EXPANDING MAPS

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It is the purpose of this note to give a short survey of some of the facts known about expanding maps and of one approach to the fundamental problem of expanding maps, which is stated below. Another approach to the problem is contained in an article by M. W. Hirsch in these same proceedings (vol. 14). Generalizations of some of these results, as well as some of the results themselves treated slightly differently, are contained in John Franks, *Anosov Diffeomorphisms*, also in these proceedings (vol. 14). Smale [7] and R. F. Williams [8] have used expanding maps and generalizations to branched manifolds to study basic sets of diffeomorphisms. Manifold will be used throughout to denote a connected C^∞-manifold without boundary.

DEFINITION 1. Let M be a complete Riemannian manifold. A C^1 endomorphism $f: M \to M$ is expanding if there exist constants $c > 0$ and $\lambda > 1$ such that $\|Tf^m v\| \geq c\lambda^m \|v\|$ for all $v \in TM$ and all $m > 0$.

If M is compact this definition is clearly independent of the metric.

DEFINITION 2. Let $f: X \to X$ and $g: Y \to Y$ be continuous maps of topological spaces. f and g are said to be topologically conjugate (semiconjugate) if there exists a homeomorphism (continuous map) $h: X \to Y$ such that $hf = gh$. h is called a conjugacy (semiconjugancy).

FUNDAMENTAL PROBLEM OF EXPANDING MAPS. Find all expanding maps of compact manifolds, up to topological conjugacy.

Examples of expanding maps. These examples are motivated by the examples of Anosov diffeomorphisms given in Smale [7].

Let G be a 1-connected Lie group with a left invariant metric and let $A: G \to G$ be an expanding Lie group automorphism. It is easy to see that G must be nilpotent. Now let $\Gamma \subset G$ be a uniform discrete subgroup of G, i.e. G/Γ is a compact manifold called a nilmanifold. If $A(\Gamma) \subset \Gamma$, A defines a map $\bar{A}: G/\Gamma \to G/\Gamma$ which is expanding, which we shall call a nil-expanding map.

EXAMPLE 1. $G = R$; $\Gamma = Z$; $R/Z = S^1$ and A is multiplication by an integer n, with $|n| > 1$.

2. $G = R^n$; $\Gamma = Z^n$; $R^n/Z^n = T^n$ and A is a matrix with integer entries and eigenvalues greater than 1 in absolute value.

3. G is the group of lower triangular matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}$$

$x, y, z \in R$ and Γ the uniform discrete subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \gamma & \beta & 1 \end{pmatrix}$$

where $\alpha, \beta, \gamma \in Z$. If $a, b, c \in Z$ with $|a|, |b|, |c| > 1$ and $ab = c$ then the map

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ ax & 1 & 0 \\ cz & by & 1 \end{pmatrix}$$

defines an expanding Lie group automorphism of G which takes Γ into itself and, hence, projects to a nil-expanding map of G/Γ.

Modification of these examples. Let G be a 1-connected nilpotent Lie group, C a compact group of automorphisms of G, $G \cdot C$ the semidirect product of G and C considering G as acting on itself by left translations, and Γ a torsion free uniform discrete subgroup of $G \cdot C$. It is a theorem of L. Auslander [1] that $\Gamma \cap G$ is a uniform discrete subgroup of G and $\Gamma \cap G$ is normal in G with $\Gamma/\Gamma \cap G$ finite. G/Γ the orbit space of G under the action of Γ is a compact manifold. Let $A: G \cdot C \to G \cdot C$ be an automorphism such that $A(\Gamma) \subset \Gamma$, $A(G) = G$ and $A|G$ is an expanding Lie group automorphism. A induces an expanding map $\bar{A}: G/\Gamma \to G/\Gamma$, which following the terminology of M. W. Hirsch we call an infranil-expanding map. $\bar{A}: G/\Gamma \to G/\Gamma$ is finitely covered by the nil-expanding map $\overline{A|G}: G/G \cap \Gamma \to G/G \cap \Gamma$.

$$\begin{array}{rccc} \overline{A|G}: & G/G \cap \Gamma & \to & G/G \cap \Gamma \\ & \downarrow & & \downarrow \\ \bar{A}: & G/\Gamma & \to & G/\Gamma \end{array}$$

Example. $G = R^n$, $C = \mathcal{O}(n)$ the orthogonal group. Then the G/Γ are the compact flat Riemannian manifolds.

Theorem (D. B. A. Epstein—Shub [3]). *If M is a compact flat Riemannian manifold, M admits an infranil-expanding map.*

The problem of finding infranil-expanding maps when the group is not abelian seems much more difficult. While such examples do exist, D. B. A. Epstein has an example of a nilmanifold which does not admit an expanding map.

Problem. Find all infranil-expanding maps up to topological conjugacy.

Question. Are all expanding maps of compact manifolds topologically conjugate to infranil-expanding maps?

Theorems about expanding maps. Proofs of these theorems may be found in [6].

