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On the Geometry and Topology of the Solution Variety for Polynomial System Solving

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Abstract We study the geometry and topology of the rank stratification for polynomial system solving, i.e., the set of pairs (system, solution) such that the derivative of the system at the solution has a given rank. Our approach is to study the gradient flow of the Frobenius condition number defined on each stratum.

Keywords Condition number · Gradient flow · Solution variety · Stratification · Homotopy methods · Condition metric

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1 Introduction

1.1 The Solution Variety and Its Stratification

For $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$ and positive integers n,d, let $\mathcal{H}_d$ denote the vector space of homogeneous polynomials of degree d with coefficients in $\mathbb{K}$ and unknowns $X_0,\dots,X_n$. So if $f\in\mathcal{H}_d$, $f:\mathbb{K}^{n+1}\to\mathbb{K}$. For a multi-index of nonnegative integers $(\alpha)=(\alpha_0,\dots,\alpha_n)$ with $|(\alpha)|=\sum_0^n\alpha_i=d$, let $x^{(\alpha)}=\prod_0^n x^{\alpha_i}$ be the monomial with multi-index (α). Give $\mathcal{H}_d$ the Hermitian product or inner product which makes the basis of monomials of $\mathcal{H}_d$ orthogonal and makes $\|x^{(\alpha)}\|^2=\binom{d}{(\alpha)}^{-1}$, where

$$\binom{d}{(\alpha)}=\frac{d!}{\alpha_0!\cdots\alpha_n!}$$

is the multinomial coefficient. This Hermitian (inner) product is frequently called the Bombieri–Weyl product or Kostlan product. For a list of positive degrees $(d)=(d_1,\dots,d_m)\in\mathbb{N}^m$, let $\mathcal{H}_{(d)}=\prod_1^m\mathcal{H}_{d_i}$. So $\mathcal{H}_{(d)}$ is the set of all systems $f=(f_1,\dots,f_m)$ of homogeneous polynomials of respective degrees $\deg(f_i)=d_i$, $1\le i\le m$, and unknowns $X_0,\dots,X_n$. Considered as a function, $f:\mathbb{K}^{n+1}\longrightarrow\mathbb{K}^m$. We consider $\mathcal{H}_{(d)}$ endowed with the Hermitian (inner) product, and the corresponding norm (denoted $\|\cdot\|$). One of the most important properties of this inner product is the unitary invariance (see for example [10, Sect. 12.1]): if $f,g\in\mathcal{H}_{(d)}$ and U is a unitary (orthogonal if $\mathbb{K}=\mathbb{R}$) matrix of size $n+1$, then

$$\langle f,g\rangle=\langle f\circ U^*,g\circ U^*\rangle,\qquad \|f\|=\|f\circ U^*\|. \tag{1.1}$$

Another nice property of the Bombieri–Weyl product is the higher derivative estimate from [24] (see also [10, Sect. 14.2]): for a degree d homogeneous polynomial $f\in\mathcal{H}_d$, $\zeta\in\mathbb{C}^{n+1}$ and for any $k\ge 0$ we have

$$\|D^kf(\zeta)\|\le d(d-1)\cdots(d-k+1)\|f\|\|\zeta\|^{d-k}, \tag{1.2}$$

where $\|D^kf(\zeta)\|$ is the usual operator norm of the k-linear operator $D^kf(\zeta)$. For $k=0$ this reads $\|f(\zeta)\|\le\|f\|\|\zeta\|^d$.

The *solution variety* $V\subseteq\mathbb{P}(\mathcal{H}_{(d)})\times\mathbb{P}(\mathbb{K}^{n+1})$ is defined as the set of pairs (f,ζ) such that $f(\zeta)=0$. Observe that V is a smooth manifold (cf. [10, p. 193]), endowed with a natural Riemannian structure (and corresponding volume form) inherited from the Bombieri–Weyl product in $\mathcal{H}_{(d)}$ and the naturally induced Riemannian structure in $\mathbb{P}(\mathbb{K}^{n+1})$, which we will refer to as the Fubini–Study metric, both in the real and complex cases. We refer to this Riemannian structure in V and the metric it defines as the Fubini–Study metric.

