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Recent Results about Stable Ergodicity

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Introduction

The ergodic theory of uniformly hyperbolic, or Axiom A, diffeomorphisms has been studied extensively, beginning with the pioneering work of Anosov, Sinai, Ruelle and Bowen [**An, Si, Rue, Bow**]. Much work in dynamical systems since then, including this paper, has been directed toward extending their results beyond Axiom A.

Three basic ways to relax the condition of uniform hyperbolicity on a diffeomorphism $f : M \to M$ are:

(a) *Nonuniform hyperbolicity.* Consider Lyapunov exponents and assume that none of them are zero. Most orbits are hyperbolic, but the hyperbolicity is nonuniform.

(b) *Partial hyperbolicity.* Permit some tangent directions on which Tf acts neutrally, but require that Tf be uniformly hyperbolic in some other tangent directions. No orbits need be completely hyperbolic, but all have uniformly hyperbolic parts.

(c) *Dominated splitting.* Keep a Tf-invariant splitting into two subbundles, but relax hyperbolicity to a domination condition: maximal expansion in one bundle is uniformly bounded by least expansion in the other.

Approach (a) is due to Pesin. For an introduction to the theory of nonuniform hyperbolicity (often called Pesin theory) we refer the reader to the survey by Barreira and Pesin in this volume. Approach (b) is due to Brin and Pesin, and is the one we follow here. Of course, it is possible to combine (a) and (b) to produce the concept of partial non-uniform hyperbolicity. Approach (c) is due to Mañé [**Mañ2**]. There is now a large and important body of work in this area. It is closely related to the results that we discuss, but lies a little outside the scope of this survey; we refer the reader to [**AlBonVi**], [**BonDí**], [**BonDíPuj**], [**BonVi**], [**Cas**], [**DíPujUr**], [**DíRo**], [**Do1**], [**He1**], [**He2**], [**PujSam**], [**Vi**].

Our central theme is, to quote [**PugSh5**], "that a little hyperbolicity goes a long way in guaranteeing stably ergodic behavior." By "a little hyperbolicity" we mean

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partial hyperbolicity, i.e. (b) above. More precisely there is a Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$, in which at least two of the subbundles are nontrivial, such that Tf uniformly expands all vectors in E^u and uniformly contracts all vectors in E^s, while vectors in E^c are neither contracted as strongly as any nonzero vector in E^s nor expanded as strongly as any nonzero vector in E^u. We believe that the following should be true:

CONJECTURE 0.1. [**PugSh5**, Conjecture 1] and [**PugSh6**, Conjecture 1] On any compact manifold, ergodicity holds for an open and dense set of C^2 volume preserving partially hyperbolic diffeomorphisms (provided E^u and E^s are both nontrivial).

This conjecture can be split into two parts using the concept of accessibility. If we have a partially hyperbolic diffeomorphism $f : M \to M$ with associated splitting $TM = E^u \oplus E^c \oplus E^s$, let us say that a point y is accessible from a point x if there is a C^1 path from x to y whose tangent vector always lies in $E^u \cup E^s$ and vanishes at most finitely many times. Accessibility is an equivalence relation. We say that f has the *accessibility property* if there is only one equivalence class. If every measurable set which is a union of equivalence classes has either full measure or measure zero, then f has the *essential accessibility property.* It is clear that accessibility implies essential accessibility.

CONJECTURE 0.2. [**PugSh5**, Conjecture 4] and [**PugSh6**, Conjecture 2] Accessibility holds for an open and dense subset of C^2 partially hyperbolic diffeomorphisms, volume preserving or not (provided E^u and E^s are both nontrivial).

CONJECTURE 0.3. [**PugSh6**, Conjecture 3] A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.

The paper presents the current evidence in favor of these conjectures. In Section 1, we introduce the concept of partial hyperbolicity and give examples of partially hyperbolic diffeomorphisms. In Section 2 we discuss the result of Pugh and Shub from [**PugSh6**], which is the closest approach so far to a general proof of Conjecture 0.3. Section 3 is devoted to a discussion of topological ideas which can be used to approach Conjecture 0.2. Section 4 surveys the applications that have been made of the Pugh-Shub theorem and these ideas. In particular, we will see that Conjecture 0.1 holds within several interesting families of examples. In Section 5, we discuss the pathological center foliation of the partially hyperbolic diffeomorphisms in [**ShWi2**]. In the last section, we raise a few questions, make some conjectures, and propose a vague idea of how one might proceed to prove them.

1. Partially hyperbolic diffeomorphisms

Throughout this paper, M is a compact, connected and boundaryless Riemannian manifold, and $f : M \to M$ is a C^2 diffeomorphism. We will focus here on the ergodic properties of f as it is perturbed inside $\mathrm{Diff}^2(M)$, the space of all C^2 diffeomorphisms of M. For a smooth measure μ on M, let $\mathrm{Diff}^2_\mu(M)$ be those f in $\mathrm{Diff}^2(M)$ that preserve μ. We say that $f \in \mathrm{Diff}^2_\mu(M)$ is *stably ergodic* if there is a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2_\mu(M)$ such that every $g \in \mathcal{U}$ is ergodic with respect to

μ.[1] More generally f is stably ... if there is a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2_\mu(M)$ such that every $g \in \mathcal{U}$ is ....

The first diffeomorphisms proven to be stably ergodic were Anosov diffeomorphisms [**An**]. Partial hyperbolicity is a weakening of the Anosov condition that allows for much more dynamical complexity, and yet, as we shall see, often provides enough structure for stable ergodicity. A diffeomorphism f is *partially hyperbolic* if the tangent bundle to M splits as a Tf-invariant sum:

$$TM = E^u \oplus E^c \oplus E^s,$$

with at least 2 of the subbundles in the sum nontrivial, and there exist constants $a < b < 1 < c < d$, and a Finsler structure $\|\cdot\|$ on M such that, for all $p \in M$ and all $v \in T_pM$,

$$\begin{aligned} v \in E^u(p) &\Rightarrow d\,\|v\| \le \|T_pfv\| \\ v \in E^c(p) &\Rightarrow b\,\|v\| \le \|T_pfv\| \le c\,\|v\| \\ v \in E^s(p) &\Rightarrow \|T_pfv\| \le a\,\|v\|. \end{aligned}$$

The bundles E^u and E^s are called the *stable* and *unstable* bundles, respectively, and E^c the *center* bundle for f. It is not necessary to assume these bundles are continuous in the definition; continuity follows from invariance and the growth conditions given above. A similar argument shows that the splitting changes continuously when f varies in the C^1 topology; see e.g. [**HiPugSh**]. Equivalent definitions of partial hyperbolicity are given by Brin and Pesin in [**BriPe1**] and by Hirsch, Pugh, and Shub in [**HiPugSh**].

Several classically-studied dynamical systems are partially hyperbolic systems, including time-t maps of Anosov flows, frame flows for negatively curved manifolds, and certain algebraic systems. We give here a list of examples. They are grouped into categories, which have some overlap.

Anosov diffeomorphisms. An Anosov diffeomorphism is by definition a partially hyperbolic diffeomorphism with a trivial center bundle. Every Anosov diffeomorphism is partially hyperbolic in at least three ways: first, the obvious way, with E^c trivial, E^u = the expanding bundle from the Anosov splitting, and E^s = the contracting bundle from the Anosov splitting; second, with E^s trivial, E^c = contracting bundle, and E^u = expanding bundle; third, with E^u trivial, E^c = expanding bundle, and E^s = contracting bundle. Some Anosov diffeomorphisms are even partially hyperbolic with *three* nontrivial invariant bundles; the center bundle can include invariant weakly stable or unstable directions. Unless otherwise specified, when we discuss Anosov diffeomorphisms, we will assume that it is the center bundle that is trivial. Anosov showed in 1967 that C^2 Anosov diffeomorphisms that preserve a smooth invariant measure μ are ergodic [**An**].

Time-t maps of Anosov flows. Closely related to the Anosov diffeomorphisms are the *time-t maps of Anosov flows.* Recall that a flow $\varphi_t : M \to M$ is *Anosov* if for every $t \neq 0$, the derivative $T\varphi_t$ leaves invariant a splitting $TM = E^u \oplus \langle\dot\varphi\rangle \oplus E^s$ with E^u and E^s uniformly expanded and contracted, respectively, by $T\varphi_t$. For every $t \neq 0$, the diffeomorphism φ_t is partially hyperbolic, with the

[1] We use the C^2 topology on $\mathrm{Diff}^2_\mu(M)$ to define stable ergodicity. We remark that, in most of the results discussed in this paper, properties that are open in $\mathrm{Diff}^r(M)$ are in fact C^1-open, and dense properties are C^r-dense.

center bundle tangent to the flow lines. Note that, because of the neutral direction $\langle\dot{\varphi}\rangle$, the time-$t$ map of an Anosov flow is not an Anosov diffeomorphism. Basic examples of Anosov flows are: geodesic flows on unit tangent bundles of closed Riemannian manifolds of negative sectional curvature, and special flows built over Anosov diffeomorphisms. Geodesic flows preserve Liouville measure, and a special flow built over an Anosov diffeomorphism f_0 preserves a smooth volume if f_0 does.

Anosov proved that for certain Anosov flows, for example the geodesic flows, the time-t map, for $t \neq 0$ is ergodic [**An**]. The precise condition that implies ergodicity of φ_t is essential accessibility, which we discuss in the next section.

Algebraic systems. Let G be a connected Lie group and let Γ be a discrete cocompact subgroup of G, i.e. a discrete subgroup such that G/Γ is compact. For $g \in G$ denote by $L_g : G/\Gamma \to G/\Gamma$ the left translation $L_g(a\Gamma) = ga\Gamma$. Let $A : G \to G$ be an automorphism such that $A(\Gamma) = \Gamma$; then A induces a diffeomorphism $A : G/\Gamma \to G/\Gamma$. An *affine diffeomorphism* of G/Γ is a map of the form

$$L_g \circ A : G/\Gamma \to G/\Gamma.$$

Since Γ is discrete and cocompact, Haar measure on G projects to a finite measure μ on G/Γ, invariant under left translations and under the action of A. Then the affine diffeomorphism $L_g \circ A$ preserves μ. Let $\mathfrak{g}$ be the Lie algebra of G and let $f = L_g \circ A : G/\Gamma \to G/\Gamma$, be an affine diffeomorphism. Then f induces the Lie algebra automorphism $l(f) : \mathfrak{g} \to \mathfrak{g}$:

$$l(f) = ad(g) \circ T_e(A),$$

where $ad(g) : \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of g on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ splits into generalized eigenspaces for $l(f)$:

$$\mathfrak{g} = \mathfrak{g}^u \oplus \mathfrak{g}^c \oplus \mathfrak{g}^s,$$

where the eigenvalues of $l(f)$ are respectively outside, on, or inside the unit circle.

The Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by $\mathfrak{g}^u \oplus \mathfrak{g}^s$ is an ideal in $\mathfrak{g}$. The following is proved[2] in [**PugSh6**].

PROPOSITION 1.1. *The affine diffeomorphism $f = L_g \circ A$ of G/Γ is partially hyperbolic if and only if $\mathfrak{h} \neq 0$.*

The ergodicity of L_g on a homogeneous space G/Γ was studied extensively in [**Mo**] and [**BreMo**]. It follows easily from their work that the condition $\mathfrak{h} = \mathfrak{g}$ implies the ergodicity of L_g on G/Γ.

Systems that fiber over partially hyperbolic diffeomorphisms. Let $f_0 : M_0 \to M_0$ be a partially hyperbolic diffeomorphism with splitting $TM_0 = E_0^u \oplus E_0^c \oplus E_0^s$, and let $\pi : M \to M_0$ be a fiber bundle over M_0. If $f : M \to M$ is a lift of f_0, then f is partially hyperbolic, provided that:

$$\|Tf|_{\ker(T\pi)}\| \cdot \|Tf_0^{-1}|_{E_0^u}\| < 1 \quad \text{and} \quad \|Tf^{-1}|_{\ker(T\pi)}\| \cdot \|Tf_0|_{E_0^s}\| < 1.$$

Indeed, the splitting $TM = E^u \oplus E^c \oplus E^s$ for f projects under $T\pi$ to the splitting $TM_0 = E_0^u \oplus E_0^c \oplus E_0^s$ for f_0. The stable and unstable bundles are found by a standard graph transform argument, and the center bundle is $T\pi^{-1}(E_0^c)$.

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[2] A more general proposition with Γ replaced by a closed cocompact subgroup appears in [**PugSh6**]. The arguments given there are incomplete in that generality, but are valid when the subgroup is discrete.

Falling into this category are direct products, compact group extensions of Anosov diffeomorphisms, and time-t maps of frame flows for negatively-curved manifolds.

Direct products. Let $f_0 : M_0 \to M_0$ be an Anosov diffeomorphism with associated splitting $TM_0 = E_0^u \oplus E_0^s$ and let $f_1 : M_1 \to M_1$ be a diffeomorphism of a Riemannian manifold. Suppose that no vector in TM_1 is contracted as strongly as any vector in E_0^s or expanded as strongly as any vector in E_0^u, i.e.

$$\|Tf_0^{-1}|_{E_0^s}\|^{-1} < \|Tf_1^{-1}\|^{-1} \leq \|Tf_1\| < \|Tf_0|_{E_0^u}\|.$$

Then $f_0 \times f_1$ is partially hyperbolic with $E^u = E_0^u \times \{0\}$, $E^c = \{0\} \times TM_1$, and $E^s = E_0^s \times \{0\}$. In this simple situation, the unstable and stable bundles through $(p_0, p_1) \in M_0 \times M_1$ are tangent to the submanifold $M_0 \times \{p_1\}$, and the center bundle is the tangent space to $\{p_0\} \times M_1$.

Compact group extensions of Anosov diffeomorphisms. Let G be a compact Lie group, and let $f_0 : M_0 \to M_0$ be an Anosov diffeomorphism. Each smooth function $\varphi : M_0 \to G$ defines a *skew product transformation* $f_\varphi : M_0 \times G \to M_0 \times G$ by the formula:

$$f_\varphi(p_0, g) = (f_0(p_0), \varphi(p_0)g).$$

Left translations are isometries of G in the bi-invariant metric, and so f_φ is partially hyperbolic. If f_0 preserves a smooth probability measure μ_0, then f_φ preserves the smooth probability measure $\mu = \mu_0 \times \nu_G$, where ν_G is (normalized) Haar measure on G. Skew products over f_0 are also called *G-extensions of f_0*. The set of all C^2 skew products over f_0,

$$\mathrm{Skew}(f_0, G) = \{f_\varphi \mid \varphi \in C^2(M_0, G)\},$$

is a closed subset of $\mathrm{Diff}^2_\mu(M_0 \times G)$; its connected components correspond to homotopy classes of φ.

The ergodic properties of such skew products were initially studied by Brin [**Bri2**]. He proved that ergodic diffeomorphisms form an open and dense subset of $\mathrm{Skew}(f_0, G)$ for any volume preserving Anosov diffeomorphism f_0 and any compact group G.

