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DYNAMICAL SYSTEMS

Morse–Smale Diffeomorphisms Are Unipotent on Homology[†‡]

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On the same day on the beaches of Rio that Steve Smale has written about in these proceedings [4], we discussed the question of what isotopy classes of diffeomorphisms admit a Morse–Smale diffeomorphism. The author guessed that if $f \in \mathrm{Diff}^r(M)$ is Morse–Smale, then all the eigenvalues of $f_* : H_*(M, R) \supsetdot$ are roots of unity, or what is the same thing $\exists n, m \in Z$ such that $(f_*{}^n - I)^m = 0$. We later learned from R. Thom that such linear maps are called unipotent.[§] M will denote a compact differentiable manifold.

THEOREM If $f: M \to M$ is homotopic to a Morse–Smale diffeomorphism $f_* : H_*(M, R) \supsetdot$ is unipotent.

Thom asked the author if a diffeomorphism $f: M \to M$ admits a nondegenerate Morse function φ as first integral, i.e., $\varphi(f(x)) = \varphi(x)$, then is f_* unipotent on homology? The proof of this fact was worked out with J. Guckenheimer at the coffee hour in Salvador, and follows almost immediately from Peixoto [3], who essentially proved that any such f is

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§ Actually quasi unipotent is the standard term for this.

on the boundary of the Morse–Smale systems and hence isotopic to a Morse–Smale diffeomorphism. We will sketch the proof of this, since Peixoto's theorem is actually for flows and a slight change is necessary for diffeomorphisms. Consider $h_t = (-\mathrm{grad}_t\,\varphi) \circ f$, for small t; then the singularities of φ are the nonwandering set of h_t and for appropriate arbitrarily small choice of t they are hyperbolic periodic points of f. Now by Palis [2], h_t may be approximated by a Morse–Smale diffeomorphism. So we have:

COROLLARY If $f \in \mathrm{Diff}^r(M)$ has a nondegenerate Morse function as first integral $f_* : H_*(M, R) \circlearrowleft$ is unipotent.

Unipotent is the best that one can do in this corollary as Bob Williams showed the author an example of an f with a nondegenerate Morse function as first integral, in the homotopy class of the linear map $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$: $T^2 \to T^2$, where T^2 is the 2-torus. Thom was interested in this corollary because with it one can give a topological proof of the monodromy theorem for an isolated singularity of a complex hypersurface. Since unipotent is the best one can do in the corollary, not periodic, it does not prove the Brieskorn conjecture.

It, of course, suffices to prove the theorem when f is Morse–Smale. For this we use a sort of filtration.

PROPOSITION† Let $f \in \mathrm{Diff}^r(M)$ be Morse–Smale. There exist compact manifolds smooth boundary M_i, $i = 0, \ldots, n$, $M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0$ such that:

a. $f(M_i)$ is contained in the interior of M_i;

b. $i : H_j(M_i, R) \to H_j(M, R)$ is an isomorphism for $j < i$ and surjective for $j = i$;

c. $H_j(M_j) = 0$ for $j > i$.

From the proposition, we prove the theorem easily by induction. We need to show that all the eigenvalues of $f_* : \bigoplus_{i \le k} H_i(M, R) \circlearrowleft$ are roots of unity assuming this fact for $f_* : \bigoplus_{i \le k-1} H_i(M, R) \circlearrowleft$. By (a) and (b) it follows that all the eigenvalues of $f_* : \bigoplus_{i < k-1} H_i(M_k, R) \circlearrowleft$ are roots of unity and by (a) and (b) again it follows that any eigenvalue of f_* : $H_k(M, R) \circlearrowleft$ is already an eigenvalue of $f_* : H_k(M_k, R) \circlearrowleft$. So, it suffices to prove that all the eigenvalues of $f_* : H_k(M_k, R) \circlearrowleft$ are roots of unity.

† *Added in proof*: The proof of the proposition seems not to be so easy. The theorem and corollary are correct and proofs will appear elsewhere.

Since f is a diffeomorphism, $f_* : H_k(M_k, R) \supset$ is an integer matrix of determinant one. Hence its characteristic polynomial is a monic polynomial over the integers; so by a result of algebra if all its roots have absolute value one then they are all roots of unity. So we may assume that $f_* : H_k(M_k, R) \supset$ has an eigenvalue which is not of absolute value one, and since its determinant is one it has an eigenvalue of absolute value greater than one. Consequently the trace of $f_*^n: H_k(M_k, R) \supset$ is unbounded, but the trace of $f_*^n: H_j(M_k, R) \supset$ for $j \neq k$ stays bounded. So the Lefshetz number

$$L(f^n \mid M_k) = \sum_{i=0}^{k} (-1)^i \text{ trace } f_*^n: H_i(M_k, R) \supset$$

is unbounded as $n \to \infty$. But $f: M_k \supset$ has a finite number of hyperbolic periodic points since f was Morse–Smale, so $L(f^n \mid M_k)$ is bounded and all the eigenvalues of $f_* : H_k(M_k, R) \supset$ are roots of unity.

The proof of the proposition follows from Meyer's energy functions for Morse–Smale systems [1], which is equally valid for diffeomorphisms (by taking suspensions say) and standard Morse theory type arguments.

It would be interesting to have necessary and sufficient conditions for a diffeomorphisms to be isotopic (homotopic) to a Morse–Smale diffeomorphism and, in particular, to know to what extent the converse to the theorem is true, even say for simply connected manifolds of high dimension so that some of the usual difficulties of differentiable topology vanish. More precisely:

PROBLEM Let $\dim M \geq 6$ and $\Pi_1(M) = 0$. Let $f \in \mathrm{Diff}^r(M)$ be unipotent on homology. Is f isotopic (homotopic) to a Morse–Smale diffeomorphism? F. Takens has pointed out to the author that this problem is more reasonable if $H_*(M, Z)$ has no torsion.

The problem seems very related to finding a good handle decomposition in Smale's theorem [4].

References

[1] K. R. Meyer, Energy functions for Morse–Smale systems, *Amer. J. Math.* **90** (1968), 1031.

[2] J. Palis, On Morse–Smale dynamical systems, *Topology* **8** (1969), 385.

[3] M. M. Peixoto, Qualitative theory of differential equations, *in* "Differential Equations and Dynamical Systems," (J. K. Hale and J. P. La Salle, eds.), Academic Press, New York, 1967.

[4] S. Smale, Isotopy and stability, this Vol.