Let $1\le k\le\min\{m,n\}$ and consider the stratum

$$\begin{aligned}W&=W^k=W^k_{(d)}\\&=\{(f,\zeta)\in\mathbb{P}(\mathcal{H}_{(d)})\times\mathbb{P}(\mathbb{K}^{n+1}):f(\zeta)=0\text{ and }\operatorname{rank}(Df(\zeta))=k\},\end{aligned}$$

and in the special case of rank 0,

$$\mathcal{W} = \left\{(f,\zeta) \in \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{P}\left(\mathbb{K}^{n+1}\right) : f(\zeta) = 0 \text{ and } Df(\zeta) = 0\right\}.$$

From (1.1), the mapping $f \to f \circ U^*$ (U a unitary matrix if $\mathbb{K} \in \mathbb{C}$, orthogonal if $\mathbb{K} = \mathbb{R}$) is an isometry both in $\mathcal{H}_{(d)}$ and $\mathbb{P}(\mathcal{H}_{(d)})$, and the mapping $(f,\zeta) \to (f \circ U^*, U\zeta)$ is an isometry in V and in each W^k.

1.2 Complexity and the Frobenius Condition Number for Polynomial Systems

In this paper we will study the geometry and topology of $W^k_{(d)}$ via the gradient flow of the Frobenius condition number $\widetilde{\mu}(f,\zeta)$ defined on this variety. We recall and introduce a few definitions before we state our principal results.

For $(a_1, \dots a_m) \in \mathbb{K}^m$ we denote by $\mathrm{Diag}(a_i)$ the $m \times m$ diagonal matrix whose ith diagonal entry is a_i. The *Frobenius condition number* in W is defined as follows:

$$\widetilde{\mu}(f,\zeta) = \|f\| \left\| Df(\zeta)^\dagger \, \mathrm{Diag}\left(\|\zeta\|^{d_i - 1} d_i^{1/2}\right)\right\|_F, \quad \forall (f,\zeta) \in W,$$

where $\|\cdot\|_F$ is the Frobenius norm (i.e. $\mathrm{Trace}(L^*L)^{1/2}$ where L^* is the adjoint of L) and † is the Moore–Penrose pseudoinverse. The Moore–Penrose pseudoinverse $L^\dagger : \mathbb{F} \to \mathbb{E}$ of a linear operator $L : \mathbb{E} \to \mathbb{F}$ of finite dimensional Hilbert spaces is defined as the composition

$$L^\dagger = i_{\mathbb{E}} \circ (L\,|_{\mathrm{Ker}(L)^\perp})^{-1} \circ \pi_{\mathrm{Image}(L)}, \tag{1.3}$$

where $\pi_{\mathrm{Image}(L)}$ is the orthogonal projection on image L, $\mathrm{Ker}(L)^\perp$ is the orthogonal complement of the nullspace of L, and $i_{\mathbb{E}}$ is the inclusion. If A is an $m \times (n+1)$ matrix and $A = UDV^*$ is a singular value decomposition of A, $D = \mathrm{Diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)$, then we can write

$$A^\dagger = VD^\dagger U^*, \qquad D^\dagger = \mathrm{Diag}\left(\sigma_1^{-1}, \dots, \sigma_k^{-1}, 0, \dots, 0\right). \tag{1.4}$$

Note that from (1.1) we also have

$$\widetilde{\mu}(f,\zeta) = \widetilde{\mu}\left(f \circ U^*, U\zeta\right), \quad \forall U \in \mathcal{U}_{n+1}, \tag{1.5}$$

where $\mathcal{U}_{n+1}$ is the set of unitary $(n+1) \times (n+1)$ matrices.

From Proposition 7 below, the mapping $A \mapsto A^\dagger$ restricted to the set $\mathcal{R}^k$ of rank k matrices is smooth, and hence $\widetilde{\mu} : W \longrightarrow \mathbb{R}$ is also smooth, for every choice of m, n, k.