Frame flows. Let V be a closed oriented manifold of negative sectional curvatures, and let $M_0 = SV$ be the unit tangent bundle of V. Let M be the space of positively oriented orthonormal n-frames in TV. This gives a fiber bundle where the natural projection $\pi : M \to M_0$ takes a frame to its first vector. The associated structure group $SO(n-1)$ acts on fibers by rotating the frames, keeping the first vector fixed. In particular, we can identify each fiber M_x with $SO(n-1)$. Let $\hat{\varphi}_t : M \to M$ denote the frame flow, which acts on frames by moving their first vectors according to the geodesic flow and moving the other vectors by parallel transport along the geodesic defined by the first vector. The projection π is a semi-conjugacy from $\hat{\varphi}_t$ to the geodesic flow φ_t; i.e. $\pi \circ \hat{\varphi}_t = \varphi_t \circ \pi$ for each $t \in \mathbf{R}$. The frame flow $\hat{\varphi}_t$ preserves the measure that is (locally) a product of Liouville measure with (normalized) Haar measure on $SO(n-1)$. The time one map of the frame flow is a partially hyperbolic diffeomorphism. The center bundle has dimension $1 + \dim SO(n-1)$ and is spanned by the flow direction and the fiber direction. The map $\hat{\varphi}_1$ is an example of a principal $SO(n-1)$-bundle extension of φ_1.

Brin together with Gromov and Karcher [**Bri1, BriGr, BriK**] studied ergodicity of the frame flow. Ergodicity holds for an open and dense set of metrics that includes all metrics with constant negative curvature. The only known examples with nonergodic frame flow are very special, e.g. quotients of complex hyperbolic spaces. These results will be described in more detail in Section 4.

Anosov-like diffeomorphisms. Let $f_0 : M \to M$ be a partially hyperbolic diffeomorphism and let $f_t : M \to M$, for $t \in [0, 1]$, be an isotopy through diffeomorphisms. Suppose that during the isotopy the expansion of the unstable bundle and the contraction of the stable bundle remain stronger than any expansion or contraction in the center bundle. Then f_1 will also be partially hyperbolic. These conditions can be verified by the construction of suitable cone fields. This construction first appears in Shub's example of non-Anosov stably topologically transitive diffeomorphisms [**Sh**]. It has also been exploited in [**Mañ1, Carva, BonVi**]. In all of these cases f_0 is an Anosov diffeomorphism and the center bundle is a sum of invariant stable and unstable directions.

Perturbations of partially hyperbolic diffeomorphisms. If f is partially hyperbolic and the C^1 distance from f to g is small enough, then g is also partially hyperbolic. Thus, perturbations of the partially hyperbolic examples are partially hyperbolic. Since perturbations don't always preserve the categories we considered (e.g. a C^1 perturbation of an algebraic system is generally not algebraic), the ergodicity results mentioned above don't apply to all volume-preserving perturbations. What is known about the counterparts of these results for stable ergodicity will be described in Section 4.

A few remarks about these examples.

- Partially hyperbolic diffeomorphisms are generally not structurally stable (take the product of an Anosov diffeomorphism with any non-structurally stable map).
- On some manifolds, there are partially hyperbolic maps arbitrarily close to the identity map (consider the time-t maps of an Anosov flow as $t \to 0$).
- The decomposition of TM into stable, unstable, and center bundles is not necessarily unique (see the discussion of Anosov diffeomorphisms).
- The center bundle E^c is not always integrable — see [**Wi**] (while, as we explain in the next section, the unstable and stable bundles are always integrable). An open question is whether a bunching condition on the central part of the spectrum implies integrability.
- Volume-preserving partially hyperbolic maps are not always ergodic (take the product of a volume-preserving Anosov diffeomorphism with the identity map).
- There are partially hyperbolic diffeomorphisms with trivial stable bundles. Examples based on the derived from Anosov construction are described in [**BonVi**]. This paper also contains examples of stably ergodic diffeomorphisms that are not partially hyperbolic (although they do possess a dominated splitting).

2. Pugh and Shub's theorem

This section will give conditions under which a partially hyperbolic diffeomorphism is ergodic. The stability of these conditions will be considered in the next section.

The most important condition is essential accessibility, which was already defined in the introduction. The definition that we give here is equivalent to the earlier one.

If f is C^k partially hyperbolic, then its stable and unstable bundles are uniquely integrable and are tangent to foliations $\mathcal{W}^u$ and $\mathcal{W}^s$, whose leaves are C^k. These foliations are sometimes referred to as the *strong unstable* and *strong stable* foliations, especially if f is the time-t map of an Anosov flow.

A set is *us-saturated* if it is a union of unstable leaves and is also the union of stable leaves. The smallest us-saturated set that contains a given point is its *us-accessible set.* These accessible sets partition M. If the partition has just one element, M, we say that f has the *accessibility property.* If each measurable set that is the union of accessible sets has measure zero or one, we say that f has the *essential accessibility property.* Accessibility is widely used in control theory. As far as we know it was first used in the dynamical systems world by Brin and Pesin [**BriPe1**]. Their work was motivated in part by a paper of Sacksteder [**Sa**] where an infinitesimal version of the idea appears (in Theorem 6.2).

A simple description of accessibility can be given using us-paths. A *us-path* is a path $\psi : [0,1] \to M$ consisting of a finite number of consecutive arcs — called *legs* — each of which is a curve that lies in a single leaf of $\mathcal{W}^u$ or $\mathcal{W}^s$. The endpoints of the legs of the us-path will be called *corner points.* It is obvious that the set of points that can be reached from a given point by travelling along us-paths is exactly the us-accessible set of the point, as defined above. This means that f has the accessibility property if and only if any two points can be joined by a us-path. It is also clear that two points can be joined by a us-path if and only if they can be joined by a a C^1 path whose tangent vector always lies in $E^u \cup E^s$ and vanishes at most finitely many times. Thus the definition of accessibility given in the introduction coincides with the definition given here.

We now recall Conjecture 0.2 from the introduction.

CONJECTURE 2.1. [**PugSh6**, Conjecture 3] Let $f \in \mathrm{Diff}^2_\mu(M)$ where M is compact. If f is partially hyperbolic and essentially accessible, then f is ergodic.

Pugh and Shub have shown that this conjecture holds under some relatively mild technical hypotheses, which we describe below.

THEOREM 2.2. [**PugSh6**, Theorem A] *Let $f \in \mathit{Diff}^2_\mu(M)$ where M is compact. If f is center bunched, partially hyperbolic, dynamically coherent, and essentially accessible, then f is ergodic.*

Theorem 2.2 also holds when f is a C^2 diffeomorphism that preserves any measure equivalent to a smooth measure on M.

All of the ergodicity and stable ergodicity results in this paper are special cases or applications of this theorem. The special case when the central bundle is trivial is the classical result that a C^2 volume preserving Anosov diffeomorphism is ergodic [**An**]; the hypotheses of center bunching, dynamical coherence and essential accessibility are automatically satisfied in this case. Brin and Pesin [**BriPe1**] proved a version of Theorem 2.2 with the additional, very powerful, hypothesis that the central foliation is Lipschitz. The first case of Theorem 2.2 without such an assumption was proved by Grayson, Pugh and Shub in [**GrPugSh**]. This result was improved by Wilkinson [**Wi**] and Pugh and Shub [**PugSh5**], and reached its current form in [**PugSh6**].

Before proceeding, we discuss the technical conditions "center bunched" and "dynamically coherent".

A partially hyperbolic diffeomorphism f is *center bunched* if the action of Tf on the center bundle E^c is close enough to isometric, i.e., $\|T_pf|_{E^c}\|$ and $m(T_pf|_{E^c})$ are close enough to 1. (Here $m(A) = \|A^{-1}\|^{-1}$ denotes the conorm of the operator A.) The details can be found in section 4 of [**PugSh6**]. This property is immediately satisfied if Tf does act isometrically on E^c.[3]

A partially hyperbolic diffeomorphism f is *dynamically coherent* if:

(i) the distributions E^c, $E^c \oplus E^u$, and $E^c \oplus E^s$ are integrable, and everywhere tangent to foliations $\mathcal{W}^c$, $\mathcal{W}^{cu}$, and $\mathcal{W}^{cs}$, called the *center*, *center-unstable*, and *center-stable* foliations, respectively; and

(ii) $\mathcal{W}^c$ and $\mathcal{W}^u$ subfoliate $\mathcal{W}^{cu}$ while $\mathcal{W}^c$ and $\mathcal{W}^s$ subfoliate $\mathcal{W}^{cs}$.

This is the definition given in [**PugSh3**, §2]. It actually follows automatically from results in [**HiPugSh**] that the strong unstable and strong stable foliations sub-foliate $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ respectively. But since the center foliation may not be uniquely integrable, we cannot avoid explicitly assuming that $\mathcal{W}^c$ sub-foliates $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$. It follows from [**HiPugSh**, Theorem 7.6] that if the center foliation $\mathcal{W}^c$ exists and is C^1, then f is dynamically coherent, and in fact stably dynamically coherent as we shall see in Proposition 3.1.

2.1. E. Hopf's argument for ergodicity. Underlying many proofs of ergodicity for systems with hyperbolic behavior is an argument originally due to E. Hopf. This argument is simple yet powerful. Let X be a compact metric space, $f : X \to X$ be a homeomorphism, and μ be an f-invariant Borel probability measure.

We denote by $\mathcal{B}$ the Borel sigma algebra on X and by $\mathcal{T}$ the trivial subalgebra consisting of Borel sets which have measure zero or one. Given a subalgebra $\mathcal{A} \subset \mathcal{B}$, its *saturation with respect to sets of measure* 0 is

$$\mathrm{Sat}_0(\mathcal{A}) = \{B \in \mathcal{B} : \exists\, A \in \mathcal{A} \text{ with } \mu(\mathrm{A}\,\Delta\,\mathrm{B}) = 0\}.$$

Thus $\mathcal{T}$ is the saturation of the two element subalgebra consisting of X and the empty set.

Using the homeomorphism f we define two equivalence relations $\sim_u, \sim_s$ on X by

$$x \sim_u y \quad \Leftrightarrow \quad d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty$$

and

$$x \sim_s y \quad \Leftrightarrow \quad d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty.$$

Correspondingly, we have Borel subalgebras $\mathcal{U}, \mathcal{S}$ that consist of whole $\sim_u$ and whole $\sim_s$ equivalence classes. Note that the definitions are purely topological; it is not assumed that f is differentiable let alone partially hyperbolic. Also the convergence in the definitions of $\sim_u$ and $\sim_s$ is not required to be exponential. Of course, if f is partially hyperbolic, and x and y are in the same unstable (resp. stable) leaf, then $x \sim_u y$ (resp. $x \sim_s y$). But there are many partially hyperbolic maps for which the $\sim_u$ and $\sim_s$ equivalence classes are larger than the unstable and stable leaves respectively. A trivial example is the case where E^c is the sum of a weakly expanding and a weakly contracting subbundle.

THEOREM 2.3. *If* $\mathrm{Sat}_0(\mathcal{U}) \cap \mathrm{Sat}_0(\mathcal{S}) = \mathcal{T}$ *then* f *is ergodic.*

[3]An incorrect description of center bunching is given in [**BuPugWi, BuWi, ShWi1**]. The statements made there that center bunching holds when $\mu_c = 1$ or E^c is one dimensional are false.

Hopf's proof of Theorem 2.3 uses the Birkhoff Ergodic Theorem. We summarize various versions of it in the next theorem.

THEOREM 2.4. *Let $\phi : X \to \mathbf{R}$ be in $L^1(X, \mu)$.*

a) $B_+(\phi)(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ *exists a.e.*
b) $B_-(\phi)(x) = \lim_{n\to-\infty} \frac{1}{|n|} \sum_{j=n+1}^{0} \phi(f^j(x))$ *exists a.e.*
c) $B_+(\phi) = B_-(\phi)$ *a.e.*
d) B_+ *and* B_- *define continuous operators on* $L^1(X, \mu)$.

In order to prove Theorem 2.3, it suffices to show that the image of B_+, or equivalently of B_-, consists of functions which are a.e. constant. Since $C^0(X)$, the space of continuous, real-valued functions, is dense in $L^1(X, \mu)$, it is enough to show that $B_+(C^0(X))$ and $B_-(C^0(X))$ are contained in the constant functions. For this we have a lemma.

LEMMA 2.5. *Let ϕ be continuous.*

a) *If* $x \sim_s y$ *then* $B_+(\phi)(x) = B_+(\phi)(y)$.
b) *If* $x \sim_u y$ *then* $B_-(\phi)(x) = B_-(\phi)(y)$.

PROOF. We prove a) only. We claim that if $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ exists, then so does $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(y))$, and they are equal.

Given $\epsilon > 0$, there exists $\delta > 0$ such that if $d(w, z) < \delta$, then $|\phi(x) - \phi(y)| < \epsilon$. Since $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$, it follows that there is an N such that for $M \geq N$,

$$\frac{1}{M}\Big|\sum_{j=0}^{M-1}(\phi(f^j(x)) - \phi(f^j(y)))\Big| < 2\epsilon.$$

We are done. □

Now we prove Theorem 2.3.

PROOF. Suppose that there is a continuous function ϕ such that $B_+(\phi)$ is not a.e. constant. For a real number r, let

$$\begin{aligned} G_+(r) &= \{x \in X : \text{ the limit } B_+(\phi)(x) \text{ exists and } B_+(\phi)(x) < r\} \\ G_-(r) &= \{x \in X : \text{ the limit } B_-(\phi)(x) \text{ exists and } B_-(\phi)(x) < r\}. \end{aligned}$$

They are f-invariant Borel sets that belong to $\mathcal{S}$ and $\mathcal{U}$ respectively, and which by Birkhoff's Theorem differ by a zero set. Thus

$$G(r) = G_-(r) \cap G_+(r) \in \mathrm{Sat}_0(\mathcal{U}) \cap \mathrm{Sat}_0(\mathcal{S}).$$

Since $B_+(\phi)$ is not constant a.e. there is a real number r such that

$$0 < \mu\{x \in X \mid B_+(\phi)(x) < r\} < 1,$$

and so $G(r) \notin \mathcal{T}$, which is a contradiction. It follows that after all $B_+(\phi)$ is a.e. constant. This is true for all continuous ϕ and hence for all integrable ϕ, which gives ergodicity of f. □

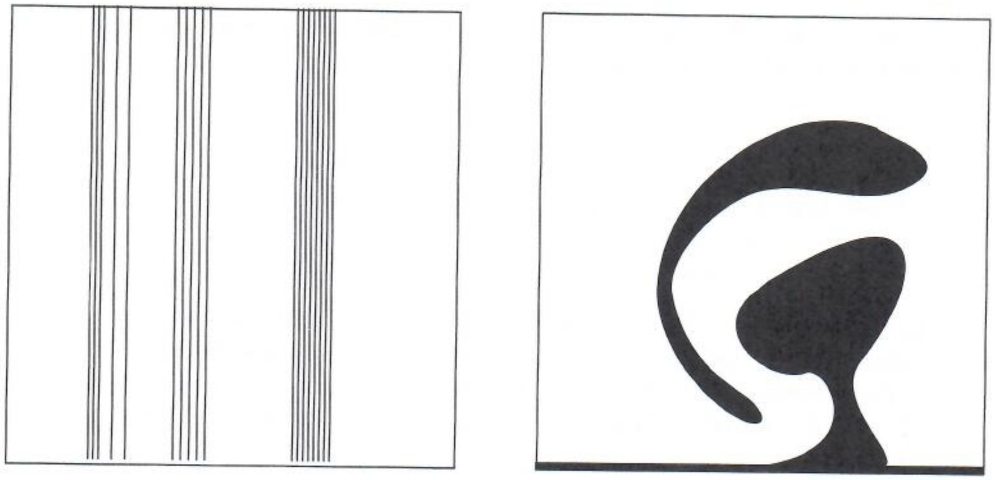

FIGURE 1. Sets in the the Borel subalgebras $\mathcal{A}$ and $\mathcal{C}$.

