs centered at $\left(\frac{\theta}{4} + \frac{\pi}{2}, \epsilon_1\right)$ and $\left(\frac{\theta}{4} + 3\,\frac{\pi}{2}, \epsilon_1\right)$ in $D\left(\frac{\theta}{2} + \pi\right)$. Applying f we get 4 discs in $D(\theta)$ of radius ϵ_1^2 centered at :

$$\left(\frac{\theta}{2}, \epsilon_1\right) + \epsilon_2\left(\frac{\theta}{4}, \epsilon_1\right), \left(\frac{\theta}{2}, \epsilon_1\right) + \epsilon_2\left(\frac{\theta}{4} + \pi, \epsilon_1\right)$$

$$\left(\frac{\theta}{2} + \pi, \epsilon_1\right) + \epsilon_2\left(\frac{\theta}{4} + \frac{\pi}{2}, \epsilon_1\right), \left(\frac{\theta}{2} + \pi, \epsilon_1\right) + \epsilon_2\left(\frac{\theta}{4} + 3\,\frac{\pi}{2}, \epsilon_1\right).$$

If we ignore the ϵ's and just keep track of the θ's we may do this as follows. For the first iterate we write pairs $(\theta_1, \theta_2) \in S^1 \times S^1$ such that

$2\theta_2 = \theta_1$. For the second iterate we write triples $(\theta_1, \theta_2, \theta_3) \in S^1 \times S^1 \times S^1$ such that $2\theta_3 = \theta_2$ and $2\theta_2 = \theta_1$. Or given θ we have the two pairs $\left(\theta, \frac{\theta}{2}\right)$ and $\left(\theta, \frac{\theta}{2} + \pi\right)$ and the four triples $\left(\theta, \frac{\theta}{2}, \frac{\theta}{4}\right)$, $\left(\theta, \frac{\theta}{2}, \frac{\theta}{4} + \pi\right)$, $\left(\theta, \frac{\theta}{2} + \pi, \frac{\theta}{4} + \frac{\pi}{2}\right)$, $\left(\theta, \frac{\theta}{2} + \pi, \frac{\theta}{4} + 3\,\frac{\pi}{2}\right)$. These centers tell us a succession of discs each one contained in the one before with size reduced by a factor of ϵ_2. If we continue we get infinite sequences

$$(\theta_1, \theta_2, \ldots, \theta_k, \theta_{k+1}, \ldots) \in S^1 \times \ldots \times S^1 \times S^1 \times \ldots$$

such that $2\theta_{k+1} = \theta_k$ for $k > 1$. These infinite sequences specify an infinite sequence of ever smaller discs and consequently each infinite sequence specifies exactly one point in $\cap f^n(\mathrm{R})$. We may identify the set $\cap f^n(\mathrm{R})$ with all the infinite sequences above, which is called the solenoid. In each disc $\mathrm{D}(\theta)$ we can see that this solenoid leaves a Cantor set as intersection.

The set of infinite sequences

$(\theta_1, \theta_2, \ldots \theta_k, \theta_{k+1}, \ldots) \in S^1 \times \ldots \times S^1 \times S^1 \times \ldots$ such that $\theta_k = 2\theta_{k+1}$, is also called the inverse limit of the map $\theta \to 2\theta$ and f under this identification becomes

$$(\theta_1, \theta_2, \ldots \theta_k, \theta_{k+1}, \ldots) \to (2\theta_1, \theta_1, \theta_2, \ldots \theta_k, \theta_{k+1}, \ldots)$$

which is a homeomorphism of this compact space. This is Williams' point of view. The thing that makes the attractor structurally stable is the expanding nature of the map on the circle part, $\theta \to 2\theta$ stretches distances locally, and the contracting nature of the map on the normal disc slices.

This contracting map on the normal disc slices leads to another observation. Let $g : \mathrm{R} \to \mathrm{E}^1$ be a real valued continuous function. If x, y are both on the same $\mathrm{D}(\theta)$ then

$$d(f^n(x), f^n(y)) \to 0 \quad \text{so} \quad |g f^n(x) - g f^n(y)| \to 0$$

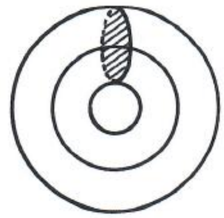
and

$$\frac{1}{n}\left| \sum_{i=0}^{n} g f^i(x) - \sum_{i=0}^{n} g f^i(y) \right| \to 0 .$$

One limit exists if and only if the other does. That is, if $x \in \mathrm{D}(\theta)$ and $\lim \frac{1}{n} \sum_{i=0}^{n-1} g f^i(x)$ exists then so does $\lim \frac{1}{n} \sum_{i=0}^{n-1} g f^i(y)$ for every $y \in \mathrm{D}(\theta)$.

The $\lim \frac{1}{n} \sum_{i=0}^{n-1} g f^i(x)$ are the time averages of the observable g for the discrete time process f. In the case of hyperbolic stable equilibria or hyperbolic stable periodic orbits the time averages converge to the value of the function at 0 or the integral around the periodic solution with respect to the normalized time parameterization on the periodic orbit.

In our case, if the function g were constant on the discs $D(\theta)$ the time averages would be exactly the time averages with respect to the transformation $\theta \to 2\theta$ on the circle. For this transformation the usual



measure is invariant and ergodic so the time averages are just the integral of g with respect to the usual measure. $\lim \frac{1}{n} \Sigma\, g f^i(\theta, \overline{r}, s) = \int g(\theta, \overline{r}, s)\, d\theta$ and the limit exists for almost every $(\theta, \overline{r}, s)$ in R.