Our main interest is the case $m = n = k$ and $\mathbb{K} = \mathbb{C}$. This is the case of n homogeneous equations in $n+1$ unknowns which we have studied in a series of papers on the complexity of Bezout's theorem. We recall a result in this case. In [27] we have bounded the number of projective Newton method steps sufficient to follow a homotopy $\Gamma_t = (f_t, \zeta_t)$ in W, by the length of the path Γ_t in the condition metric, using the condition number of [24, 27]. Recall that if we are given a Riemann structure on a manifold M, the length of a piecewise $\mathcal{C}^1$ curve Γ_t in M is then given by

$$\mathrm{Length}(\Gamma_t) = \int \|\dot{\Gamma}_t\| \, \mathrm{dt},$$

and a metric d on M is defined by $d(x,y) = \inf \text{Length}(\gamma)$ over piecewise $\mathcal{C}^1$ differentiable paths γ in W joining x to y. Recall that in the case $k = m = n$ the normalized condition number $\mu_{\text{norm}}(f,\zeta)$ is defined for $(f,\zeta) \in W$ by

$$\mu_{\text{norm}}(f,\zeta) = \|f\| \left\| \left(Df(\zeta)|_{\zeta^\perp}\right)^{-1} \text{Diag}\left(\|\zeta\|^{d_i - 1} d_i^{1/2}\right) \right\|_{\text{op}},$$

or ∞ if $\det(Df(\zeta)|_{\zeta^\perp}) = 0$. Here, $\|\cdot\|_{\text{op}}$ denotes the operator norm of a linear map. The *Condition Riemann Structure* $\|(\dot{f}_t, \dot{\zeta}_t)\|_\kappa$ on W is defined by $\|(\dot{f}_t, \dot{\zeta}_t)\|_\kappa = \mu(f_t,\zeta_t)\|(\dot{f}_t, \dot{\zeta}_t)\|$. We denote by $d = \max\{d_i : 1 \le i \le m\}$ the maximum of the degrees. The Main Theorem of [27] (see also the explicit versions in [2, 12, 13]) is the following.

Theorem 1 *Let $m = n = k$. There is a constant $C > 0$, such that if $\Gamma_t = (f_t, \zeta_t)$ $t_0 \le t \le t_1$ is a C^1 path in W with the Condition Riemann Structure, then*

$$Cd^{3/2}\text{Length}_\kappa(\Gamma_t)$$

steps of the projective Newton method are sufficient to continue an approximate zero x_0 of f_{t_0} with associated zero ζ_0 to an approximate zero x_1 of f_{t_1} with associated zero ζ_1.

The projective Newton method is Newton's method adapted to homogeneous problems in projective space. An approximate zero x with associated zero ζ is a point for which the projective Newton method converges quadratically to the zero ζ, see [27].

This theorem makes the geometry of W in the Condition Riemann Structure and the distance function a central object of study with potential application to the understanding of algorithms for solving square systems of polynomial equations. There is a difficulty in that the condition number μ_{norm} is not a smooth mapping; thus, we have introduced the Frobenius condition number, which is smooth. The corresponding smooth *Frobenius Condition Riemann Structure* on W is defined by $\|(\dot{f}_t, \dot{\zeta}_t)\|_{F,\kappa} = \widetilde{\mu}(f_t,\zeta_t)\|(\dot{f}_t, \dot{\zeta}_t)\|$. Note that $\mu_{\text{norm}}(f,\zeta) \le \widetilde{\mu}(f,\zeta) \le \sqrt{n}\mu_{\text{norm}}(f,\zeta)$. Thus the Main Theorem of [27] can now be rephrased with at most an extra factor of $\sqrt{n}$ in the estimate as follows.

Theorem 2 *Let $m = n = k$. There is a constant $C > 0$, such that if $\Gamma_t = (f_t, \zeta_t)$ $t_0 \le t \le t_1$ is a C^1 path in W with the Frobenius Condition Riemann Structure, then*

$$Cd^{3/2}\text{Length}_{F,\kappa}(\Gamma_t)$$

steps of the projective Newton method are sufficient to continue an approximate zero x_0 of f_{t_0} with associated zero ζ_0 to an approximate zero x_1 of f_{t_1} with associated zero ζ_1.

Besides the Main Theorem of [27] all the known results about μ_{norm} can be stated, up to some factors of $\sqrt{n}$, in terms of $\widetilde{\mu}$. For example, the analysis of good initial pairs for homotopy methods in [6–8] can be done using $\widetilde{\mu}$ instead of μ_{norm}.