2.2. Ergodicity of partially hyperbolic systems. To prove the ergodicity of a volume preserving partially hyperbolic diffeomorphism f with the essential accessibility property would now seem to be a simple matter. By Hopf's argument in Theorem 2.3, it suffices to show that each f-invariant Borel set which, up to a zero set, is the union of whole strong unstable manifolds and which is also, up to a possibly different zero set, the union of whole strong stable manifolds, has measure zero or one. But there is a problem. Essential accessibility gives a different, weaker, and hence more easily satisfied condition. It requires the two zero sets to be *the same.*

In terms of subalgebras this can be expressed as follows. Let $\mathcal{M}^u$ and $\mathcal{M}^s$ be the subalgebras of Borel sets that consist of whole strong unstable and whole strong stable leaves respectively. Essential accessibility gives

$$\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}} \cap \mathcal{M}^{\mathrm{s}}) = \mathcal{T}. \tag{2.1}$$

Since $\mathcal{U} \subset \mathcal{M}^u$ and $\mathcal{S} \subset \mathcal{M}^s$, Hopf's hypothesis, $\mathrm{Sat}_0(\mathcal{U}) \cap \mathrm{Sat}_0(\mathcal{S}) = \mathcal{T}$, is implied by

$$\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}}) = \mathcal{T}. \tag{2.2}$$

Clearly the saturate of an intersection is contained in the intersection of the saturates; hence $\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}} \cap \mathcal{M}^{\mathrm{s}}) \subset \mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}})$ and

$$(2.2) \Rightarrow (2.1).$$

The fact that the converse is false is shown by the following example.

EXAMPLE 2.6. Let X be the unit square I^2 equipped with Lebesgue measure, and let $\mathcal{A}$ be the Borel subalgebra of sets of whole vertical segments $x \times I$, while $\mathcal{C}$ is the Borel subalgebra of sets that either contain the horizontal $I \times 0$ or are disjoint from it.

One easily checks that $\mathrm{Sat}_0(\mathcal{C}) = \mathcal{B}$, while $\mathcal{A} \cap \mathcal{C} = \{\emptyset, X\}$. Thus

$$\mathrm{Sat}_0(\mathcal{A} \cap \mathcal{C}) = \mathcal{T} \neq \mathcal{A} \subset \mathrm{Sat}_0(\mathcal{A}) = \mathrm{Sat}_0(\mathcal{A}) \cap \mathrm{Sat}_0(\mathcal{C}).$$

In essence, what Pugh and Shub show in [**PugSh6**] is that under the additional hypotheses on f of dynamical coherence and center bunching,

$$(2.1) \Rightarrow (2.2),$$

so Hopf's idea works after all. Namely, they prove

THEOREM 2.7. *If f is C^2, partially hyperbolic, dynamically coherent, and center bunched then*

$$\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}}) \subset \mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}} \cap \mathcal{M}^{\mathrm{s}}).$$

Equality of the two sets under the hypotheses of the theorem is an immediate corollary, and we obtain

COROLLARY 2.8. *If f is C^2, volume-preserving, partially hyperbolic, dynamically coherent, center bunched, and essentially accessible then f is ergodic.*

Proof. Since $\mathcal{U} \subset \mathcal{M}^u$ and $\mathcal{S} \subset \mathcal{M}^s$, we have

$$\mathrm{Sat}_0(\mathcal{U}) \cap \mathrm{Sat}_0(\mathcal{S}) \subset \mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}}) = \mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}} \cap \mathcal{M}^{\mathrm{s}}),$$

which, by essential accessibility, is $\mathcal{T}$. Then Theorem 2.3 implies that f is ergodic. □

REMARK 2.9. This argument applies whenever f preserves a measure equivalent to a smooth measure.

2.3. The Kolmogorov Property. The conclusion of Theorem 2.3 can be strengthened when f is partially hyperbolic. An invertible μ-preserving map $f : X \to X$ has the *Kolmogorov property* if there is a sub-σ-algebra $\mathcal{A}$ of the Borel σ-algebra $\mathcal{B}$ such that $f^{-1}\mathcal{A} \subseteq \mathcal{A}$, $\bigcup_{n=-\infty}^{\infty} f^{-n}\mathcal{A}$ generates $\mathcal{B}$, and $\bigcap_{n=0}^{\infty} f^{-n}\mathcal{A}$ is the trivial σ-algebra $\mathcal{T}$. An equivalent definition is that f has the Kolmogorov property if and only if f has no nontrivial factors of zero entropy. A diffeomorphism f that has the Kolmogorov property is also called a *K-system*. K-systems are mixing of all orders. See [**Par1**].

THEOREM 2.10. *Let f be partially hyperbolic. If $\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}}) = \mathcal{T}$ then f has the Kolmogorov property.*

Proof: Let $\mathcal{P}$ be the Pinsker subalgbra, the largest subalgebra for which $f|_{\mathcal{P}}$ has zero entropy. According to Proposition 5.1 of [**BriPe1**], if f is partially hyperbolic, then $\mathcal{P}$ is contained mod 0 in $\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}})$. If $\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}}) = \mathcal{T}$, then $\mathcal{P}$ is trivial, and f is a K-system. □

Theorem 2.10 implies the following strengthening of Corollary 2.8.

COROLLARY 2.11. *If f is C^2, volume-preserving, partially hyperbolic, dynamically coherent, center bunched, and essentially accessible, then f has the Kolmogorov property.*

Notice that the hypothesis of Theorem 2.10 is $\mathrm{Sat}_0(\mathcal{M}^{\mathrm{u}}) \cap \mathrm{Sat}_0(\mathcal{M}^{\mathrm{s}}) = \mathcal{T}$, rather than the (possibly weaker) condition of Hopf that $\mathrm{Sat}_0(\mathcal{U}) \cap \mathrm{Sat}_0(\mathcal{S}) = \mathcal{T}$. This is because the results in [**BriPe1**] which we use in the proof apply only to partially hyperbolic maps. We do not know if there is a general proof along the lines of the Hopf argument that $\mathrm{Sat}_0(\mathcal{U}) \cap \mathrm{Sat}_0(\mathcal{S}) = \mathcal{T}$ implies the Kolmogorov property. The closest approach is provided by Theorem B of [**LeYou**]. This theorem applies to any C^2 diffeomorphism and any invariant Borel probability measure. It says that $\mathcal{P} = \mathrm{Sat}_0(\mathcal{B}^u) = \mathrm{Sat}_0(\mathcal{B}^s)$, where $\mathcal{P}$ is the Pinsker subalgebra, and $\mathcal{B}^s$ and $\mathcal{B}^u$ are the subalgebras of Borel sets that are unions of entire Pesin stable and unstable manifolds respectively. Note that $\mathcal{S} \subset \mathcal{B}^s$ and $\mathcal{U} \subset \mathcal{B}^u$, but the containments may be proper. Theorem 2.10 is a special case of this result because $\mathcal{B}^s \subset \mathcal{M}^s$ and $\mathcal{B}^u \subset \mathcal{M}^u$ when f is partially hyperbolic.

It is natural to ask whether the Kolmogorov property can be replaced by the Bernoulli property in Corollary 2.11. We suspect that the answer is "no", but have not been able to adapt any of the established methods for constructing examples that are K but not Bernoulli in order to prove this. Other stochastic properties have been studied by Dolgopyat in [**Do3**].

2.4. Absolute continuity. Even in the case when the center bundle is trivial and f is an Anosov diffeomorphism, the proof of Theorem 2.2 uses a major idea: absolute continuity of foliations.

Let $\mathcal{W}$ be a topological foliation with smooth d dimensional leaves in an n dimensional Riemannian manifold. A transversal to $\mathcal{W}$ is a smooth submanifold with dimension $n - d$ that intersects all leaves of $\mathcal{W}$ transversely. If T_1 and T_2 are two nearby transversals, there is a natural map, known as the holonomy or Poincaré map, defined by moving from T_1 to T_2 along short paths in the leaves of $\mathcal{W}$. Let μ_1 and μ_2 be the measures on T_1 and T_2 by the Riemannian structure. The foliation $\mathcal{W}$ is *absolutely continuous* if the measure μ_1^* obtained by using the holonomy map to transport μ_1 from T_1 to T_2 is absolutely continuous with respect to μ_2 for any choice of T_1 and T_2. We say that $\mathcal{W}$ is absolutely continuous *with bounded jacobians* if in addition the Radon-Nikodym derivative $d\mu_1^*/d\mu_2$ is always essentially bounded on compact subsets of T_2.

It is easily seen that a Lipschitz foliation is absolutely continuous with bounded jacobians. But continuous foliations are usually not absolutely continuous, no matter how smooth their leaves are. One of the fundamental properties of partially hyperbolic diffeomorphisms, first obtained by Anosov [**An**] in the uniformly hyperbolic context and independently by Brin and Pesin [**BriPe1**] and Pugh and Shub [**PugSh2**] in the partially hyperbolic setting, is:

THEOREM 2.12. *The stable and unstable foliations of a C^2 partially hyperbolic diffeomorphism are absolutely continuous with bounded jacobians.*

Ergodicity of C^2 volume preserving Anosov diffeomorphisms is an easy consequence of this theorem and the Hopf argument. The general proof of Theorem 2.2, however, uses a second major idea, namely juliennes and julienne density points, which we describe in the next subsection.

The main difficulty is that the center foliation $\mathcal{W}^c$, in contrast to $\mathcal{W}^u$ and $\mathcal{W}^s$, is not always absolutely continuous. Explicit examples of the this phenomenon have been constructed. They are described in Section 5. In the appendix to this paper we give a proof of Theorem 2.2 under the additional assumption that $\mathcal{W}^c$ is absolutely continuous.[4]

[4]The argument in the appendix is a corrected version of the "first false proof" in [**GrPugSh**]. The julienne argument in the next subsection is a corrected version of the "second false proof" in [**GrPugSh**].

2.5. Juliennes and julienne quasi-conformality. As we saw in Section 2.2, proving Theorem 2.2 reduces to proving Theorem 2.7. The proof of Theorem 2.7 proceeds via julienne density points. It is carried out in [**PugSh6**]. We give here a brief indication of what is involved.

Given a smooth Riemannian metric on M and a measurable subset X of M, a point $y \in M$ is a *density point* of X if

$$\frac{\mu(D(y,r)\cap X)}{\mu(D(y,r))} \to 1 \qquad \text{as } r \to 0.$$

Here $D(y,r)$ is the ball of radius r around y in M. Let $D(X)$ be the set of density points of X.

The Lebesgue-Vitali theorem says that $D(X) = X \bmod 0$. We would like to assert that if X is mod 0 a union of strong stable manifolds, then $D(X)$ is a union of whole strong stable manifolds and similarly that if X is mod 0 a union of strong unstable manifolds, then $D(X)$ is a union of whole strong unstable manifolds. This would prove Theorem 2.7.

The trouble is that strong stable and strong unstable holonomy maps don't quasi preserve the family of discs; the distortion in shape produced by the holonomy can become worse and worse as the size of the disc shrinks. This makes it difficult to conclude that y is a density point of X if z is a density point of X for some $z \in W^s(y)$ or $W^u(y)$.

Instead of using discs to define density points, we define for each $p \in M$ a family of neighborhoods of p, $J_n(p)$, for $n \in \mathbf{N}$. They are called juliennes because their shapes are reminiscent of julienned vegetables. The juliennes are dynamically defined so that they are not too badly distorted by the holonomy maps; the long thin shape is essential for this purpose. For each p we have $J_1(p) \supset J_2(p) \supset \cdots$ and $\operatorname{diam} J_n \to 0$ exponentially as $n \to \infty$. As $n \to \infty$ the shape of $J_n(p)$ changes: it becomes relatively longer in the center direction and relatively shorter in the stable and unstable directions.

The volume of $J_n(p)$ shrinks as n increases in a uniform way. For each $j \in \mathbf{N}$

$$\mu(J_n(p))/\mu(J_{n+j}(p)) \textit{ is uniformly bounded}$$

for all $p \in M$ and, more significantly, for all $n \in \mathbf{N}$.

A point $y \in M$ is a *julienne density point* of a measurable set $X \subset M$ if

$$\frac{\mu(J_n(y)\cap X)}{\mu(J_n(y))} \to 1 \qquad \text{as } n \to 0.$$

We let $D_J(X)$ be the set of julienne density points of X. The family of juliennes forms what is called a differentiation basis. An analogous differentiation basis that defines density points in the ordinary sense is the family of discs $D_n(p) = D(p, \delta^n)$, for $p \in M$ and $n \in \mathbf{N}$, where $\delta < 1$ is a constant.

The analogue of the Lebesgue-Vitali theorem holds for julienne density points: $D_J(X) = X \bmod 0$ for any measurable X. This fact is non-trivial because in general there are counter-examples to the prevalence of density points when one strays too far from the use of round or quasi-round sets like balls and cubes as differentiation bases. The main step in proving Lebesgue-Vitali for juliennes is to show that there is a constant $l \in \mathbf{N}$ such that (for all $p, q \in M$ and $n \in \mathbf{N}$)

$$J_{n+l}(p) \cap J_{n+l}(q) \neq \emptyset \quad \Rightarrow \quad J_{n+l}(q) \subset J_n(p).$$

The corresponding statement for the discs $D_n(p)$ is easy to prove and implies the usual Lebesgue-Vitali theorem in the same way that this statement implies the prevalence of julienne density points.

Unlike ordinary density points, julienne density points of a set X are preserved under strong stable (resp. strong unstable) holonomy provided that X is mod 0 a union of complete strong stable (resp. strong unstable) manifolds. The basis for this is the following quasi-conformality property, which we formulate in the case of strong unstable holonomy (the stable case is completely analogous). Let $J_n^{cs}(p)$ be the component of p in $J_n(p) \cap \mathcal{W}^{cs}(p)$. Suppose $q \in \mathcal{W}^u(p)$ and there is a curve from p to q with length at most 1 that lies in $\mathcal{W}^u(p)$. Let $\theta : \mathcal{W}^{cs}(p) \to \mathcal{W}^{cs}(p)$ be the holonomy along leaves of $\mathcal{W}^u$. Then there is a $k \in \mathbf{N}$ such that for all large enough n we have

$$J_{n+k}^{cs}(q) \subset \theta(J_n^{cs}(p)) \subset J_{n-k}^{cs}(q).$$

The constant k can be chosen so that this holds for all p and q with the above properties. The corresponding property does not hold for the discs $D_n(p)$.

