On the Entropy Conjecture : a report on conversations among R. Bowen, M. Hirsch, A. Manning, C. Pugh, B. Sanderson, M. Shub and R. Williams.

recorded by

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The topological entropy of a map $f : M \to M$, h_f, measures how much f mixes up the point set topology of M while $f_* : H_*(M ; \mathbb{R}) \circlearrowleft$ measures how much f mixes up the algebraic topology of M. For the past few years it has seemed likely that h_f dominates f_*. Precisely

Entropy Conjecture. If M is compact and f is any diffeomorphism then $\lambda_f \leq h_f$ where λ_f is the logarithm of the largest modulus of the eigenvalues of f_*; i.e., $\exp(\lambda_f)$ = the spectral radius of f_*.

There is a fair amount of evidence in favour of this conjecture. For example, those diffeomorphisms for which it holds form a C^0-dense set in Diff(M) [see 7]. It holds for Anosov diffeomorphisms and for all known structurally stable diffeomorphisms [8]. Finally, Anthony Manning has proved it for all homeomorphisms if M has dimension ≤ 3 [6]. Besides, he proved that h_f is always $\geq \lambda_{1f}$ = the log of the spectral radius of $f_* \mid H_1(M ; \mathbb{R})$. Here we point out that the Entropy Conjecture fails for some homeomorphisms of high dimensional manifolds, and that H_1 cannot be replaced by H_2 in Manning's Theorem.

Theorem. There exists a homeomorphism f of some smooth M^8 with $0 = h_f < \lambda_f$. In fact $f_* \mid H_2(M ; \mathbb{R})$ has a real eigenvalue > 1.

Proof. Let A be an Anosov diffeomorphism of the 2 - torus, T^2. On $H_1(T^2)$, A_* has an eigenvalue $\mu > 1$ so $\lambda_A > 0$. Let $g : [-1, 1] \to [-1, 1]$ be a monotone homeomorphism fixing only ± 1 and having a source at -1, and a sink at $+1$. Let K be the two-point suspension of T^2, $K = T^2 \times [-1, 1]$ with $T^2 \times \{\pm 1\}$ pinched to points $P_\pm$ and define $B : K \to K$ by

$$(x, t) \mapsto (Ax, gt) .$$

B is a homeomorphism whose nonwandering set, $\Omega(B)$, is exactly the two "poles" $P_{\pm}$. Therefore, the topological entropy of B is zero [1]. On homology, B_* is just A_* with the dimensions increased by 1. Hence $\lambda_B > 0$.

Since K is not a manifold, we are not finished. Let $i : K \to \mathbb{R}^8$ be a PL - embedding. Any two PL - embeddings of K in $\mathbb{R}^8$ are equivalent by an ambient PL - homeomorphism of $\mathbb{R}^8$ (see [4] and [5], actually $\mathbb{R}^6$ would suffice for this) so there exists $\overline{B}$ making

$$\begin{array}{ccc} K & \xrightarrow{\ i\ } & \mathbb{R}^8 \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle \overline{B}} \\ K & \xrightarrow{\ i\ } & \mathbb{R}^8 \end{array}$$

commute. Let N be the star neighbourhood of iK in the second barycentric subdivision of a triangulation of $\mathbb{R}^8$ which includes iK. Then N and $\overline{B}N$ are regular neighbourhoods of K. Any two such are PL - equivalent [3], so there is a PL - homeomorphism $h : \overline{B}N \to N$ fixing all points of K. The composition $h \circ \overline{B}$ is a homeomorphism $C : N \to N$ extending B to N.

Take two copies of (N, K), say (N_-, K_-) and (N_+, K_+). Identify them across ∂N, glueing by the identity map. This produces a compact combinatorial 8 - manifold M containing the compact set $L = K_- \cup K_+$. By [3], M has a compatible smooth structure. On M there is a homeomorphism E which is just C on each copy of N. To make the sought-after f, we shall compose E with a deformation D of M which "dominates" E.