Now if we let $\Lambda = \bigcap_{i=1}^{\infty} f^i R$ then we may define a measure μ on Λ as follows. Consider a continuous function $g : \Lambda \to E^1$. Let

$$\hat{g}(\theta) = \min_{(\theta, \overline{r}, s) \in D(\theta)} g(\theta, \overline{r}, s).$$

Let $\int g d\mu = \int \hat{g} d\theta$. Then μ is an invariant measure which is ergodic and $\lim \frac{1}{n} \sum_{i=0}^{n-1} g f^i(\theta, \overline{r}, s) = \int g d\mu$ for almost all $(\theta, \overline{r}, s)$ in R. This construction is related to the inverse limit construction above. This measure has entropy log 2 which happens to be the logarithm of the degree of the map $\theta \to 2\theta$, by no accident.

The map f has some fairly pleasant properties. A point $(\theta, \overline{r}, s) \in R$ will be said to have period n if $f^n(\theta, \overline{r}, s) = (\theta, \overline{r}, s)$. f has exactly 2^n points of period n, which one can see by observing that $D(\theta)$ must be mapped into itself by f^n for $(\theta, \overline{r}, s)$ to be periodic of period n and this disc is contracted, so it suffices to observe that $\theta \to 2\theta$ has 2^n points of period n. Not only does every nearby f' to f have 2^n periodic points of period 2^n, but every map $f' : R \to R$ homotopic to f must have at least 2^n periodic points of period n.

The general theory proceeds as follows. Suppose M is a compact manifold perhaps with boundary, or say a compact region in Euclidean space with smooth boundary and $f : M \to$ Interior M is a diffeomorphism.

If $\Lambda \subset M$ is an invariant set for f, that is $f(\Lambda) = \Lambda$, then we say that Λ has a hyperbolic structure if the tangent boundle of M. $TM|\Lambda$ may be written as a sum of two sub-bundles $E^S \oplus E^U$ where E^S and E^U are invariant by Tf the derivative of f and where Tf contracts E^S and expands E^U. More precisely, there are constants $C > 0$ and $\lambda < 1$ such that $\| Tf^n v \| \leqslant C\lambda^n \| v \|$ for any tangent vector $v \in E^S$ any $n > 0$ and $\| Tf^{-n} v \| \leqslant C\lambda^n \| v \|$ for any tangent vector $v \in E^U$ and any $n > 0$.

This formal definition corresponds to the expanding and contracting features of our example. Smale's Axiom A imposes a hyperbolic structure on a set of weakly recurrent points. A point $x \in M$ is called wandering if there is a neighborhood U of x in M such that $f^n(U) \cap U = \phi$ for some $n > 0$. The wandering set is open and invariant and its complement the non-wandering set, $\Omega(f)$, is a closed invariant set.

Axiom A : a) $\Omega(f)$ has a hyperbolic structure ;

b) the periodic points of f are dense in $\Omega(f)$. Under these hypotheses Smale proves :

Theorem. If $f : M \to M$ satisfies Axiom A then $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_k$ where Ω_i are closed and f invariant and each Ω_i contains dense orbits.

$W^S(\Omega_i)$ is defined to be those points $x \in M$ such that $f^n(x) \to \Omega_i$ as $n \to \infty$. That Ω_i be an attractor means that $W^S(\Omega_i)$ contains a neighborhood of Ω_i.

Theorem (Bowen). If f is a C^2 Axiom A diffeomorphism then Ω_i is an attractor if and only if the measure of $W^S(\Omega_i)$ in M is bigger than 0.

Thus every C^2 Axiom A diffeomorphism, f, must have an attractor and almost every point must tend to an attractor under iterates of f. Also there is a measure μ on a C^2 attractor Ω_i such that

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} gf^i(x) \to \int_{\Omega_i} gd_\mu$$

for almost every x in $W^S(\Omega_i)$ and any continuous g defined on $W^S(\Omega_i)$. This measure was constructed by Ruelle, and has interesting ergodic properties.

Given an Axiom A attractor Ω_i there is a manifold $M_i \subset M$ with dimension M_i = dimension M, $f(M_i) \subset$ Interior M_i and $\bigcap_{i=0}^{\infty} f^n(M_i) = \Omega_i$.

f is structurally stable on M_i. That is if $g : M_i \to M_i$ is C^2 close enough to f there is a homeomorphism $h : M_i \to M_i$ such that $gh = hf$. This result is contained in the works of Robbin and Robinson.

Similar theorems are known for vector fields, that is time independent ordinary differential equations. The study of time dependent time periodic equations is like the study of diffeomorphisms in the component of the identity. How the regions of attraction are divided requires studying the global features of the systems.

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DISCUSSION

The discussion, introduced by Ivar Ekeland, has singled out three main points.

First, the type of attractor described in the paper, which also exists for continuous flows, should not be considered as some untractable object : on the contrary it is rather pleasant because it carries an ergodic measure.

Next, if the economists have not yet encountered such attractors, for instance in optimal growth, it is because they mainly work in a two dimensional capital-labour framework ; then Poincaré's theorem : "any invariant compact set contains a fixed point or a closed orbit", excludes the type of situation peresented in the paper.

But, last remark, in higher dimensions, even the case where some dynamics are Hamiltonian does not exclude the appearance of the described situation.