In summary, the following is proved in [**PugSh6**]:

THEOREM 2.13. *(a)* $D_J(X) = X$ mod 0 *for any measurable set* X.

(b) If X *is* mod 0 *a union of strong stable manifolds, then* $D_J(X)$ *is a union of whole strong stable manifolds.*

(c) If X *is* mod 0 *a union of strong unstable manifolds, then* $D_J(X)$ *is a union of whole strong unstable manifolds.*

Theorem 2.7 is an immediate corollary of this theorem and, as we saw earlier, Theorem 2.2 follows from Theorem 2.7.

Finally a few words about the juliennes. Note that a homeomorphism within ϵ of the identity may violently distort the shape of small discs, but does not distort large discs. The image of a disc of radius 1 is contained in a disc of radius $1+\epsilon$ and contains a disc of radius $1-\epsilon$. Suppose we are interested in holonomy at distance 1 along a strong unstable leaf. Here is a very simplified version of what is done.

Let p, q be points of a common strong unstable leaf such that their distance apart along that leaf is 1. See Figure 2.

Choose n large so that the distance $d(f^{-n}(p), f^{-n}(q))$ is small. Take sets in $W^{cs}(f^{-n}(p))$, $W^{cs}(f^{-n}(q))$ which are small but large compared to this distance and such that the unstable holonomy quasi preserves their shape. These sets are essentially small product rectangles in the center stable leaves at $f^{-n}(p)$ and $f^{-n}(q)$ with respect to the sub-foliations of the center stable leaves by the center and strong stable foliations. See Figure 3.

Then we define the n'th center stable juliennes at p and q, $J_n^{cs}(p)$ and $J_n^{cs}(q)$, as the image of these sets. See Figure 4.

By construction their "shapes" are quasi preserved. Do the same for $W^{cu}(p)$ to define $J_n^{cu}(p)$. Then define $J_n(p)$ to be the "product" of $J_n^{cu}(p)$ and $J_n^{cs}(p)$.

3. Stable accessibility and stable ergodicity

Theorem 2.2 gives stable ergodicity in any situation where its hypotheses are stably satisfied, i.e. they hold in an open subset of $\mathrm{Diff}^2_\mu(M)$. We now consider what happens to the hypotheses when f is perturbed.

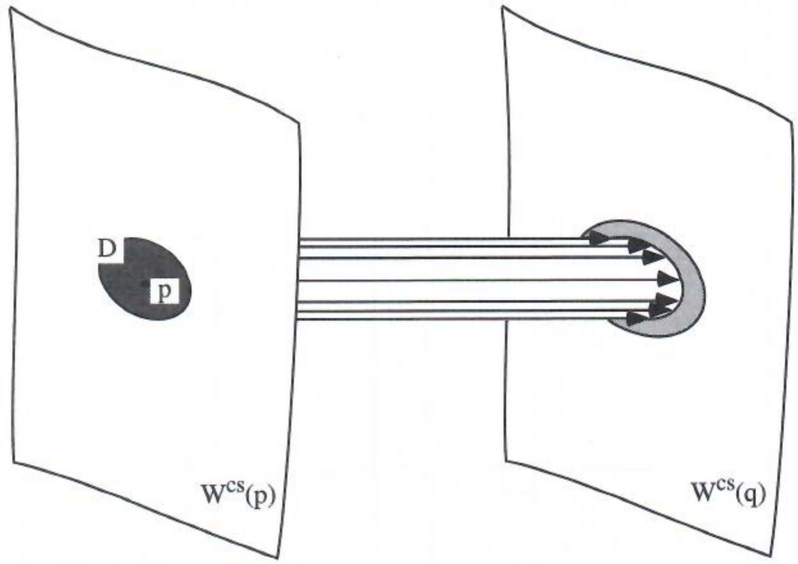

FIGURE 2. Unit holonomy along the strong unstable foliation.

Partial hyperbolicity is an open property in the C^1 topology on diffeomorphisms of M, and so any diffeomorphism g of M that is sufficiently C^1-close to the partially hyperbolic diffeomorphism f has stable and unstable foliations $\mathcal{W}_g^u$ and $\mathcal{W}_g^s$.

The property of center bunching is obviously C^1 open.

We suspect that dynamical coherence is also always C^1 open. But the state of the art is:

PROPOSITION 3.1. [**PugSh5**, Proposition 2.3] *If the center foliation $\mathcal{W}^c$ exists and is of class C^1, then f is dynamically coherent, as is any g close enough to f in the C^1 topology.*

Combining this proposition with Theorem 2.2 and the above remarks gives:

THEOREM 3.2. *Let $f \in Diff^2_\mu(M)$ where M is compact. Suppose f is*

(i) *partially hyperbolic,*
(ii) *center bunched, and*
(iii) *f has a central foliation and this foliation is C^1.*

If f is stably essentially accessible, then f is stably ergodic.

Another form of this result is useful in situations where one wishes to show that f is approximated by stably ergodic diffeomorphisms.

THEOREM 3.3. *If f satisfies (i)–(iii) and there are stably essentially accessible diffeomorphisms arbitrarily close to f, then f is in the closure of the stably ergodic diffeomorphisms.*

Applying these theorems reduces to verifying stable essential accessibility. We now discuss this point. Essential accessibility is not always stable. A skew product example that demonstrates this was constructed by Brin in the 1970's [**Bri1**, end of §2]. It will be described in more detail at the end of this section.

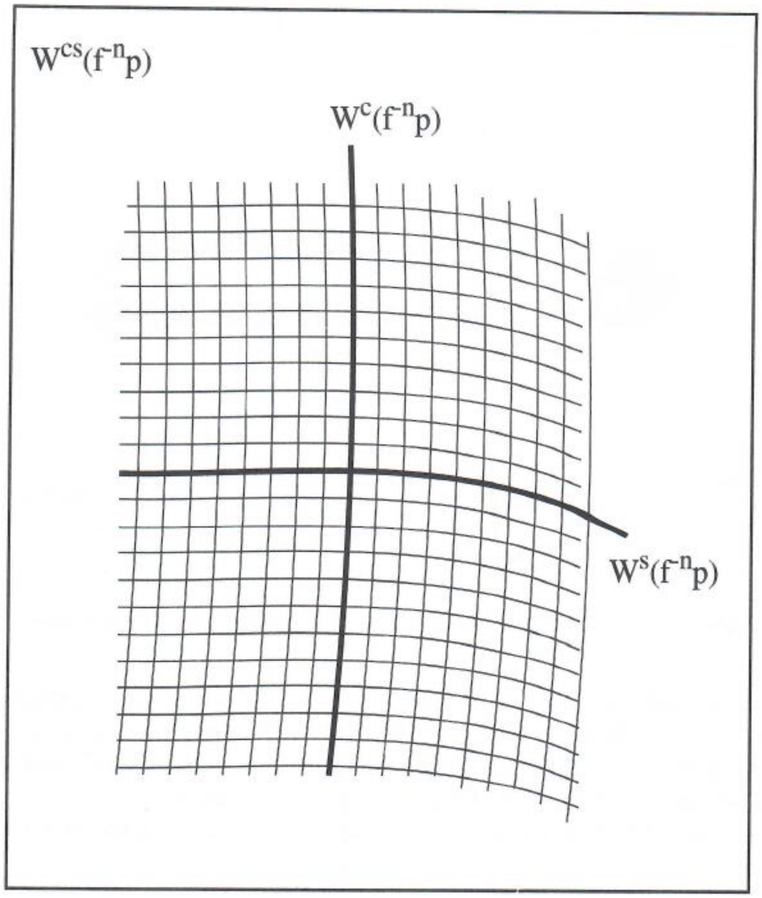

FIGURE 3. The center-stable julienne at the point $f^{-n}(p)$.

Essential accessibility is implied by accessibility, a topological property, whose stability can be verified with the use of topological tools. It is possible that accessibility is always stable, but this is an open question. There is one important but very exceptional case in which stability is automatic:

PROPOSITION 3.4. [**PugSh5**, Corollary 4.2] *If $\mathcal{W}^u$ and $\mathcal{W}^s$ are C^1 foliations and have the accessibility property, then f is stably accessible.*

Even if accessibility is not always stable, it seems quite likely that it should be a dense property:

CONJECTURE 3.5. [**PugSh5**, Conjecture 4] Stable accessibility should hold for a dense set of C^2 partially hyperbolic diffeomorphisms, volume preserving or not.

This is just a reformulation of Conjecture 0.3. In the case when the leaves of the center foliation $\mathcal{W}^c$ are one dimensional, the conjecture has been proved by Niţică and Török [**NiTö**] under some mild additional assumptions; see Section 5. The conjecture has been proven in full generality by Dolgopyat [**Do2**] in the case when E^u, E^c and E^s are all one dimensional; in particular he does not assume

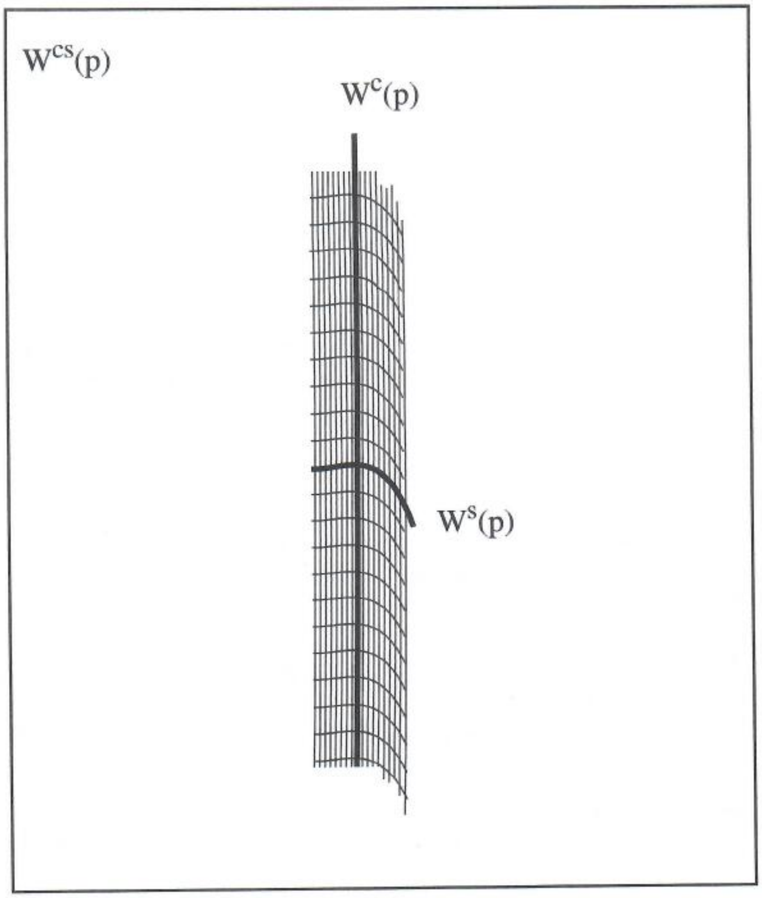

FIGURE 4. The center-stable julienne at the point p.

dynamical coherence. As we shall see in Section 4, the conjecture holds within several interesting special classes of maps.

We now describe some topological ideas which have been useful in verifying stable accessibility. This discussion is based on [**BuPugWi**] and [**BuWi**, §5].

Recall that the accessibility property means that any two points can be joined by a us-path. This leads us to consider conditions under which the fact that there is a us-path from a point p_0 to a point q_0 will be preserved under C^1 small changes of the map, in other words under small perturbations of the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$. It is enough to have a continuous, dim M-parameter family of us-paths emanating from p_0 whose endpoints surround q_0. One way to make this idea precise is the following:

DEFINITION 3.6. A point q_0 can be *engulfed* from a point p_0 if there is a continuous map $\Psi : Z \times [0, 1] \to M$ such that:

(1) Z is a compact, connected, orientable, manifold with boundary (e.g. a disc) whose dimension is $n = \dim M$.
(2) For each $z \in Z$, the curve $\psi_z(\cdot) = \Psi(z, \cdot)$ is a us-path with $\psi_z(0) = p_0$.
(3) There is a constant C such that every path ψ_z has at most C legs.

(4) $\psi_z(1) \neq q_0$ for all $z \in \partial Z$.
(5) The map $(Z, \partial Z) \to (M, M \setminus \{q_0\})$ defined by $z \mapsto \Psi(z, 1)$ has nonzero degree. The degree is the unique integer $l \geq 0$ such that a generator for $H_n(Z, \partial Z)$ is mapped to $\pm l$ times a generator for $H_n(M, M \setminus \{q_0\})$.

Recall two properties of degree: it does not change under small perturbations of the map; and nonzero degree implies that the map is surjective. Thus q_0 can be reached from p_0 along a us-path if q_0 can be engulfed from p_0. Moreover engulfing is stable under small perturbations of p_0, q_0 and the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$. Another useful property, which is minor extension of [**BuWi**, Proposition 5.1][5], is:

PROPOSITION 3.7. *Suppose that q_0 can be engulfed from p_0 and there are us-paths from p_1 to p_0 and from q_0 to q_1. Then q_1 can be engulfed from p_1.*

Engulfing can be reduced to a simpler condition.

DEFINITION 3.8. A point q_0 can be *centrally engulfed* from a point p_0 if there is a continuous map $\Psi : Z \times [0, 1] \to M$ such that:

(1) Z is a compact, connected, orientable manifold with boundary whose dimension c is the dimension of the fibers of $\mathcal{W}^c$.
(2) For each $z \in Z$, the curve $\psi_z(\cdot) = \Psi(z, \cdot)$ is a us-path with $\psi_z(0) = p_0$ and $\psi_z(1) \in \mathcal{W}^c(q_0)$.
(3) There is a constant C such that every path ψ_z has at most C legs.
(4) $\psi_z(1) \neq q_0$ for all $z \in \partial Z$.
(5) The map $(Z, \partial Z) \to (\mathcal{W}^c(q_0), \mathcal{W}^c(q_0) \setminus \{q_0\})$ defined by $z \mapsto \Psi(z, 1)$ has nonzero degree.

It is easy to extend a central engulfing to a full engulfing by adding one unstable and one stable leg to each of the paths from the central engulfing. Thus we have

LEMMA 3.9. [**BuWi**, Lemma 5.2][6] *Suppose q_0 can be centrally engulfed from p_0. Then q_0 can be engulfed from p_0.*

An immediate consequence of the above results is

COROLLARY 3.10. *Let $f : M \to M$ be a dynamically coherent, partially hyperbolic diffeomorphism. Suppose that there is a point p_0 such that any point of M can be reached from p_0 along a us-path and p_0 can be centrally engulfed from p_0. Then f is stably accessible.*

The construction of central engulfings is based on the *quadrilateral argument* first used by Brin in [**Bri1**, Proposition 2.3]. This argument produces a strong conclusion, namely central engulfing, from the surprisingly mild hypotheses that $E^u \oplus E^s$ is not integrable and E^c is one dimensional. Thus we suppose that the central foliation has one dimensional leaves and there is a short us-path with 4 legs that begins at some point p_0, ends in $\mathcal{W}^c(p_0)$ and is *not* closed. Then it is easy to create a central engulfing of p_0 from itself. First choose a second us-quadrilateral that passes through the same central fibers as the first path but does so in the opposite order; its endpoint will be in $\mathcal{W}^c(p_0)$, on the other side of p_0 from the endpoint of the first path. Then contract each of these us-quadrilaterals down to

[5] We take the opportunity to point out that the proof given in [**BuWi**] used but did not state the fact that foliations in question have differentiable leaves.