1.3 The Set $\mathcal{B}$ of Best-Conditioned Pairs

We now wish to study the geometry of W with the Frobenius Condition Riemann Structure and the corresponding distance function. We will do this by studying the gradient of $\widetilde{\mu}$ on W. It turns out that our analysis extends to all m, n, k, so we have made our definitions and stated our theorems with this generality and we now return to considering general W. The topology of W is given to us by the gradient of $\widetilde{\mu}$. It is remarkably simple! The varieties V and W would seem to be so natural in algebraic geometry that we would not be surprised if some or even most of our results are known. But we are not aware of any references.

We start with the following basic result, whose proof may be found in Sect. 3.

Proposition 1 *Let* $\mathbb{K} = \mathbb{C}$ (*resp.* $\mathbb{R}$). *For any choice of* $m, n, k, (d)$, *the set* W *is a complex* (*real*) *submanifold of* V *of complex* (*real*) *codimension* $(m-k)(n-k)$. *Moreover*, $\mathcal{W}$ *is a complex* (*real*) *submanifold of* V *of complex* (*real*) *codimension* mn.

The following definition will play an important role in the study of W.

Definition 1 Let $\mathcal{B} = \mathcal{B}^k \subseteq W$ be the set of pairs (f, ζ) such that for any choice of affine representatives of f, ζ there exists an $m \times (n+1)$, rank k matrix B with all its nonzero singular values equal to 1 and such that

$$\mathrm{Diag}\big(d_i^{-1/2}\|\zeta\|^{1-d_i}\big)Df(\zeta) = \frac{\|f\|}{\sqrt{k}}B.$$

Here $Df(\zeta)$ is considered to be an $m \times (n+1)$ matrix in the standard bases

One can easily see that if the defining property of $\mathcal{B}$ holds for *some* choice of representatives of f, ζ, then it holds for every other choice. The following proposition characterizes $\mathcal{B}$. The proof is given in Sect. 4.

Proposition 2 *Given* $(f, \zeta) \in W$ *the following three are equivalent*:

(1) $(f, \zeta) \in \mathcal{B}$.
(2) *For any representatives of* f *and* ζ, *the* k *nonzero singular values of* $A = \mathrm{Diag}(d_i^{-1/2}\|\zeta\|^{d_i-1})Df(\zeta)$ *are equal and* $\|f\| = \|A\|_F$.
(3) *There are representatives with* $\|f\| = \|\zeta\| = 1$ *and a rank* k *matrix* B *with all its singular values equal to* 1 *such that*

$$f(z) = \frac{1}{\sqrt{k}}\mathrm{Diag}\big(d_i^{1/2}\langle z, \zeta\rangle^{d_i-1}\big)Bz, \qquad B\zeta = 0.$$

Again, note that if property (2) in Proposition 2 holds for some choice of representatives of f, ζ, then it holds for every other choice.

Let $\mathcal{G}_n$ denote $\mathcal{U}_n$, the group of $n \times n$ unitary matrices in the case $\mathbb{K} = \mathbb{C}$ and $\mathcal{O}_n$, the group of $n \times n$ orthogonal matrices in the case $\mathbb{K} = \mathbb{R}$. There is a natural action of the group $\mathcal{G}_{n+1}$ on V which leaves each W invariant, so we consider the action restricted to W:

$$\big(U, (f, \zeta)\big) \mapsto \big(f \circ U^*, U\zeta\big) \tag{1.6}$$

When all the d_i's are equal, the group $\mathcal{G}_m \times \mathcal{G}_{n+1}$ also acts on $\mathcal{B}$ by

$$\begin{aligned} &\rho\big((U, V)\big)\left(\frac{1}{\sqrt{k}} \operatorname{Diag}\big(d_i^{1/2} \langle z, \zeta\rangle^{d_i-1}\big) Bz, \zeta\right) \\ &\quad = \left(\frac{1}{\sqrt{k}} \operatorname{Diag}\big(d_i^{1/2} \langle z, V\zeta\rangle^{d_i-1}\big) UBV^* z, V\zeta\right). \end{aligned} \tag{1.7}$$

Note that, if not all the d_i's are equal, then the element in W represented by this last formula depends on the chosen representative of ζ, and thus the action is not well defined.

The geometrical structure of $\mathcal{B}$ can be analyzed using the following result, which is proved in Sect. 5.