Theorem 1. *Let M be a compact manifold and $f: M \to M$ an expanding map. Then*

(a) *f has a fixed point, m_0.*

(b) *The universal covering space of M is diffeomorphic to R^n.*

(c) *The periodic points of f are dense in M.*
(d) *f has a dense orbit.*
(e) *M is a $K(\Pi_1(M, m_0), 1)$.*
(f) *$\Pi_1(M, m_0)$ is torsion free.*
(g) *$x(M) = 0$ where $x(M)$ is the Euler characteristic of M.*

THEOREM 2. *If M is compact, $f, g: M \to M$ are expanding maps and f is homotopic to g then f is topologically conjugate to g.*

COROLLARY. *An expanding endomorphism of a compact manifold is structurally stable.*

THEOREM 3. *Let N be compact and $g: N \to N$ continuous with $g(n_0) = n_0$. Let $f: M \to M$ be expanding with $f(m_0) = m_0$. Then if $A: \Pi_1(N, n_0) \to \Pi_1(M, m_0)$ is a homomorphism such that*

$$\begin{array}{ccc} \Pi_1(N, n_0) & \xrightarrow{A} & \Pi_1(M, m_0) \\ g_\# \big\downarrow & & \big\downarrow f_\# \\ \Pi_1(N, n_0) & \xrightarrow{A} & \Pi_1(M, m_0) \end{array}$$

commutes, there exists a unique semiconjugacy $h: N \to M$ such that $h(n_0) = m_0$, $fh = hg$, and $h_\# = A$. Moreover, if M is compact, g is expanding, and A is an isomorphism then h is a topological conjugacy, i.e. h is a homeomorphism.

Applications of Theorem 3. Let M be compact, $f: M \to M$ expanding, and m_0 a fixed point of f. If $\Pi_1(M, m_0)$ has a nilpotent subgroup of finite index, it follows that the Hirsch-Plotkin radical of $\Pi_1(M, m_0)$, i.e. the maximal locally nilpotent normal subgroup of $\Pi_1(M, m_0)$, is of finite index in $\Pi_1(M, m_0)$. Recalling that $\Pi_1(M, m_0)$ is torsion free, $\Pi_1(M, m_0)$ is in the terminology of L. Auslander and E. Schenkman [2] a nil-admissible group. Thus, $\Pi_1(M, m_0)$ embeds as a uniform discrete subgroup of a nilpotent group N semidirect product with a finite group of automorphisms F. Moreover, by L. Auslander [1] $f_\#: \Pi_1(M, m_0) \to \Pi_1(M, m_0)$ extends uniquely to an automorphism $A: N \cdot F \to N \cdot F$ with $A(N) = N$.

THEOREM 4 (JOHN FRANKS [4]). *$A|N: N \to N$ is expanding. Thus, A induces an infranil-expanding map $\bar{A}$ of $N/\Pi_1(M, m_0)$ with $A|\Pi_1(M, m_0) = f_\#$.*

Theorem 3 now says that $\bar{A}$ and f are topologically conjugate, so we have

THEOREM 5. *An expanding map $f: M \to M$ of a compact manifold M is topologically conjugate to an infranil-expanding map if and only if $\Pi_1(M, m_0)$ has a nilpotent subgroup of finite index.*

ALGEBRAIC FACTS ABOUT $\Pi_1(M, m_0)$. Let $f: M \to M$ be expanding. We state some facts about $\Pi_1(M, m_0)$ not stated above.

PROPOSITION [6]. *Let m_0 be a fixed point of f. Then $f_{\#} : \Pi_1(M, m_0) \to \Pi_1(M, m_0)$ is a monomorphism,* $\mathrm{Im}(f_{\#})$ *is of finite index in $\Pi_1(M, m_0)$ and $\bigcap_{i=0}^{\infty}\mathrm{Im}(f_{\#}^{i}) = e$ the identity of $\Pi_1(M, m_0)$.*

DEFINITION. A finitely generated group is said to have polynomial growth if given a set of generators of the group the function $\gamma(S)$, the number of distinct elements of the group which can be written as a word of length $\leq S$ in the generators, is majorized by a polynomial.

The following Theorem is proven in J. Wolf [9] and is a combination of Theorems of J. Wolf [9] and Milnor [5].

THEOREM. *A finitely generated solvable group, either is polycyclic and has a nilpotent subgroup of finite index and is thus of polynomial growth, or has no nilpotent subgroup of finite index and is not of polynomial growth.*

THEOREM 6 (JOHN FRANKS [4]). *Let $f: M \to M$ be an expanding map with $\Pi_1(M, m_0)$ finitely generated; then $\Pi_1(M, m_0)$ has polynomial growth and thus nilpotent may be replaced by solvable in Theorem 5.*

The question of whether an expanding map of a compact manifold M is topologically conjugate to an infranil-expanding map is thus equivalent to the question of whether $\Pi_1(M, m_0)$ has a solvable subgroup of finite index.

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