[6] We have corrected the typographical error in [**BuWi**].

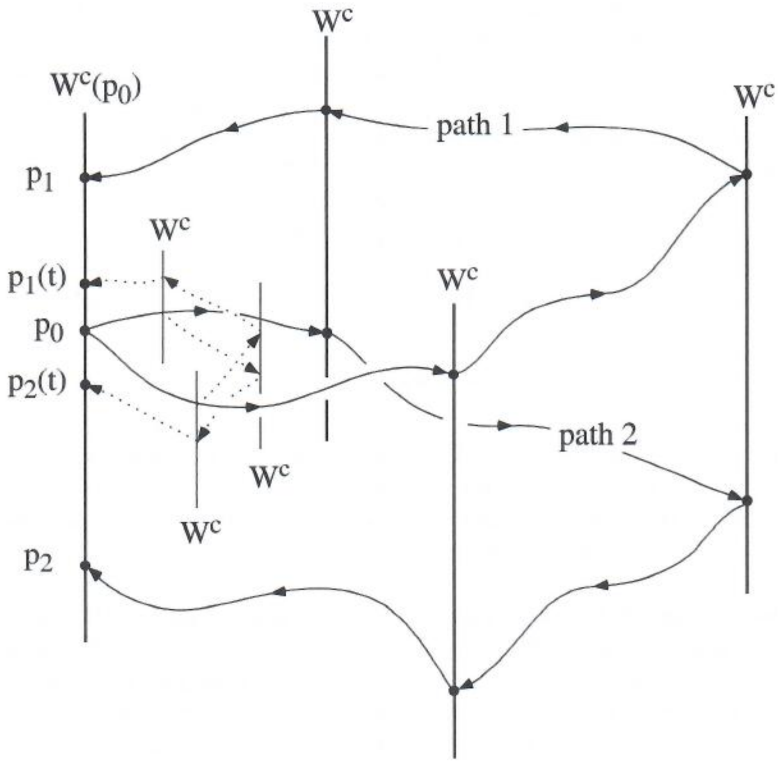

FIGURE 5. A Brin quadrilateral in which the arrows indicate the orientations of the two paths.

the trivial quadrilateral, as shown in Figure 5. See [**BuPugWi, KaKo**] for a more detailed description.

The argument is well suited to showing that stable accessibility is a dense property, because it is easy to introduce small perturbations that break up closed us-quadrilaterals. When the leaves of the central foliation have dimension greater than one, it is necessary to consider several different families of loops. This introduces significant technical difficulties, which have been overcome in several cases; see [**ShWi1**], [**BuWi**, §9] and the discussion in this paper about Theorems 4.8, 4.9, and 4.11.

Brin's skew product example. We now describe the example of Brin [**Bri1**, end of §2] mentioned earlier in this section. Let $f_0 : M_0 \to M_0$ be a C^2 volume preserving Anosov diffeomorphism of a compact manifold. We can choose a C^2 map $\varphi : M_0 \to \mathbf{R}$ such that the quadrilateral argument can be applied to the skew product $f_\varphi : M_0 \times \mathbf{R} \to M_0 \times \mathbf{R}$. Indeed the set of such φ is open and dense; see Proposition 2.3 of [**Bri1**] or the discussion at the end of §9 in [**BuWi**]. Since the base map f_0 obviously has the accessibility property, it is easy to see that f_φ also has the accessibility property (in fact any point in $M_0 \times \mathbf{R}$ can be centrally engulfed from any other).

Now let $i : \mathbf{R} \to \mathbf{T}^2$ be a group homomorphism whose image is a line with irrational slope. Consider the skew product $f_\psi : M_0 \times \mathbf{T}^2 \to M_0 \times \mathbf{T}^2$, where

$\psi = i \circ \varphi$. Then f_ψ does not have the accessibility property because the set of points that can reached from a given point (p, g) intersects each fiber in an irrational line. But f_ψ does have essential accessibility because the Lebesgue measurable subsets of the torus that are unions of translates of a given irrational line have either full measure or measure zero. It is immediate from the results in Section 2 that f_ψ is ergodic and K.

The essential accessibility of f_ψ is not stable, because one can rotate the immersion i through an arbitrarily small angle and make its image become a line of rational slope.

It is possible to replace $\mathbf{R}$ by $\mathbf{R}^k$ and $\mathbf{T}^2$ by $\mathbf{T}^l$ with $l > k$ in the construction.

4. Applications

For most of the examples discussed in Section 1, the Pugh-Shub Theorem, more specifically Theorems 3.2 and 3.3, can be applied to obtain stable ergodicity results. We survey these applications in this section. In all of these applications, the verification of center bunching and dynamical coherence is a simple matter. Stable accessibility is the central issue.

Time-t maps of Anosov flows. In 1994, Grayson, Pugh and Shub introduced the idea of juliennes in [**GrPugSh**] and used it to show that if S is a surface of constant negative curvature and φ_t is the geodesic flow on the unit tangent bundle of S, then the time-t map φ_t is stably ergodic, for $t \neq 0$. In 1995, Wilkinson [**Wi**] generalized this result to surfaces of variable negative curvature.

It is not difficult to see that the time-t map of an Anosov flow, for any $t \neq 0$, is stably dynamically coherent and center bunched; dynamical coherence follows from the fact that the center foliation by orbits is C^1, and center bunching is immediate, since the center foliation is 1-dimensional. Thus, the Pugh-Shub Theorem reduces stable ergodicity of φ_t to stable accessibility.

Stable accessibility was proven for contact Anosov flows, a class which includes geodesic flows for metrics with negative curvature, by Katok and Kononenko [**KaKo**, Proposition 5.2] . In the contact case, the distribution $E^u \oplus E^s$ is smooth and nowhere integrable. This means that *any* point of M centrally engulfs itself, and Brin's quadrilateral argument described in the previous section can be applied to prove stable accessibility. It turns out that for Anosov flows, non-integrability of $E^u \oplus E^s$ *at a single point* is sufficient for stable accessibility; this was proven by Burns, Pugh and Wilkinson.

THEOREM 4.1. [**BuPugWi**] *Let φ_t be a C^1 Anosov flow. Suppose that the strong stable and unstable foliations for the flow are not jointly integrable. Then the time-t map φ_t, for $t \neq 0$, is stably accessible.*

For codimension-1 Anosov flows, the non-integrability condition in Theorem 4.1 is equivalent to topological mixing. Since Plante [**Pl**] showed that the only non-mixing codimension-1 Anosov flows are special flows under constant height functions, there is the corollary:

COROLLARY 4.2. [**BuPugWi**] *Let $\varphi_t : M \to M$ be a C^2, volume-preserving Anosov flow on a compact 3-manifold. The time-t map φ_t is stably ergodic ($t \neq 0$) if and only if the flow φ_t is not a special flow under a constant height function.*

Thus:

Virtually all volume-preserving 3-dimensional Anosov flows have stably ergodic time-1 map.

Algebraic systems. Let $f : G/\Gamma \to G/\Gamma$ be a volume-preserving affine diffeomorphism, and let $\mathfrak{h}$ be the hyperbolic ideal in $\mathfrak{g}$ defined in Section 1. By Proposition 1.1, f is partially hyperbolic if and only if $\mathfrak{h} \neq 0$. A simple argument shows that such systems are always dynamically coherent and center bunched. Because the stable and unstable bundles are smooth, accessibility implies stable accessibility, by Proposition 3.4. Accessibility then can be reduced to an algebraic condition:

PROPOSITION 4.3. [**PugSh6**] *The affine diffeomorphism $f = L_g \circ A$ of G/Γ is partially hyperbolic, dynamically coherent, center bunched and has the stable accessibility property if and only if $\mathfrak{h} = \mathfrak{g}$.*

In view of the Pugh-Shub Theorem, this has as a corollary:

COROLLARY 4.4. *The affine diffeomorphism $f = L_g \circ A$ of G/Γ is stably ergodic if $\mathfrak{h} = \mathfrak{g}$.*

Here are some special cases of Corollary 4.4:

PROPOSITION 4.5. *Suppose that G is a simple Lie group (its Lie algebra has no non-trivial ideals), and that Γ is a discrete cocompact subgroup of G. Let $g \in G$ be given. Then L_g is stably ergodic if $Ad(g)$ has at least one eigenvalue of modulus different from 1.*

COROLLARY 4.6. *Let Γ be a discrete cocompact subgroup of $SL(n, \mathbf{R})$ and let $g \in SL(n, \mathbf{R})$ be given. Left multiplication by g on $SL(n, \mathbf{R})/\Gamma$ is stably ergodic if g has at least one eigenvalue of modulus different than 1.*

In [**BreS**] it was shown that some hyperbolicity is necessary for left multiplication by g to be stably ergodic even among the left translations. The results in [**BreS**], combined with Corollary 4.5 completely classify the stably ergodic affine transformations of compact homogeneous spaces of simple Lie groups. For these spaces,

stable ergodicity is equivalent to stable ergodicity among affine transformations.

Nonetheless, there are very basic examples which are not covered by this corollary, and for which stable ergodicity is not known to hold. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}.$$

The matrix A induces a volume-preserving diffeomorphism of the 4-torus $f_A : \mathbf{T}^4 \to \mathbf{T}^4$. The map $l(f_A) = A$ has eigenvalues $\{\exp \pm 2\pi\alpha, \lambda^{\pm 1}\}$, where $\alpha \approx .206827$ is irrational and $\lambda \approx 7.32763 > 1$; since none of these are roots of 1, it follows from a classical theorem that f_A is ergodic. Alternatively, one can observe that f_A is center bunched, is dynamically coherent because it has a C^1 center foliation, and has the essential accessibility property because the stable and unstable eigenspaces of A span an irrational plane. Ergodicity now follows from Theorem 2.2.

For this example, $G = \mathbf{R}^4$, $\Gamma = \mathbf{Z}^4$, and the hyperbolic subalgebra $\mathfrak{h}$ is two dimensional, and so f_A does not have the accessibility property. We cannot apply Corollary 4.4 to obtain stable ergodicity. Stable ergodicity would follow from Theorem 3.2 if the essential accessibility of f_A were stable.

QUESTION 4.7. Is either the ergodicity or the essential accessibility of f_A stable?

We find this a fascinating but difficult problem. In fact f_A was the first example examined by Pugh and Shub with respect to stable ergodicity, beginning in 1969.

Although we do not know that f_A is stably ergodic, it does lie in the closure of the stably ergodic diffeomorphisms.

THEOREM 4.8. [**ShWi1**] *f_A can be approximated arbitrarily well (in the C^∞ topology) by a stably accessible, stably ergodic diffeomorphism.*

We sketch the proof of Theorem 4.8 here. Since f_A is center bunched and stably dynamically coherent, stable ergodicity in Theorem 4.8 follows if we can approximate f_A by a stably accessible diffeomorphism. Fix x, and let $h_0, k_0 : \mathcal{W}^c_{loc}(x) \to \mathcal{W}^c(x)$ be the holonomy maps defined by two different us-quadrilaterals γ_1 and γ_2 that start at x and end in $\mathcal{W}^c_{loc}(x)$. Integrability of E^u and E^s implies that the holonomy around any sufficiently small us-path beginning and ending in the same local center leaf is the identity map. Thus $h_0 = id = k_0$. The main idea in the proof of Theorem 4.8 is to perturb f_A in a careful way so that these holonomy maps become nontrivial.

More precisely, there is a one parameter family f_ϵ of perturbations of f_A for which the corresponding holonomy maps $h_\epsilon, k_\epsilon : \mathcal{W}^c_{loc}(x) \to \mathcal{W}^c(x)$ have the following strong engulfing property: there is a $\delta_0 > 0$ such that, for each $\epsilon > 0$,

$$\{h_\epsilon^m \circ h_\epsilon^n(x) \mid |n|, |m| < \delta_0/\epsilon\}$$

is an ϵ-dense net in $\mathcal{W}^c_{\delta_0}(x)$ (the δ_0-ball around x in $\mathcal{W}^c_{loc}(x)$). This is possible because h_0 and k_0 are the identity map; one perturbs f_A in neighborhoods of γ_1 and γ_2 so that the diffeomorphisms h_ϵ and k_ϵ are approximately ϵ-translations of $\mathcal{W}^c_{\delta_0}(x)$ in orthogonal directions inside $\mathcal{W}^c(x)$.

Once this is arranged, it is not difficult to construct a center engulfing from x to any point in $\mathcal{W}^c_{\delta_0}(x)$. Since $f_\epsilon \to f_A$ as $\epsilon \to 0$ and stable and unstable leaves for f_A are dense, the stable and unstable leaves are δ_0-dense for ϵ sufficiently small. This proves stable accessibility.

Systems that fiber over partially hyperbolic diffeomorphisms. Suppose that f fibers over the partially hyperbolic diffeomorphism f_0. If the center foliation for f_0 is C^1, then so is the center foliation for f, and f is stably dynamically coherent. In the examples discussed here, this is always the case. Center bunching is also an easy condition to verify for these examples.

Direct products. As was remarked in Section 1, partially hyperbolic diffeomorphisms are not always ergodic. A dramatic illustration of this fact is obtained by taking the product $f \times g_\lambda$, where f is an area-preserving Anosov diffeomorphism of the 2-torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ and $g_\lambda : \mathbf{T}^2 \to \mathbf{T}^2$ is the *standard map:*

$$g_\lambda(z, w) = (z + w, w + (\lambda \sin(2\pi(z + w)))).$$

By KAM theory, for all values of λ near 0, g_λ has a positive-measure set of invariant circles. For such parameter values, this map is persistently not ergodic; any sufficiently nearby C^∞ symplectic map will fail to be ergodic.

Since g_λ is not ergodic, the product $f \times g_\lambda$ is not ergodic; it has a positive measure set of invariant, codimension-1 tori. But while g_λ is stably non-ergodic, $f \times g_\lambda$ may be approximated by a stably accessible, stably ergodic diffeomorphism, and all of these invariant tori disappear. The hyperbolicity of the base map f dominates the KAM center behavior of g_λ.

THEOREM 4.9. [**ShWi1**] *Let $f : \mathbf{T}^{2n} \to \mathbf{T}^{2n}$ be a C^r symplectic Anosov diffeomorphism, $r \geq 2$, and let $G : \mathbf{T}^{2m} \to \mathbf{T}^{2m}$ be a symplectic linear map whose eigenvalues lie on the unit circle in $\mathbf{C}$.*

Then there is a neighborhood $\mathcal{U}$ of G in the space of symplectic C^r diffeomorphisms $Diff^r_\omega(\mathbf{T}^{2m})$ such that for every $g \in \mathcal{U}$, the diffeomorphism $f \times g : \mathbf{T}^{2(m+n)} \to \mathbf{T}^{2(m+n)}$ is partially hyperbolic, dynamically coherent and center bunched. Further, for every neighborhood $\mathcal{V}$ of $f \times g$ in $Diff^r_\omega(\mathbf{T}^{2m})$, there exists an $h \in \mathcal{V}$ such that such that h is stably accessible and stably ergodic.