Proposition 3

(1) *$\mathcal{B}$ is the quotient of a homogeneous space of $\mathcal{G}_m \times \mathcal{G}_{n+1}$ by a further free $S^1 \times S^1$ action in the complex case or a further free $\mathbb{Z}_2 \times \mathbb{Z}_2$ action in the real case.*
(2) *$\mathcal{B}$ has a structure of a fiber bundle over $\mathbb{P}(\mathbb{K}^{n+1})$ with the fiber a homogeneous space of $\mathcal{G}_m \times \mathcal{G}_n$.*
(3) *If $\mathbb{K} = \mathbb{C}$, $\mathcal{B}$ has real dimension $2mk + 2nk + 2n - 3k^2 - 1$.*
(4) *If $\mathbb{K} = \mathbb{R}$, $\mathcal{B}$ has real dimension $mk + nk + n - \frac{3}{2}k^2 - \frac{1}{2}k$.*
(5) *For every $(f, \zeta) \in W$, we have that $\widetilde{\mu}(f, \zeta) \geq k$, and $\mathcal{B} = \{(f, \zeta) : \widetilde{\mu}(f, \zeta) = k\}$.*
(6) *If $m = k \leq n$, then $\mathcal{G}_{n+1}$ acts transitively on $\mathcal{B}$.*
(7) *If all the d_i's are equal, then $\mathcal{G}_m \times \mathcal{G}_{n+1}$ acts transitively on $\mathcal{B}$.*

Note that $\widetilde{\mu}$ is equivariant under the action (1.6) and hence by Proposition 3(5), $\mathcal{B}$ is also invariant for this action of $\mathcal{G}_{n+1}$.

Using Proposition 3 we may identify $\mathcal{B}$ topologically. The dependence of $\mathcal{B}$ on (d) is minor! We work out some examples when $m = k \leq n$ and their homotopy groups in Sect. 8. An example is the following result, which is immediate from the precise description of the homogeneous spaces of Proposition 3, described in Sect. 5.

Corollary 1 *If $\mathbb{K} = \mathbb{C}$ then $\mathcal{B}$ is connected. If $\mathbb{K} = \mathbb{R}$ and k, m, n are not all equal, then $\mathcal{B}$ is connected. In the case $\mathbb{K} = \mathbb{R}$ and $k = m = n$, $\mathcal{B}$ may have one or two connected components* (*see Sect.* 8 *for a complete description*).

The connected components of the set $\mathbb{P}(\mathcal{H}_{(d)}) \setminus \Sigma$, where $\Sigma \subseteq \mathbb{P}(\mathcal{H}_{(d)})$ is the subset of systems which have some zero such that the derivative does not have maximal rank, are studied in [11].

2 The Topology of W

Our main theorem describes W in terms of $\mathcal{B}$.

For $b \in \mathcal{B}$, let $W^s(b) = \{x \in W : \phi_t(x) \mapsto b, t \mapsto \infty\}$, where ϕ_t is the solution flow of the vector field $V(x) = -\operatorname{grad} \widetilde{\mu}(x)$ on W. Note that the definition of $W^s(b)$

is not very sensitive to the parametrization of ϕ_t: we may multiply $V(x)$ by a smooth positive function without changing $W^s(b)$.

Theorem 3 (Main)

(1) *$\widetilde{\mu}$ is equivariant for the action of $\mathcal{G}_{n+1}$ on W.*
(2) *$\mathcal{B}$ is the set of minima of $\widetilde{\mu}$ and the set of critical points of $\widetilde{\mu}$.*
(3) *The Hessian of $\widetilde{\mu}$ is positive definite on the normal bundle of $\mathcal{B}$.*
(4) *The $W^s(b)$ form a $\mathcal{C}^\infty$ foliation of W.*
(5) *The normal bundle $N(\mathcal{B})$ is $\mathcal{C}^\infty$ diffeomorphic to W, by a diffeomorphism $\sigma : N(\mathcal{B}) \longrightarrow W$ such that*
   - $\sigma|_{\mathcal{B}} = \mathrm{Id}_{\mathcal{B}}$,
   - $D\sigma|_{\mathcal{B}} = \mathrm{Id}_{T\mathcal{B}}$,
   - *σ maps the fibers of $N(\mathcal{B})$ to the $W^s(b)$ leaves of the foliation of W.*

Remark 1 The three first items of our Main Theorem can be summarized by saying that $\widetilde{\mu} : W \to \mathbb{R}$ is an equivariant Morse function for the action of $\mathcal{G}_{n+1}$ on W, with one orbit of nondegenerate minima and no other critical points.

