Book Reviews

Gary Cornell* and Ian Stewart**

Geometric Theory of Dynamical Systems: An Introduction

by Jack Palis, Jr., and Wellington de Melo

New York: Springer-Verlag, 1982. xii + 198 pp.

Reviewed by Michael Shub

When I was an undergraduate at Columbia College, I used to do odd jobs around the math department to pay the rent, etc. If it wasn't me, then it must have been the graduate student who had the job just before I did who mimeographed Smale's paper on stable manifolds. Chapter 3 of the *Geometric Theory of Dynamical Systems* is concerned with one of the principal results of this paper, the Kupka–Smale theorem, and much of Chapter 2 is concerned with the local stable manifold theorem itself. It is this theory of stable manifolds and generalizations which provides the underlying structure of much of the successful analysis of dynamical systems both in the abstract and qualitative sense and in the applied and quantitative sense.

If we are given a compact differentiable manifold M, which we take without boundary for simplicity, and a function f: $M \to R$ we have the gradient flow of f, ϕ_t: $M \to M$, where $(d/dt)\,\phi_t(x) = -\mathrm{grad}\, f_x$. Note the minus sign so that the flow flows downhill, i.e., $(d/dt)\, f[\phi_t(x)] = -\|\mathrm{grad}\, f_x\|^2 \leq 0$. Morse theory proves that, for an open and dense (in the C^r topology) set of functions f, the vector field $X = -\mathrm{grad}\, f$ has only finitely many singularities, say $p_1, \ldots, p_m$, where $x(p_i) = 0$, and moreover near any of the critical points p_i there is a local chart so that f has the form

$$f(x) = f(p_i) - x_1^2 - x_2^2 - \cdots - x_u^2 + x_{u+1}^2 + x_{u+2}^2 + \cdots + x_n^2$$

where $x = (x_1, \ldots, x_n)$.

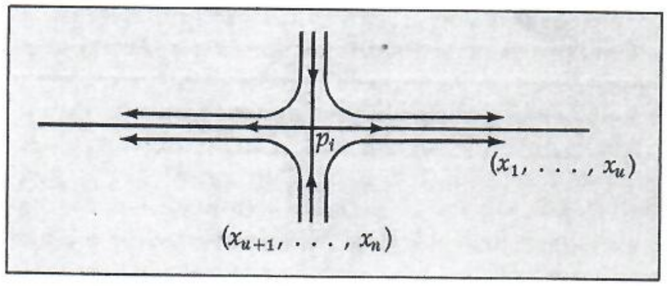
Thus for any $x \in M$, $\phi_t(x)$ converges to some p_i as $t \to +\infty$ and another as $t \to -\infty$. Near p_i, $-\mathrm{grad}\, f$ takes the form

$$(+2x_1, +2x_2, \ldots, +2x_u, -2x_{u+1}, -2x_{u+2}, \ldots, -2x_n)$$

and

$$\phi_t(x_1, \ldots, x_n) = (e^{+2t}x_1, e^{+2t}x_2, \ldots, e^{+2t}x_u, e^{-2t}x_{u+1}, e^{-2t}x_{u+2}, \ldots, e^{-2t}x_n)\ .$$

This gives the standard picture:



The points in the $(x_1, \ldots, x_u)$ space tend to p_i as t approaches $-\infty$, and the points in the $(x_{u+1}, \ldots, x_n)$ space tend to p_i as t approaches $+\infty$. Locally these are discs of dimension u called the index of the point p, and $s = n - u$. These discs are denoted by $W^u_{loc}(p_i)$ and $W^s_{loc}(p_i)$, respectively, the local unstable and local stable manifolds of p_i. . The set of $x \in M$ such that $\phi_t(x) \to p_i$ as $t \to -\infty$ is denoted by $W^u(p_i)$, and the set $x \in M$ such that $\phi_t(x) \to p_i$ as $t \to +\infty$ is denoted by $W^s(p_i)$, the (global) unstable and stable manifolds of P_i. Here $W^u(p_i)$ and $W^s(p_i)$ are 1-1 immersed discs of dimension u and s, respectively. The manifold M is the disjoint union of theses unstable (or stable) manifolds:

$$M = \bigcup_{i=1,\ldots,m} W^s(p_i) = \bigcup_{i=1,\ldots,m} W^u(pi)$$

This far Morse theory takes us. Facts about the homology theory of M can be calculated and the Morse inequalities deduced. Now add another condition, that all these manifolds $W^s(p_i)$, $W^u(p_i)$ are all transversal wherever they meet. The set of such f remains open and dense. The vector fields $X = -\mathrm{grad}\, f$ are called Morse–Smale. The underlying structure of M is more exposed, and in particular a full-chain complex for M with boundary maps is geometrically calculated. These Morse–Smale gradient vector fields are also structurally stable; i.e., if $X = -\mathrm{grad}\, f$ is Morse–Smale, then for any vector field Y sufficiently C^1 close to X there is a homeomorphism of M mapping the trajectories of X onto the trajectories of Y. That is, the structure is rigid or rough or robust.