COROLLARY 4.10. [**ShWi1**] *For any C^∞ symplectic Anosov diffeomorphism f, the map $f \times g_\lambda$ can be C^∞ approximated arbitrarily well by a symplectic, stably ergodic diffeomorphism if λ is sufficiently close to 0.*

The word "symplectic" may be changed to "volume-preserving" in Theorem 4.9.

Since they have C^1 central foliations, partially hyperbolic direct products are stably dynamically coherent. To approximate a center bunched direct product by a stably ergodic diffeomorphism, it suffices, by the Pugh-Shub Theorem, to approximate it by a stably accessible diffeomorphism. The apparent difficulty in doing this is that direct products are never accessible, or even essentially accessible (unless the second factor is trivial). Indeed, it is not hard to see that the accessibility classes for $f \times g$ are the compact leaves of the "horizontal foliation" formed by products of the first factor with points of the second factor — far from the whole manifold! But it is precisely the triviality of the unperturbed system that gives enough control over the perturbation to obtain stable accessibility. The perturbation proceeds much as in the proof of Theorem 4.8 outlined above. The argument in the proof of Theorem 4.9 uses nothing about the dynamics of the second factor g and works for all partially hyperbolic direct products. Thus:

Partially hyperbolic direct products are in the closure of the stably accessible diffeomorphisms. Center bunched direct products are in the closure of the stably ergodic diffeomorphisms.

Skew products. We have seen in the sections on time-t maps of Anosov flows and affine diffeomorphisms that stable ergodicity is often dense within a family of partially hyperbolic diffeomorphisms. In particular, stable ergodicity is dense among time-1 maps of Anosov flows of three manifolds, and among left-translations of compact homogeneous spaces of simple lie groups. These results are a baby version of Conjecture 0.1: they show density of stable ergodicity within a (nowhere-dense) submanifold of the volume preserving diffeomorphisms. Another natural family to study in this context is the space Skew(f_0, G) of C^2 extensions of an Anosov diffeomorphism f_0 by a compact Lie group G.

In 1975, Brin showed that ergodicity is open and dense in Skew(f_0, G). Brin studied the holonomy subgroup H of G obtained by following us-paths beginning at a given point (x_0, g_0) and ending in $\mathcal{W}^c(x_0)$. More precisely

$$H = \{g \in G \mid \text{there is a } us\text{-path from } (x_0, g_0) \text{ to } (x_0, gg_0)\}.$$

Let H^0 be the connected component of the identity in H and let $\overline{H}$ be the closure of H in G. Then H^0 is the subgroup of H corresponding to us-paths whose projections to M_0 are contractible. It follows that H^0 is path connected and hence is a Lie subgroup of G by the theorem of Kuranishi and Yamabe. It is not difficult to see that $\overline{H} = G$ if and only if the skew product is essentially accessible.

Brin showed that $\overline{H} = G$ is a necessary and sufficient condition for a skew product to be a K-system. He also showed by a direct construction that $H^0 = G$ for an open and dense set in Skew(f_0, G). Similar results, using different techniques, have subsequently been obtained by Adler, Kitchens and Shub [**AdKiSh**], Parry and Pollicott [**ParPo**], Field and Parry [**FiPa**], Field and Nicol [**FiNi1, FiNi2**], and Walkden [**Wa**]; in this work the base map f_0 is often a basic set for an Axiom A diffeomorphism.

The ergodicity results for partially hyperbolic systems at the time of Brin's work did not apply to perturbations of skew products, and Brin did not investigate whether $H^0 = G$ (or equivalently $H = G$) implies stable accessibility. Burns and Wilkinson showed that $H^0 = G$ does imply stable accessibility, and thus obtained the result:

THEOREM 4.11. [**BuWi**] *Stable ergodicity is open and dense in Skew(f_0, G).*

Another question is whether the notions of "stable ergodicity" and "stable ergodicity among skew products" coincide (certainly the first implies the second). Burns and Wilkinson show that under an additional assumption on f_0, they are equivalent:

THEOREM 4.12. [**BuWi**] *Suppose $f_0 : N_0 \to N_0$ is an Anosov diffeomorphism of a compact infranilmanifold. Then $f_\varphi \in$ Skew(f_0, G) is stably ergodic (in Diff$^2_\mu(N_0 \times G)$) if and only if f_φ is stably ergodic among skew products (that is, inside of Skew(f_0, G)).*

This theorem is based on a classification of skew products that are not stably ergodic.

THEOREM 4.13. [**BuWi**] *Suppose $f_0 : N_0 \to N_0$ is an Anosov diffeomorphism of a compact infranilmanifold and $f_\varphi \in$ Skew(f_0, G) is not stably ergodic (in Diff$^2_\mu(N_0 \times G)$). Then f_φ has a factor that is either:*

(1) *$f_0 \times id_{S^1}$;*
(2) *$f_0 \times R$, where R is a rational rotation of S^1; or*
(3) *a skew product of the type described at the end of Section 3.*

Frame flows. The frame flow for a manifold with negative sectional curvatures is not always ergodic. Kähler manifolds with negative curvature and real dimension at least 4 — in particular quotients of the complex and quaternionic hyperbolic spaces — have nonergodic frame flows because the complex structure is invariant under parallel translation [**BriGr**]. The curvature tensor in these examples is invariant under parallel translation and, if the metric is suitably scaled, every vector lies in both a plane with curvature -1 and a plane with curvature $-1/4$. These seem to be the only known examples with negative curvature and nonergodic frame flow. It has been conjectured that if the sectional curvatures are strictly pinched between -1 and $-1/4$, then the frame flow is ergodic [**BriK, Bri4**]. Brin also conjectures in [**Bri4**] that the frame flow should be ergodic unless the holonomy of the manifold is a proper subgroup of $SO(n)$.

Here are the cases in which the frame flow is known to be ergodic:

THEOREM 4.14. *The frame flow of a compact n-dimensional Riemannian manifold V with negative sectional curvatures pinched between $-\Lambda^2$ and $-\lambda^2$ is ergodic:*

(1) *if V has constant negative curvature* (Brin [**Bri1**]);
(2) *for a set of metrics with negative curvature that is open and dense in the C^3 topology* (Brin, [**Bri1**]);
(3) *if n is odd and $n \neq 7$* (Brin and Gromov [**BriGr**]);
(4) *if n is even, $n \neq 8$, and $\lambda/\Lambda > 0.93$* (Brin and Karcher [**BriK**]).
(5) *if $n = 7$ and $\lambda/\Lambda > 0.999714\dots$* (Burns and Pollicott [**BuPo**]), *or*
(6) *if $n = 8$ and V and $\lambda/\Lambda > 0.999785\dots$* (Burns and Pollicott [**BuPo**]).

Ergodicity of the frame flow and stable ergodicity of the time one map of the frame flow are very closely related. It is not difficult to see that both properties hold if there is one point in the frame bundle M which can be centrally engulfed from itself. The central direction is spanned by the flow direction and the fiber direction. The fact that the geodesic flow in negative curvature is a contact Anosov flow makes engulfing in the flow direction automatic. Central engulfing therefore reduces to engulfing in the fiber direction. The study of fiber engulfing is completely analogous to the study of central engulfing for skew products. The original proofs by Brin, Gromov and Karcher in [**Bri1, BriGr, BriK**] do not explicitly construct fiber engulfings but they can all be adapted to do so; these adaptations are described in [**BuPo**]. The proofs of 5. and 6. in [**BuPo**] are also based on the construction of a fiber engulfing.

Thus in all known examples where the frame flow is ergodic, its time one map is stably ergodic. In particular we have another partial verification of Conjecture 0.1: the time one map of the frame flow is stably ergodic for an open and dense set of metrics with negative curvature. We conjecture that the frame flow is ergodic if and only if its time one map is stably ergodic.

Perturbations of partially hyperbolic systems. There are a few stable ergodicity results that apply to a dense set of diffeomorphisms in a neighborhood of a *non*ergodic diffeomorphism. By combining the Brin quadrilateral argument with the dynamics on the center manifolds, Niţică and Török were able to show:

THEOREM 4.15. [**NiTö**] *Let $f_0 : N_0 \to N_0$ be an Anosov diffeomorphism, and let S^1 be the circle. Then there is a neighborhood $\mathcal{U}$ of $f_0 \times id_{S^1} : N_0 \times S^1 \to N_0 \times S^1$ in $Diff^2(N_0 \times S^1)$ such that stable accessibility is open and dense in $\mathcal{U}$.*

They also prove, under some additional assumptions, that stable accessibility is dense among partially hyperbolic diffeomorphisms with one-dimensional center. As a corollary, they obtain:

COROLLARY 4.16. [**NiTö**] *Assume that f_0 preserves the volume μ_0, and let μ be the product of μ_0 with any smooth measure on S^1. Then there is a neighborhood $\mathcal{U}$ of $f_0 \times id_{S^1} : N_0 \times S^1 \to N_0 \times S^1$ in $Diff^2_\mu(N_0 \times S^1)$ such that stable ergodicity is open and dense in $\mathcal{U}$.*

Dolgopyat [**Do2**] has shown that the techniques of Niţică and Török can be generalized to show that among *all* partially hyperbolic diffeomorphisms of 3-manifolds (with both stable and unstable bundles nontrivial), stable accessibility is dense. Then, using other techniques, he shows that stable ergodicity (in fact, stable Bernoullicity) is dense. He does not assume dynamical coherence.

5. Pathological foliations

Shub and Wilkinson [**ShWi2**] have found an open set of partially hyperbolic diffeomorphisms of $\mathcal{T}^3$ that are ergodic and Bernoulli with respect to Lebesgue measure and have non absolutely continuous center foliation. Their examples are small perturbations of the map $f_0 = A \times id_{S^1}$, where A is the Anosov diffeomorphism of $\mathcal{T}^2$ induced by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Any diffeomorphism f that is close enough to f_0 in the C^1 topology is partially hyperbolic. The three invariant subbundles E^s, E^c and E^u into which $T\mathcal{T}^3$ splits all have one dimensional fibers. The leaves of the corresponding foliations are all C^1 and the leaves of the center foliation $\mathcal{W}^c$ are circles, which are small perturbations of the fibers of the product foliation. If f preserves Lebesgue measure, there will be Lyapunov exponents λ^s, λ^c and λ^u associated to these subbundles and defined Lebesgue almost everywhere. It is clear that $\lambda^s < 0 < \lambda^u$ almost everywhere. Since the leaves of the foliation $\mathcal{W}^c$ are all C^1 embedded circles, whose lengths are uniformly bounded from above and from below, it is clear that that the length of a finite arc of a central leaf cannot grow exponentially under iteration by f. Nevertheless, *it is possible for f to be ergodic with respect to μ and have $\lambda^c > 0$ almost everywhere.*

Such paradoxical behavior is possible only if the set $P = \{p \in \mathcal{T}^3 : \lambda^u(p) > 0\}$, which has full Lebesgue measure, intersects each leaf of $\mathcal{W}^c$ in a set that has measure 0 with respect to arclength along the leaf. Thus we cannot use Fubini's theorem to compute the Lebesgue measure of P by integrating the length measures of its intersections with the leaves of $\mathcal{W}^c$; this means that the foliation $\mathcal{W}^c$ is not absolutely continuous. (Of course the foliation $\mathcal{W}^c$ does not satisfy the hypotheses of Fubini's theorem — even though its leaves are C^1, they vary in a way that is only Hölder continuous.) The existence of such non absolutely continuous foliations has been called "Fubini's nightmare" by Flaminio. Katok had previously constructed a dynamically invariant foliation that is not absolutely continuous; it was presented by Milnor [**Mi**] under the title *Fubini foiled*. The important features of Shub and Wilkinson's example persist under small perturbations. Thus Fubini's nightmare can arise for an open set of diffeomorphisms, and one must face the fact that it is a natural, perhaps even typical, phenomenon.

QUESTION 5.1. How typical is it?

A partial answer is given by the work of Dolgopyat [**Do2**] described at the end of Section 4. His results show, in particular, that the time one map of the geodesic flow of a surface of negative curvature has a neighborhood within which all maps in an open dense subset have nonzero Lyapunov exponent in the center direction. It then follows from an argument of Mañé [**Mañ4**], which exploits the fact that the center is one dimensional, that these maps have non absolutely continuous center foliations. The idea is that if the center foliation is absolutely continuous, then iteration of the map will exponentially expand a typical leaf. But this is impossible since the map is close to the time one map of the geodesic flow, which uniformly translates all center leaves.

Katok's example in [**Mi**] has the property that there is a set which has full measure in the ambient space but meets each leaf of the foliation in at most one point. Ruelle and Wilkinson [**RueWi**] and Katok have shown that the Shub-Wilkinson examples have a similar property: for each such example there is a finite number k such that the set P described above meets almost every leaf of the center foliation exactly k points. Katok has pointed out that it is possible to have $k > 1$; this is achieved by versions of the Shub-Wilkinson example that commute with a finite group of symmetries.

Ruelle and Wilkinson obtain their result by applying the following theorem to the inverses of the Shub-Wilkinson examples.

THEOREM 5.2. [**RueWi**] *Let $F : X\times M \to X\times M$ be a skew product whose base X is a compact metric space and whose fiber M is a closed Riemannian manifold. Assume that F is invertible, ergodic with respect to a Borel probability measure ν and has all Lyapunov exponents in the M direction negative. Then there exist an integer k and a set S such that $\nu(S) = 1$ and $S \cap (\{x\} \times M)$ has precisely k points for every x such that S intersects $\{x\} \times M$.*

It is essential that F be invertible; an example of Kifer [**Ki1**], which is described in [**RueWi**], shows that Theorem 5.2 is false if F is not invertible.

We now outline some of the ideas from Shub and Wilkinson's construction. They give an explicit example of a one parameter variation f_a of f_0 with the property that the average value of the center exponent λ_a^c for f_a is positive unless $a = 0$. The average value of λ_a^c is the integral of the function $r_a^c(p)$ that is defined by $\|D_pf_a(v)\| = r_a^c(p)\|v\|$ for all $v \in E_{f_a}^c(p)$; the exponent λ_a^c is the Birkhoff average of r_a^c. Estimating the integral of r_a^c is a nontrivial but feasible calculation, whereas pointwise estimates on λ_a^c would be hopelessly difficult.

By the result of Niţică and Török described in Section 4, these f_a lie in the closure of the stably ergodic diffeomorphisms for any small enough a. Thus if a is small enough, f_a is approximated by a stably ergodic diffeomorphism f with the property that the exponent for the central direction has positive average[7].

The diffeomorphism f is stably ergodic and has the average value of the exponent λ^c positive. The average values of the exponents change continuously when we perturb f, so all diffeomorphisms close enough to f are ergodic and have λ^c positive on average. Since exponents are invariant functions, all diffeomorphisms close enough to f have λ^c positive almost everywhere and thus are nonuniformly hyperbolic on a set of full measure (since we certainly have λ^u and λ^s a.e. non zero). It follows from Pesin theory that they are Bernoulli. Thus f is stably Bernoulli. As explained above the center foliation $\mathcal{W}^c$ must be non absolutely continuous when $\lambda^c > 0$ almost everywhere.