The first item of this theorem has already been pointed out (1.5). The second one is proved in Proposition 4 below. The third one is proved in Sect. 6. The last two items, as well as other facts concerning conjugacies of the flow ϕ_t, are proved in Appendix B.

Remark 2 Without reference to the $W^s(b)$ foliation, Theorem 3 is the simplest case of an application of the Morse–Bott Lemma (once the regularity results of Sect. 6 below are stated). See [29] for an equivariant version. Because we are interested in the $W^s(b)$ foliation corresponding to a particular class of conformally equivalent Riemannian structures on W, we use stable manifold techniques in Appendix B.

Corollary 2 *The inclusion $i : \mathcal{B} \to W$ is a homotopy equivalence.*

Thus the homogeneous space structure of $\mathcal{B}$ given by Proposition 3 which we elaborate on in Sect. 8 rather simply describes the diffeomorphism type of $\mathcal{B}$ and the homotopy type of W.

2.1 The Geometry of W

The proof of Proposition 6, comparisons of the distance between points in W and lengths of gradient curves, and other useful information about the geometry of W which we prove in Sect. 7 follow from the next proposition (see Sect. 6 for a proof).

Proposition 4 (compare to [14]) *For any pair $(f, \zeta) \in W$, the following holds*:

$$\frac{\widetilde{\mu}(f,\zeta)^2}{k}\sqrt{1-\frac{k^2}{\widetilde{\mu}(f,\zeta)^2}} \le \left\|\operatorname{grad}\widetilde{\mu}(f,\zeta)\right\| \le d^{3/2}\widetilde{\mu}(f,\zeta)^2.$$

In particular, $(f, \zeta) \in W$ is a regular point of $\widetilde{\mu}$ unless $(f, \zeta) \in \mathcal{B}$.

The last claim of this proposition can be viewed in the opposite way.

Corollary 3 *A point $(f,\zeta)\in W$ is a critical point of $\widetilde{\mu}$ if and only if $(f,\zeta)\in\mathcal{B}$.*

Any function satisfying Proposition 4 has many nice properties. Using Proposition 4 alone, in Sect. 7 below we use simple arguments to prove some results for $\widetilde{\mu}$ and all rank strata that generalize results known to be true (and sometimes more difficult to prove) for the nonsmooth condition number μ_{norm} and $m=n=k$. For example, we prove a smooth version of Proposition 1 of [9] (see Corollary 7 below), a sharp version of Theorem 1 of [25] (see Corollary 9 below), and a version of Theorem 1 of [27] (see Corollary 6 below). All these results are valid for $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. The next proposition states some of these results; there are more in Sect. 7.

Proposition 5 (See Corollaries 5, 6, 7, and 9 below) *Let $(f,\zeta)\in W$. Then,*

1. *The distance $d_{F,\kappa}((f,\zeta),\mathcal{B})$ in the Frobenius condition metric from (f,ζ) to $\mathcal{B}$ satisfies*:
$$\frac{1}{d^{3/2}}\ln\frac{\widetilde{\mu}(f,\zeta)}{k}\le d_{F,\kappa}\big((f,\zeta),\mathcal{B}\big)\le k\ln\frac{\widetilde{\mu}(f,\zeta)+\sqrt{\widetilde{\mu}(f,\zeta)^2-k^2}}{k}.$$
2. *Let $d((f,\zeta),\Sigma')$ be the Fubini–Study distance from (f,ζ) to the set of ill-posed problems Σ'. Then*[1]
$$\frac{1}{d^{3/2}\widetilde{\mu}(f,\zeta)}\le d\big((f,\zeta),\Sigma'\big)\le\arcsin\frac{k}{\widetilde{\mu}(f,\zeta)}\le\frac{\pi}{2}\frac{k}{\widetilde{\mu}(f,\zeta)}.$$
3. *If $(f,\zeta),(h,\eta)\in W$ and $\varepsilon=d((f,\zeta),(h,\eta))\widetilde{\mu}(f,\zeta)d^{3/2}$ satisfies $\varepsilon<1$,*
$$\frac{\widetilde{\mu}(f,\zeta)}{1+\varepsilon}<\widetilde{\mu}(h,\eta)<\frac{\widetilde{\mu}(f,\zeta)}{1-\varepsilon}.$$
4. *If $k=m\le n$, $(f,\zeta)\in W$, and $h\in\mathbb{P}(\mathcal{H}_{(d)})$ is another system such that $\varepsilon=d(f,h)\widetilde{\mu}(f,\zeta)^2d^{3/2}\sqrt{1+\frac{1}{m^2}}$ satisfies $\varepsilon<1$, then ζ can be continued to a zero η of h and*
$$\frac{\widetilde{\mu}(f,\zeta)}{\sqrt{1+\varepsilon}}<\widetilde{\mu}(h,\eta)<\frac{\widetilde{\mu}(f,\zeta)}{\sqrt{1-\varepsilon}}.$$