Column Editors: *Dept. of Mathematics, University of Connecticut, Storrs, CT 06268; **Mathematics Institute, University of Warwick, Coventry CV4 7AL, England.

This is the happiest state of affairs that one could have in dynamical systems. On every manifold M we have an open set of vector fields, the Morse–Smale vector fields, and among the gradients this set of vector fields is also dense; the dynamics are relatively simple, robust and intimately linked to the topology of M. In fact, these vector fields stand at an important juncture of differentiable dynamics and differentiable topology. Andronov and Pontryagin introduced the notion of structural stability in 1937. Peixoto proved the density of structurally stable vector fields on the disc in 1958 and on two manifolds in general. To generalize Peixoto's conditions to higher dimensions Smale introduced the notion of the transversality of the stable and unstable manifolds. He exploited this structure in his work on the generalized Poincaré conjecture, the H-cobordism theorem and the structure of manifolds.

Gradient flows are particularly simple. For general vector fields the recurrence becomes much more complicated, especially in higher dimensions, and the relationships to the topology of M more difficult to see. Even in dimension two one has to consider more general singularities and periodic solutions. For the generic vector field the singularities and periodic orbits are hyperbolic and have stable and unstable manifolds (in the case of periodic solutions these are cylinders instead of discs) which meet transversally. This is the Kupka–Smale theorem. There is a notion of a Morse–Smale vector field for general vector fields as well as gradients. On two manifolds the Morse–Smale vector fields are open and dense and structurally stable. This is Peixoto's theorem.

Palis and de Melo begin with an introduction to calculus on manifolds and ordinary differential equations. They go on to linear vector fields, local stability and the stable manifold theorem in Chapter 2, the Kupka–Smale theorem in Chapter 3 and Peixoto's theorem in Chapter 4. They end with a survey of more advanced results. (The Morse–Smale vector fields are open and structurally stable, as are the geodesic flow on manifolds of negative curvature, Anosov systems and Axiom A and strong transversality systems, etc.) There are many excellent exercises and examples in this book, and the 114 illustrations are very instructive and very well done. The book is thoroughly geometric in flavor; it works in general when it can and in low dimensions when the going might get tough for a student. I think this book is an excellent introduction to the qualitative theory of differential equations even at the masters level. I don't know why Springer did not put it in their graduate text series; it clearly seems to belong there. Just the other day I received a mailing from Columbia College. Once a year the alumni are invited back for Dean's Day. Lectures of general interest are presented. At 11:15, "Order and Chaos in Physical and Chemical Dynamics" will be presented by Professor Philip Pechukas of the chemistry department. The announcement begins, "For those who take the long view, the question, 'Is the solar system stable?' has a certain urgency. Attempts to answer this question have uncovered a diversity of dynamical behavior, ranging from extreme order to absolute chaos, in systems as different in size as the galaxy and the molecule." Poincaré's problem and the origins of the qualitative theory for general college-educated audiences! And I'll bet the stable manifold theorem lurks somewhere behind this order and chaos in chemistry! We need good texts!

It seems that it is a rare topologist who knows the stable manifold theorem and a rare dynamicist with a constructive interest in topology. Palis and de Melo's book, while very geometric in spirit, tends to ignore the underlying manifold. The student should also be made aware of the Poincaré–Hopf theorem, the Morse inequalities and the general connections of dynamics and topology which are exemplified by gradient systems. Recently, for example, Bob Williams and John Franks have proven that every smooth flow on the three sphere of positive entropy has infinitely many distinct knots as periodic orbits. Of course, their proof uses stable manifold theory.

The Graduate School
City University of New York
New York, New York 10036

Several Complex Variables and Complex Manifolds, Vols. I and II
by Mike Field

London Mathematical Society Lecture Note Series, 65–66 New York: Cambridge University Press, 1982

Reviewed by Michael Eastwood

Suppose U is an open subset of $\mathbb{C}^n$, the space of n complex variables $z_1, z_2, \ldots, z_n$. A continuous function $f\colon U \to \mathbb{C}$ is said to be *analytic* or *holomorphic* if it is analytic as a function of one complex variable in each variable separately. (It is remarkable theorem of Hartogs that continuity is actually automatic from separate analyticity!) By iterating the one variable Cauchy integral formula

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_j - z_j| = \text{constant}} \frac{f(\zeta)\, d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}$$

and, just as for one variable complex analysis, we can conclude that f has a convergent power series expansion. There are other results, such as the uniqueness of analytic continuation, which generalize immediately to several variables.

Complex analysis in several variables, however, is far from being just a follow-your-nose transcription