ON A THEORY OF COST FOR EQUATION SOLVING

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We study algorithms for finding a zero of a single complex polynomial, from the point of view of computational complexity. Thus the problem is to give comparative assessment of the cost of various methods for equation solving. The case of a single polynomial is focussed on as the prototype of a non-linear system of equations.

A class of algorithms which we call incremental algorithms is studied systematically, and a subclass is distinguished for its speed. Criteria are developed to minimize the cost relative to the degree of the polynomial and probability of success.

The incremental algorithms are iterative ones and include the classic and fast methods that have proved successful in practice. A measure of efficiency is described and incremental algorithms of efficiency k are characterized. Among these are a special series, incremental $k^{\underline{th}}$ Schroeder algorithms, $k = 1, 2, 3, \ldots, \infty$, which seem simplest to administer. For a method based on these algorithms we show that the number of steps required is linear, and the number of multiplications quadratic. In view of the robustness, and that one has quadratic order or higher near the solution, these estimates are quite strong. The speed is comparable to Schur-Cohn methods, but for Schur-Cohn, round-off errors are serious and convergence at a solution is only linear.

Background for this paper is S. Smale, "The Fundamental Theorem of Algebra and Complexity Theory," Bull. Amer. Math. Soc,., 1981, p. 1-36. As there, we use heavily the theory of shlicht functions. A full detailed account is in the process of being written. We finish this brief announcement by stating one result with

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An <u>incremental algorithm</u> is an analytic endomorphism of the complex numbers of the form

$$z' = I(z) = z + FR(h, f, z)$$

where $F = \frac{-f(z)}{f'(z)}$, $0 \leqslant h \leqslant 1$, f is a complex polynomial and $R(0, f, a) \equiv 0$. Thus, h, f could be thought of as parameters. The most famous example is the <u>incremental Newton method</u> where $R \equiv h$, or simply Newton's method where $h = 1$ as well.

An important generalization is <u>incremental $k^{\underline{th}}$ Schroeder's</u> method $S_k(z) = z + FR$ where

$$R(h, f, z) = \frac{1}{F} T_k \left[\sum_{\ell=1}^{\infty} \frac{(f_z^{-1})^{(\ell)}}{\ell!} f(z)(-hf(z))^{\ell} \right] .$$

In this expression T_k is truncation at the $k^{\underline{th}}$ pwer of h , and $(f_z^{-1})^{(\ell)}(f(z))$ is the $\ell^{\underline{th}}$ derivative at $f(z)$ of the branch of f^{-1} which sends $f(z)$ to z . Here k is an integer between 1 and ∞ inclusive; S_1 is incremental Newton and

$$E_2(z) = z - \frac{f(z)}{f'(z)} (h - \sigma_w h^2) , \text{ where } \sigma_2 = \frac{f^{(2)}(z)F}{2!f'(z)} .$$

An <u>approximate zero</u> of a complex polynomial f is defined as in the above reference (i.e., so that Newton's method $(k = 1)$ converges rapidly). Let the space of initial points z_0 , be the set $S_R^1 = \{z \in \mathbb{C} \mid |z| = R\}$ with $R = 3$. Let $P_d(1)$ be the space of all complex polynomials $f(z) = \sum_{i=0}^{d} a_i z^i$, $a_d = 1$, $|a_i| \leqslant 1$, $i = 0, \ldots, d - 1$. Take normalized Lebesgue measure on the product $S_R^1 \times P_d(1)$ for probability measure.

<u>Theorem</u>. There exist universal positive constants K_1 , K_2 , (probably less than 10) with the following true. Given $(z_0, f) \in S_R^1 \times P_d(1)$, take $k = [\log d]$. Then for any μ , $0 < \mu < 1$, with probability $1 - \mu$, z_s will be an approximate zero of f where

$$z_{n+1} = S_k(z_n) , \quad n = 0, 1, \ldots, s - 1$$

and

$$s = K_1 d \frac{|\log \mu|}{\mu} + K_2 \text{ , (greater integer) .}$$

Here μ is interpreted in terms of probability in $S^1_R \times P_d(1)$.