In Lemma 2.3 of [2], Moe Hirsch shows that there is a transverse field across ∂N. In fact, through each point $x \in \partial N$ he finds a unique segment in N from x to $y \in K$. This gives a PL - surjection $R : [-1, 1] \times \partial N \to M$ such that

$R \mid (-1, 1) \times \partial N$ is a homeomorphism onto $M - L$

$R \mid \{0\} \times \partial N$ is the inclusion $\partial N \hookrightarrow M$

$R \mid \{\pm 1\} \times \partial N$ is a surjection to $K_{\pm}$.

Lift E to $(-1, 1) \times \partial N$ by R, $\overline{E} = R^{-1} \circ E \circ R$, and define

$$e(t) = \inf\{\overline{E}_1(t, w) \; ; \; w \in \partial N\} \qquad -1 < t < 1$$

where $\overline{E} = (\overline{E}_1, \overline{E}_2)$ respecting $\mathbb{R} \times \partial N$. Since E is a homeomorphism which leaves $L = K_- \cup K_+$ invariant, it is clear that $-1 < e(t) < 1$, $e(t) \to \pm 1$ as $t \to \pm 1$, and that e is continuous. Let $\tau : [-1, 1] \to [-1, 1]$ be any homeomorphism $< e$

$$\tau(t) < e(t) \qquad -1 < t < 1 .$$

Consider

$$\overline{D} : [-1, 1] \times \partial N \longrightarrow [-1, 1] \times \partial N \qquad (t, w) \mapsto (\tau(t), w)$$

which covers the homeomorphism $D : M \to M$. The composition $\overline{D} \circ \overline{E}$ has the property

$$\overline{D}_1 \circ \overline{E}(t, w) = \tau \circ \overline{E}_1(t, w) \geq \tau \circ e(t) > t$$

for $-1 < t < 1$ and $\overline{D} = (\overline{D}_1, \overline{D}_2)$ respecting $\mathbb{R} \times \partial N$. Hence $f = D \circ E$ has the property that

$$f^n(x) \to K_{\pm} \quad \text{as} \quad n \to \pm \infty \qquad x \in M - L .$$

Therefore $\Omega(f) \subset L = K_- \cup K_+$ and since $f \mid K_{\pm}$ is just B, $\Omega(f)$ is finite. Therefore f has zero entropy [1]. In $H_*(T^2 ; \mathbb{R})$, A_* sends some non-zero 1-cycle a onto some multiple μa, $\mu > 1$, and B_* sends its suspension, $b \in H_*(K ; \mathbb{R})$, to the multiple μb. Think of b as a 2-cycle lying in K_+. We claim that $b \neq 0$ in $H_*(M ; \mathbb{R})$. Suppose b bounds some 3-chain c in M. Since M is smooth, we can assume c is transverse to K_-. Since c and K_- have total dimension < 7, this means $c \cap K_- = \emptyset$. But $M - K_-$ retracts to K_+, so

$$b = \partial c \text{ in } M - K_- \Longrightarrow b = 0 \text{ in } H_*(K_+)$$

a contradiction. Thus, $f_*(b) = B_*(b) = \mu b$ for some $\mu > 1$ and non-zero $b \in H_*(M;\mathbb{R})$. Since f_* has this eigenvalue $\mu > 1$, the log of its spectral radius, λ_f, is > 1, completing the proof of our theorem.

Remark 1. The construction of f can be done in the PL category. For A, g, h, R exist as PL maps, so $B, C, E, \overline{E}$ are PL. Near $t = \pm 1$, $e(t)$ measures how sharply E propels points away from K_+ and toward K_-. Since E is PL, e is differentiable at $t = \pm 1$, and $0 < e'(\pm 1) < \infty$. Hence $\tau, \overline{D}, D$, and f exist as PL maps.

Remark 2. f has only four periodic points and yet $\sum_{i=0}^{8} (-1)^i \text{trace } f^n_{*i} \to \infty$ as $n \to \infty$. Thus by the Lefschetz Trace Formula, f provides an example of an isolated fixed point p of a PL homeomorphism with the property that

$$\text{Index}(f^n \text{ at } p) \to \infty \text{ as } n \to \infty .$$

Moreover, there is no C^1 g homologous to f on M^8 with a finite Ω[9], so this example cannot be smoothed. We could have done the same construction on a seven-manifold, M^7. On M^7 the Lefschetz formula does not eliminate the possibility of finding a smooth g homologous to f with a finite Ω. The existence of such a g would contradict the entropy conjecture.

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