These items all have ready interpretations. Item 1 gives fairly tight bounds for the number of steps of homotopy methods starting in $\mathcal{B}$ for optimal or near optimal paths. A comparable upper bound estimate in the condition metric is the main result of [9]. This states a similarity between solution curves of the gradient flow and geodesics

[1] The upper bound in the following equation is not optimal: it is known from the Condition Number Theorem of [24] that $d((f,\zeta),\Sigma')\le\arcsin\frac{1}{\mu_{\mathrm{norm}}(f,\zeta)}$, which is a better upper bound because $\mu_{\mathrm{norm}}(f,\zeta)\ge\widetilde{\mu}(f,\zeta)/\sqrt{k}$. However, we include our upper bound to give an idea of how sharp results can be obtained with our approach.

in the Frobenius condition metric, which makes us wonder if these solution curves are actually geodesics in the Frobenius condition metric. We have not pursued this question so far. It might help to consider the convexity properties of these geodesics, whose study has been started in [3, 4]. Item 2 establishes that the Frobenius condition number is comparable to the reciprocal of the distance to the ill-posed problems. Comparing the condition number to the reciprocal of the distance to the ill-posed problems is a recurring theme [14, 15, 24]. Demmel uses an estimate as Proposition 4 and differential inequalities, as we do below. The analog of item 3 for μ_{norm} was the principal new ingredient in the proof of Theorem 1. With item 3 one can prove Theorem 2 directly exactly as in the proof of Theorem 1, even for the cases $n \geq m = k$. Item 4 gives a lower bound for the radius of a ball where the projection on the first coordinate $W \to \mathbb{P}(\mathcal{H}_{(d)})$ can be inverted.

Let $\overline{W} = \overline{W^k_{(d)}}$ be the closure of $W \subset V$. So $\overline{W^k_{(d)}} = \mathcal{W} \cup \bigcup_{1 \leq j \leq k} W^j_{(d)}$, and if $k = \min(m, n)$ then $\overline{W} = V$. Denote

$$\overline{W^k_{(d)}} - W^k_{(d)} = \Sigma'^k_{(d)} = \Sigma' = \Sigma'^k = \big\{(f, \zeta) \in V : \operatorname{rank}\big(Df(\zeta)\big) \leq k - 1\big\}. \tag{2.1}$$

The following proposition describes how W is attached to Σ' to form $\overline{W}$. Its proof is given in Sect. 7.1.

Proposition 6 *Let* $X(f, \zeta) = \operatorname{grad} \widetilde{\mu}(f, \zeta)/\|\operatorname{grad} \widetilde{\mu}(f, \zeta)\|^2$ *be the smooth vector field defined in* W, *and let* $\alpha_{(f,\zeta)}(t)$ *be the integral curve of this vector field such that* $\alpha_{(f,\zeta)}(0) = (f, \zeta)$. *If* $p = (f, \zeta) \in W - \mathcal{B}$ *then the* $\lim_{t\to\infty} \alpha_p(t)$ *exists and is an element of* Σ'. *The function* $\psi : W - \mathcal{B} \to \Sigma'$ *defined by* $\psi(p) = \lim_{t\to\infty} \alpha_p(t)$ *is continuous.*

2.2 Starting Points

Now we center our attention on the case $m = n = k$. Theorems 1 and 2 above give estimates for the number of steps required by path-following methods to solve systems of polynomial equations, once a path in the space of systems is fixed. The simplest choice for this path is the linear segment. That is, given $f, g \in \mathbb{P}(\mathcal{H}_{(d)})$ we consider $f_t = (1 - t)g + tf$, or the shorter portion of the great circle joining two representatives of g and f.