6. Further questions and conjectures

In this section we try to put our work in a broader perspective. A very optimistic goal for dynamical systems theory would be to understand most dynamical systems using a few basic properties which hold quite generally. Various unsuccessful attempts were made starting around 1960 to achieve this goal via uniform hyperbolicity and the concepts of structural and Ω stability [**Sm2**].

One of the major achievements in the theory of uniformly hyperbolic dynamical systems was the theorems of Sinai, Ruelle and Bowen about invariant measures on

[7]Shub and Wilkinson actually proceed in a different way and extend f_a to a 2-parameter family.

the attractors of a dynamical system. These measures and attractors are now called SRB measures and SRB attractors.

Much of the work in dynamical systems since the mid-seventies has been concerned with extending the domain of existence of SRB measures beyond the uniformly hyperbolic case. By now we have quite a bit of evidence coming from low dimensions such as the quadratic family and Hénon maps, and scattered examples in higher dimension where SRB measures do in fact generally exist [**BonVi, AlBonVi, Carva, Cas, Do1, FiNi2**]. The evidence is good enough so that by now various conjectures have been made. The earliest precise version of one is in [**Pal**]. This is preceded by [**PugSh4**] and followed by [**ShWi2**] which have variants. We attempt to explain the latter two.

We would like to be able to say that the typical orbit of the typical dynamical system has such and such dynamical behavior, even though it may be difficult to verify that a given system or point is typical. Stable ergodicity is a good property from this point of view. Almost every orbit is equi-distributed. Moreover, this fact is true stably or robustly. That is to say nearby systems have the same property.

SRB measures are an analog of ergodic measures in the general or non-volume preserving case. So their existence begins to tell us about the behavior of typical orbits. The proofs in low dimensions of the existence of these measures are a major achievement. The proofs give credence to the existence of SRB measures in general, but they are so particular to low dimensions and rely on so detailed an understanding of the local geometric properties of the iterates of the mappings, that they are discouraging of ever finding a proof in higher dimensions of the general existence of these SRB measures. In particular, it appears hopeless to attempt to find a proof by exhaustion of cases.

Instead we look for a few dynamical properties

- which might be proven to hold for the typical system, without first undertaking an extensive understanding of the dynamics of the system,
- which yet might allow us to conclude useful and even robust information about the dynamics, and
- which might eventually imply the existence of SRB measures.

The simple idea is to relax hyperbolicity in various ways and see what conclusions we may still make. We began in the volume preserving case with partially hyperbolic systems. An example of the type of property we have in mind is stable accessibility of partially hyperbolic systems. As we have conjectured in the introduction, this property should be open and dense among partially hyperbolic systems. We have not proven the conjecture in general and have only done some cases, described in Section 4, but we imagine that the conjecture might be proven by some transversality argument or hands on engulfing argument not too terribly tied to dynamics.

The general plan is to relax uniform partial hyperbolicity to non-uniform partial hyperbolicity, then in the discussion below to consider a variant of accessible sets. There are two frameworks: the volume preserving and the general or dissipative case. Several questions are naturally raised in each framework. Quite a bit of the following discussion is taken from [**ShWi2**].

We discuss the volume preserving case first. While uniformly hyperbolic systems enjoy strong mixing properties, they are not dense among C^1 diffeomorphisms [**Sm1**], [**AbSm**]. Using the concept of Lyapunov exponents, Pesin introduced a weaker form of hyperbolicity, which he termed *nonuniform hyperbolicity.* At almost all points, all the Lyapunov exponents are non-zero. Nonuniformly hyperbolic diffeomorphisms share several mixing properties with uniformly hyperbolic ones.

Pesin [**Pe**] proved that if $f : M \to M$ is a volume-preserving C^2 nonuniformly hyperbolic diffeomorphism, then M may be written as the disjoint union of countably many invariant sets of positive measure on which f is ergodic. He asked if nonuniform hyperbolicity is generic in $\mathrm{Diff}^r_\mu(M)$, the space of C^r, volume-preserving diffeomorphisms of M.

Pesin's question is answered in the negative for large r by Cheng-Sun [**ChSu**], Herman and Xia ([**He1**], [**Xia**]; see also [**Yoc**]). In particular, on any manifold M of dimension at least 2, and for sufficiently large r, there are open sets of volume preserving C^r diffeomorphisms of M all of which possess positive measure sets of codimension one invariant tori; on each such torus, the diffeomorphism is C^1 conjugate to a diophantine translation. In these examples *all* of the Lyapunov exponents are 0 on the invariant tori.

We use the following terminology to discuss Lyapunov exponents. An invariant measure ν is *neutral* if all its Lyapunov exponents are ν-a.e. zero, *partially non-uniformly hyperbolic* if some of them are ν-a.e. non-zero, and *(fully) non-uniformly hyperbolic* if all of them are ν-a.e. non-zero.

The example of Shub and Wilkinson described in Theorem 4.9 gives some hope that a variant of Pesin's original vision holds true for volume-preserving diffeomorphisms: either all exponents are zero (Lebesgue almost everywhere) or, as with our examples, the system may be perturbed to become stably nonuniformly hyperbolic. We express this in the following two questions, the second being a special case of the first.

QUESTION 6.1. Is it a generic property of f in $\mathrm{Diff}^r_\mu(M)$, $r \geq 1$, that almost every ergodic component is either neutral (all Lyapunov exponents zero) or fully non-uniformly hyperbolic (all Lyapunov exponents non-zero)?

QUESTION 6.2. Is the answer to Question 6.1 affirmative for the generic *ergodic* f?

Mañé [**Mañ2, Mañ5**] has outlined an affirmative answer to Question 6.1 when M is a surface and $r = 1$: the generic diffeomorphism in $\mathrm{Diff}^1_\mu(M^2)$ either has all of its Lyapunov exponents zero or is an Anosov diffeomorphism. But Mañé only sketches the proofs and as far we know nobody has filled in all the details.[8]

In another direction, Theorem 4.9 recalls [**BriFeKa**] in which Brin, Feldman and Katok construct diffeomorphisms on every manifold that are Bernoulli. Brin [**Bri3**] went on to show that such examples can be made to have all but one Lyapunov exponent not equal to zero.

QUESTION 6.3. [Brin] Does every manifold of dimension greater than or equal to two admit a *fully* nonuniformly hyperbolic Bernoulli diffeomorphism?

[8]A proof of Mañé's result was recently announced by Jairo Bochi.

If we consider general dynamical systems, i.e. $\mathrm{Diff}^r(M)$ with no invariant measure assumed, then we replace ergodicity by Sinai-Ruelle-Bowen (SRB) measures and attractors. They are also called ergodic attractors.[9]

Given $f \in \mathrm{Diff}^r(M)$ (not necessarily preserving μ), a closed, f-invariant set $A \subset M$ and an f-invariant ergodic measure ν on A, we define $B(A, \nu)$, the *basin* of A, to be the set of points $x \in M$ such that $f^n(x) \to A$ as $n \to \infty$, and for every continuous function $\phi : M \to \mathbf{R}$

$$\lim_{n\to\infty} \frac{1}{n}[\phi(x) + \cdots + \phi(f^n(x))] = \int_A \phi(x)\, d\nu.$$

Definition: ν is an *SRB measure* and A is an *SRB attractor* (we also refer to it as an *ergodic attractor*) if the Lebesgue measure of $B(A, \nu)$ is positive.

It follows from the definition that a diffeomorphism has at most countably many SRB measures. Sinai, Ruelle and Bowen proved that for $r \geq 2$ and f an Axiom A, no-cycle diffeomorphism (see [**Si**], [**Rue**], [**Bow**]), the number of SRB attractors is finite and the union of their basins has full Lebesgue measure in M.

QUESTION 6.4. For the generic f in $\mathrm{Diff}^r(M)$, $r \geq 2$, does the union of its SRB basins have full Lebesgue measure in M?

It is also conjectured in [**Pal**] that for almost every element in a generic finite dimensional family of diffeomorphisms there are only finitely many such SRB attractors. Conjecture 3 of [**PugSh4**] was intended to yield the same.

As alluded to above, this natural question is on the minds of quite a few people. See [**PugSh4**], [**Pal**], [**BonVi**] for discussions and examples. As we have suggested it might be approached along the lines of [**Pe**], [**PugSh3**], [**PugSh6**] by considering weak limits of averages of push-forwards

$$\nu = \lim_{n\to\infty} \frac{1}{n} \sum_0^n f_*^j \mu.$$

QUESTION 6.5. For the generic f in $\mathrm{Diff}^r(M)$, $r \geq 1$, is almost every ergodic component of ν partially non-uniformly hyperbolic, or even fully non-uniformly hyperbolic?

If Question 6.5 has a positive answer then one could proceed to use Lyapunov exponents, amalgamating points as in [**PugSh4**] via a variant of the unstable-stable accessibility relation. For $x, y \in M$ we say that $x \geq_1 y$ if $\mathcal{W}^u(x) \cap W^s(y) \neq \emptyset$. Let $\geq$ be the order obtained by transitivizing $x \geq_1 y$. We say that $x \sim y$ if $x \geq y$ and $y \geq x$. Now the equivalence classes themselves are ordered and minimal elements are candidates to be ergodic attractors.

Note that in the last question we have not included the possibility that all exponents be zero. There are no persistent examples we know of in $\mathrm{Diff}^r(M)$ analogous to the examples of Cheng-Sun and Herman, i.e. persistent neutral invariant measures with positive Lebesgue measure basins.

Here is a vague idea of how some of these genericity questions on Lyapunov exponents might be attacked. Instead of proving theorems about the properties of

[9] We take some of the conclusions of the theorems of Sinai, Ruelle, and Bowen as a definition and warn the reader that the use of SRB measure or attractor is not uniform in the literature. For a survey of SRB measures (using a different definition) see [**You**].

the iterates of a single diffeomorphism we would like to deduce them from properties of a family of diffeomorphisms.

Let $\mathcal{F}$ be a family of C^r diffeomorphisms of a compact manifold M. We would like to deduce that most elements of $\mathcal{F}$ have nonzero exponents. We formulate our strategy in the following principle: *if the family $\mathcal{F}$ is sufficiently rich, then good properties of $\mathcal{F}$ are inherited by many of the elements of $\mathcal{F}$.* The good property we have in mind is positive Lyapunov exponents. To make this strategy more precise, we need to formulate what it means for the family $\mathcal{F}$ to have the "good property," and then find an appropriate notion of richness of $\mathcal{F}$.

A theorem that exploits this strategy is the Abraham transversality theorem (see [**AbRo**]), where the good property is transversality. Earlier proofs of the transversality theorems proceed by perturbing individual elements. Instead, Abraham deduces transversality of most individual mappings in a family $\mathcal{F}$ whenever the joint evaluation map $(x, f) \mapsto f(x)$ has a transversality property. The family $\mathcal{F}$ must be rich in the sense that it forms a smooth (possibly infinite dimensional) manifold along which the evaluation map has the transversality property.

Further illustrations of the principle are provided by ergodic theory. The Mautner phenomenon [**Mo**] quite generally concludes the ergodicity of individual elements of a transitive semi-simple Lie group action. If $\mathbf{R}^n$ acts ergodically then almost all individual elements are ergodic [**PughSh1**]. In these examples, the good property is ergodicity (or transitivity) of the G-action, and the richness of the family comes from the Lie group structure.

We propose a notion of richness that takes into account not only the elements of $\mathcal{F}$ but their distribution, given by a probability measure ν supported on $\mathcal{F}$. Using this measure, there is a natural way to define the exponents of the family $\mathcal{F}$; namely, we consider the exponents of random products of elements of $\mathcal{F}$.

In some situations the behavior of random products is easier to understand than the behavior of the powers of individual elements. One example of this phenomenon is the relative simplicity of Franks' theorem [**Fr**] characterizing structural stability (defined with respect to random products) in comparison to Mañé's [**Mañ3**] (where structural stability is defined the standard way, using powers). Other contexts in which the behavior of random products is understood are the exponents of matrices [**GoMar**], SRB measures for diffeomorphisms [**Ar**], and the exponents of volume preserving diffeomorphisms [**Carve**]. We now describe the latter result in more detail.

Let ν be a probability measure supported on $\mathcal{F}$ and let $f_1, f_2, \ldots$ be a sequence of diffeomorphisms in $\mathcal{F}$, independent and identically distributed with respect to ν. Let $f^{(1)} = f_1$ and let $f^{(n)} = f_n \circ f^{(n-1)}$. Then the *maximal random Lyapunov exponent for $\mathcal{F}$ at x*:

$$R(\nu, x) = \lim_{n\to\infty} \frac{1}{n} \ln \|T_x f^{(n)}\|$$

exists almost surely and is positive unless ν is fairly degenerate [**Carve**]. This degeneracy condition is easily violated. Hence we *expect* random Lyapunov exponents to be nonzero.

While it is fairly easy to construct families with nonzero random Lyapunov exponents, it is more difficult to have individual iterates inherit the property. Part of the problem may be the need for quantitative estimates. Part of the problem might have to do with the richness of the family.

Here is a simple example to illustrate what happens when the family is not sufficiently rich. In this example, random exponents are nonzero but individual elements have only zero exponents. Let $M = \mathbf{T}^2$, let $\mathcal{F}$ consist of two elements, the automorphisms given by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let ν assign positive probability to each of the elements of $\mathcal{F}$. These matrices preserve the positive vectors in $\mathbf{R}^2$. A random product of the elements of $\mathcal{F}$ will almost surely contain the matrix AB with positive frequency, and this matrix expands the first coordinate of any positive vector by a factor ≥ 2. Hence, by the ergodic theorem, the random exponents are nonzero. On the other hand, the exponents of both elements of $\mathcal{F}$ are zero. The family is not very rich.

We introduce now a notion of richness of $\mathcal{F}$ which might, in some situations, be sufficient to deduce properties of the exponents of elements of $\mathcal{F}$ from those of the random exponents. This notion was suggested to us by some preliminary numerical experiments and by actual results in the setting of random matrix products [**DeSh**].

We focus on the problem for $M = S^n$, the n-sphere. Let μ be Lebesgue measure on S^n normalized to be a probability measure, and let m be Liouville measure on $T_1(S^n)$, the unit tangent bundle of S^n, similarly normalized to be a probability measure. The orthogonal group $O(n+1)$ acts by isometries on the n-sphere and so induces an action on the space of μ-preserving diffeomorphisms by

$$f \mapsto O \circ f, \qquad \text{for } O \in O(n+1).$$

Let ν be a probability measure supported on $\mathcal{F} \subset \mathrm{Diff}^r_\mu(S^n)$. We say that ν is *orthogonally invariant* if ν is preserved by every element of $O(n+1)$ under the action described above.

For example, let

$$\mathcal{F} = O(n+1)f = \{O \circ f \mid O \in O(n+1)\},$$

for a fixed $f \in \mathrm{Diff}^r_\mu(S^n)$. Defining ν by transporting Haar measure on $O(n+1)$ to $\mathcal{F}$, we obtain an orthogonally-invariant measure. Because $O(n+1)$ acts transitively on $T_1(S^n)$, a random product of elements of $\mathcal{F}$ will pick up the behavior of f in almost all tangent directions — the family is reasonably rich in that sense.

Let ν be an orthogonally invariant measure on $\mathcal{F}$. From orthogonal invariance and the ergodic theorem, it is easy to see that the mean maximal random Lyapunov exponent for $\mathcal{F}$, which we will denote by $R(\nu)$, can be expressed as an integral:

$$R(\nu) = \int R(\nu, x)\, d\mu = \int_{\mathrm{Diff}^r_\mu(S^n)} \int_{T_1(S^n)} \ln \|Df(x)v\|\, dm\, d\nu.$$

We define the *mean Lyapunov exponent* to be

$$\Lambda(\nu) = \int_{\mathrm{Diff}^r_\mu(S^n)} \int_{S^n} \lambda(f, x)\, d\mu\, d\nu$$

where $\lambda(f, x)$ is the maximal Lyapunov exponent of f at x.

QUESTION 6.6. Is there a positive constant $C(n)$ — perhaps 1 — depending on n alone such that $\Lambda(\nu) \geq C(n)R(\nu)$?

If the answer to Question 6.6 were affirmative, then a positive measure set of elements of $\mathcal{F}$ would have areas of positive exponents, assuming a nondegeneracy condition on ν. We add here that this type of question has been asked before and has been the subject of a lot of research. What is new is the notion of richness which allows us to express the relation between exponents as an inequality in integrals.

The question is already interesting for S^2. Express S^2 as the sphere of radius $1/2$ centered at $(1/2, 0)$ in $\mathbf{R} \times \mathbf{C}$, so that the coordinates $(r, z) \in S^2$ satisfy the equation

$$|r - 1/2|^2 + |z|^2 = 1/4.$$

In these coordinates define a twist map $f_\epsilon : S^2 \to S^2$, for $\epsilon > 0$, by

$$f_\epsilon(r, z) = (r, \exp(2\pi i r\epsilon)z).$$

Let $\mathcal{F}$ be the orbit $O(3)f$ and let ν be the push forward of Haar measure on $O(3)$. A very small and inconclusive numerical experiment seemed to indicate that for ϵ close to 0 the inequality may hold with $C(n) = 1$. It seemed the constant may decrease as the twist increases speed.

Michel Herman has pointed out the references [**Neĭ, Neh, Lo**] to us. In light of these references, he suggests that it is extremely likely that Question 6.6 has a negative answer, precisely for the twist map example f_ϵ, for ϵ very small. If he is correct, it would be interesting to know if some similar lower bound estimate is available with an appropriate concept of richness of the family.

There is a somewhat less speculative version of this question in linear algebra. Now we let ν be an orthogonally invariant measure on the space of $n \times n$ real matrices, $M(n, n)$. Then we let

$$R(\nu) = \int_{M(n,n)} \int_{S^{n-1}} \ln \|M(x)\| \, d\mu \, d\nu$$

and

$$\Lambda(\nu) = \int_{M(n,n)} \ln \rho(M) \, d\nu,$$

where $\rho(M)$ is the spectral radius of M. Now we repeat the question, with some more confidence that the constant is 1.

QUESTION 6.7. Is there a positive constant $C(n)$ — perhaps 1 — depending on n alone such that $\Lambda(\nu) \geq C(n)R(\nu)$?[10]

It is a pleasant exercise to prove the inequality in Question 6.7 for $n = 2$ with $C(2) = 1$. See [**Ki1, Ki2**] and [**BouLa**] for background on random diffeomorphisms and random matrix products.

We end with one last question of a very different nature. We have used both the strong unstable and strong stable foliations in our proof of ergodicity. We don't know an example where this is strictly necessary.

QUESTION 6.8. For a partially hyperbolic C^2 ergodic diffeomorphism f with the essential accessibility property, is the strong unstable foliation already uniquely ergodic?

Unique ergodicity of $\mathcal{W}^u$ was proved by Bowen and Marcus [**BowMar**] in the case where f is the time one map of a hyperbolic flow. For homogeneous space examples the question is related to the Mautner phenomenon discussed in [**Mo**] and [**Par2**].

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[10] The answer to this question is "yes" if $M(n, n)$ is the space of *complex* matrices and ν is invariant under the unitary group $U(n)$ [**DeSh**].

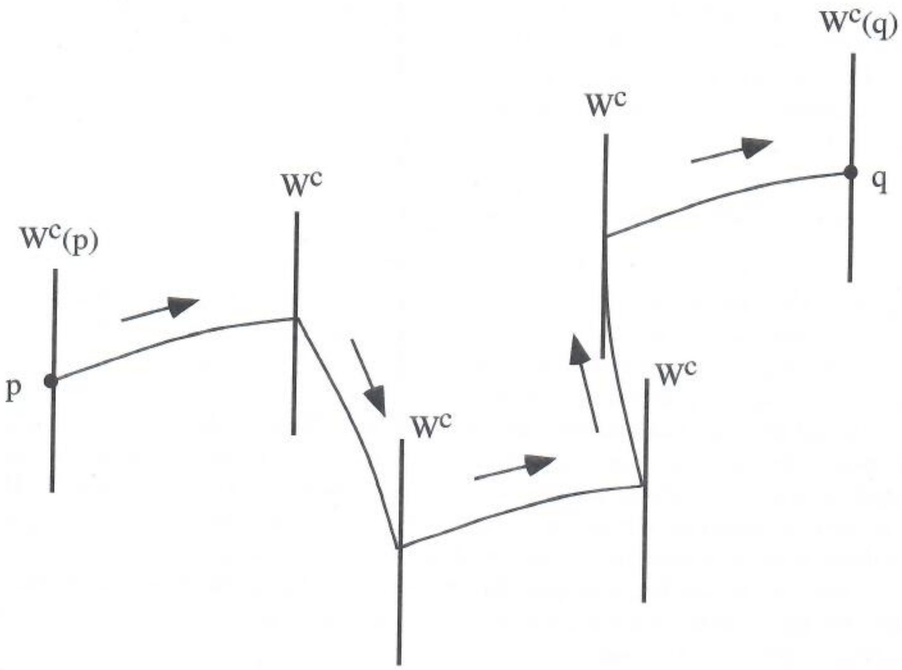

FIGURE 6. It is implausible that a is a density point of A and b is a density point of $B = M \setminus A$.

7. Appendix: Turning the corner

An intentionally over-simplified proof of ergodicity is outlined in [**GrPugSh**]. It is based on Figure 6 which purports to show that if p is a center density point (see below for the definition) of an essentially us-saturated set A then so is any point q which can be connected to p by a us-path. It is assumed that the center foliation is absolutely continuous, since otherwise the set of center density points can be empty, as is the case in the examples of Katok and Shub-Wilkinson described in Section 5. The obstacle to making Figure 6 into a proof of ergodicity is that the property of being a center density point of A may be lost at the corners of the path. We get around this obstacle, constructing us-paths with "good corners", and proving the following result.

THEOREM 7.1. *If $f \in \mathit{Diff}^2_\mu(M)$ is partially hyperbolic, dynamically coherent, center bunched, and essentially accessible then absolute continuity of the center foliation implies ergodicity of f.*

Proof. M is compact and Riemannian. Its Riemann structure induces smooth measures $\mu_u, \mu_{cu}, \mu_c, \mu_{cs}, \mu_s$ on the leaves of the invariant foliations, and induces the smooth measure μ on M. The key property of an absolutely continuous foliation is Cavalieri's Principle: if $Z \subset M$ is a μ-zero set then almost every leaf meets Z in a set of leaf measure zero. See [**PugSh2**]. The unstable and stable foliations are always absolutely continuous, and now by assumption the center foliation is absolutely continuous. This makes the center unstable and center stable foliations absolutely continuous too.

As we saw in Section 2.2, the assertion of Theorem 7.1 amounts to the inclusion

$$\mathrm{Sat}_0(\mathcal{M}^u) \cap \mathrm{Sat}_0(\mathcal{M}^s) \subset \mathrm{Sat}_0(\mathcal{M}^u \cap \mathcal{M}^s). \tag{7.1}$$

Let $A \in \mathrm{Sat}_0(\mathcal{M}^u) \cap \mathrm{Sat}_0(\mathcal{M}^s)$ be given. This means that A is measurable and

(a) Modulo a μ-zero set A is the union of almost whole unstable leaves.

(b) Modulo a μ-zero set A is the union of almost whole stable leaves.

We must show that A is sandwiched as $A_0 \subset A \subset A_1$ such that $A_1 \setminus A_0$ is a zero set and us-paths starting in A_0 stay in A_1.

The union of the unstable leaves that meet A in sets of full leaf measure is denoted A^u. The unstable holonomy maps preserve A^u, and absolute continuity of the unstable foliation implies that A^u differs from A by a zero set. Corresponding definitions and facts hold for A^s. The complement $B = M \setminus A$ also belongs to $\mathrm{Sat}_0(\mathcal{M}^u) \cap \mathrm{Sat}_0(\mathcal{M}^s)$, and we have the corresponding sets B^u, B^s.

Absolute continuity of the center foliation implies that almost every point of $p \in A$ is a *center density point* of A. That is,

$$\lim_{r \to 0} \frac{\mu_c(W^c(p,r) \cap A)}{\mu_c(W^c(p,r))} = 1.$$

The corresponding fact is true for B.

Definition: A center leaf W^c is *good* if

(a) $A^u \cap W^c = A \cap W^c = A^s \cap W^c$ modulo center leaf zero sets.

(b) $B^u \cap W^c = B \cap W^c = B^s \cap W^c$ modulo center leaf zero sets.

A us-path can safely turn the corner at a good leaf: center density of A is preserved.

Let G_1 be the union of the good center leaves. Since the symmetric differences $A\Delta A^u$, $A\Delta A^s$, $B\Delta B^u$, $B\Delta B^s$ are zero sets, absolute continuity of the center foliation implies that $\mu(M \setminus G_1) = 0$. Discard from G_1 those center leaves $W^c p$ for which $G_1 \cap W^{cu}p$ fails to have full leaf measure in $W^{cu}p$. (For example, if there are only countably many good center leaves in a center unstable leaf, we discard all of them.) Discard also those center leaves $W^c p$ for which $G_1 \cap W^{cs}p$ fails to have full leaf measure in $W^{cs}p$. The union of the remaining good center leaves, G_2, has full measure. Repeat the process, and discard from G_2 those center leaves $W^c p$ for which $G_2 \cap W^{cu}p$ or $G_2 \cap W^{cs}p$ fails to have full leaf measure. Induction gives $G_1 \supset G_2 \supset \cdots$. Then $E = \bigcap G_k$ is a set of full measure that consists of good center leaves $W^c p$ for which $G \cap W^{cu}p$ and $G \cap W^{cs}p$ have full leaf measure. We refer to the points and center leaves in E as *excellent.* Excellent center leaves are good, and moreover if $W^{cu}p$ contains one excellent center leaf then it consists almost entirely of excellent center leaves. The same holds for center stable leaves.

To arrive at a contradiction, suppose that A and $B = M \setminus A$ both have positive measure. Then

$$A^* = \{a \in A : W^c a \text{ is excellent and } a \text{ is a center density point of } A\}$$

$$B^* = \{b \in B : W^c b \text{ is excellent and } b \text{ is a center density point of } B\}$$

differ from A and B by zero sets, and so have positive measure. Essential accessibility implies that there is a us-path γ from $p \in A^*$ to $q \in B^*$. Let n be the number of legs of γ. The path γ can be approximated by a us-path γ' that starts at p, ends at a point q' arbitrarily near q, and whose corners occur at excellent center leaves. (This is what makes excellent center leaves excellent: turning a corner at an excellent center leaf permits turning the next corner also at an excellent center leaf.) The holonomy maps $\theta : W^c p \to W^c q$ and $\theta' : W^c p \to W^c q'$ are composites of n diffeomorphisms of class C^1. For the unstable and stable foliations restricted to the center unstable and center stable leaves respectively are of class C^1. This follows from Theorem B in **[PugShWi]** and the center bunching assumption. In fact, θ' C^1-approximates θ. Thus there is an $r > 0$ that depends on n but is independent of q' as $q' \to q$, such that the concentration of A in $W^c(q', r)$ is $\geq .9$ and the concentration of B in $W^c(q, r)$ is also $\geq .9$.

The stable holonomy map $h : W^{cu}q' \to W^{cu}q$ preserves center leaves and is absolutely continuous. Thus $W^{cu}q'$ consists almost entirely of excellent center leaves, the same is true of $W^{cu}q$, and h sends almost every excellent center leaf in $W^{cu}q'$ to an excellent center leaf in $W^{cu}q$. In particular, the path γ' extends to a us-path $\gamma' \cup \epsilon$ where ϵ is an arbitrarily short three legged path with excellent corners that starts at q' and ends at a point $q'' \in W^c(q, r)$. As $q' \to q$, the path ϵ shrinks to q, the point q'' converges to q, and the us-holonomy map $W^c q' \to W^c q$ around ϵ converges in the C^1 sense to the identity. Thus A has concentration $\geq .8$ in $W^c(q'', r)$. Since r is fixed while $q'' \to q$, this contradicts the fact that B has concentration $\geq .9$ in $W^c(q, r)$.

Here is the upshot. We have

$$A^* = A_0 \subset A \subset A_1 = M \setminus B^*,$$

where $A_1 \setminus A_0$ is a zero set and us-paths starting in A_0 stay in A_1. Thus $A \in \mathrm{Sat}_0(\mathcal{M}^u \cap \mathcal{M}^s)$, which completes the proof of (7.1) and hence of the theorem. $\square$

We now describe very briefly a second approach to Theorem 7.1, which is in the spirit of the julienne density point argument used to prove Theorem 2.2. The strategy is to prove that a partial analogue for Lebesgue density points of Theorem 2.13 holds when the center foliation is absolutely continuous. Theorem 7.1 then follows in the same way that Theorem 2.2 follows from Theorem 2.13. Recall that $D(X)$ is the set of Lebesgue density points for a measurable set X.

THEOREM 7.2. *Let $f \in \mathit{Diff}^2_\mu(M)$ where M is compact. Suppose f is partially hyperbolic, dynamically coherent, and center bunched. Assume that the center foliation $\mathcal{W}^c$ is absolutely continuous with bounded jacobians. Let $A \in \mathrm{Sat}_0(\mathcal{M}^u) \cap \mathrm{Sat}_0(\mathcal{M}^s)$. Then $D(X) \in \mathcal{M}^u \cap \mathcal{M}^s$.*

In this theorem, we assume the hypotheses of parts (b) and (c) of Theorem 2.13 and obtain the conclusions of both (b) and (c). The proof uses arguments similar to those in the proof of Theorem 7.1.

We conclude with a technical remark. The bunching assumption that is needed in Theorems 7.1 and 7.2 is weaker than the bunching assumption in Theorem 2.2. All that is used in the arguments in this appendix is that the stable and unstable holonomies between center leaves are C^1. With the notation of Section 4 of **[PugSh6]**, it is sufficient to have $\nu < \gamma^2$, whereas Theorem 2.2 assumes that $\nu < \gamma^{2+2\theta}$.

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