Given a starting pair $(g, \zeta_0) \in V - \Sigma'$ and a system $f \in \mathbb{P}(\mathcal{H}_{(d)})$, let $\mathcal{C}_0(f, g, \zeta_0)$ be the value of $\text{Length}_\kappa(\Gamma)$ (as in Theorem 1) for the linear homotopy f_t. This is thus, up to a small factor $cd^{3/2}$, a bound for the number of homotopy steps needed for finding a root of system f. Integrating over $f \in \mathbb{P}(\mathcal{H}_{(d)})$ gives an average cost function for solving the general system starting from (g, ζ_0),

$$\mathcal{A}_1(g, \zeta_0) = \mathrm{E}_{f \in \mathcal{P}(H_{(d)})}\big(\mathcal{C}_0(f, g, \zeta_0)\big),$$

where E denotes expectation. Then, $\mathcal{A}_1(g, \zeta_0)$ is, up to a small factor $cd^{3/2}$, an upper bound for the average number of steps required by a path-following method starting at the pair (g, ζ_0) to produce an approximate zero of a system f with probability 1.

Using the integral of Theorem 2 instead of that of Theorem 1 gives a similar function which we denote by $\tilde{\mathcal{A}}_1(g,\zeta_0)$.

In [6, Theorem 5] an upper bound cnN for the expectation of $\mathcal{A}_1(g,\zeta_0)$ is given, proving that if ζ_0 is a randomly chosen (uniform distribution) zero of a randomly chosen $g \in \{f \in \mathcal{H}_{(d)} : \|f\| = 1\}$ then (g,ζ_0) is a good starting point with high probability. Then a method to choose such a random (g,ζ_0) is given, thus producing a randomized algorithm for Smale's 17th problem which runs in average polynomial time. In [12], a certain initial system h (containing all the roots of unity) is proved to be an almost-good starting point for the linear homotopy. Here, "almost-good" means that the resulting average complexity is almost polynomial in the size of the input, $N^{\mathrm{O}(\log\log N)}$.

One would like to find deterministically a starting point that would produce one zero of almost all systems with the cost as good as or at least close to the bound cnN mentioned above, that is, a (g,ζ_0) with $\mathcal{A}_1(g,\zeta_0)$ small. It is conjectured in [26] that a particular starting point would accomplish this, and thus resolve Smale's 17th problem with a deterministic algorithm. Moreover, according to experimental results [5], it would be a good starting point in practice. The conjectured system g_{good} is given by $(g_{\text{good}})_i(x_0,\dots,x_n) = \sqrt{d_i}x_0^{d_i-1}x_i$ and the root is $\zeta = (1,0,\dots,0)^{\mathrm{T}}$. This is a best possibly conditioned pair, i.e., it belongs to $\mathcal{B}$. According to Proposition 3, $\mathcal{B}$ is the orbit of g_{good} and, using the unitary invariance of equations (1.1) and (1.5), one can easily prove that $\mathcal{A}_1$ is constant along $\mathcal{B}$.

One way to attack the problem of finding a deterministic starting point is to study the properties of the function $\mathcal{A}_1(g,\zeta_0)$, or those of $\tilde{\mathcal{A}}_1(g,\zeta_0)$.

Conjecture 1 $\mathcal{A}_1(g,\zeta)$ *(and/or* $\tilde{\mathcal{A}}_1(g,\zeta)$*) is an equivariant Morse function for the action of the unitary group on* $V - \Sigma'$. *It has one orbit of nondegenerate minima and no other critical points. This orbit is precisely* $\mathcal{B}$.

This would prove resolve the conjecture in [26]. It would also give a deterministic answer to Smale's 17th problem and more! Moreover, the conjecture is consistent with the topology of $V - \Sigma'$ as proven in this paper. It is the analog of Theorem 3 for $\mathcal{A}_1$ ($\tilde{\mathcal{A}}_1$).

3 Proof of Proposition 1

We prove the proposition for $\mathbb{K} = \mathbb{C}$, the proof being identical for $\mathbb{K} = \mathbb{R}$ substituting orthogonal groups for unitary groups. Let $\hat{W}^k = \{(f,\zeta) \in \mathcal{H}_{(d)} \times \mathbb{C}^{n+1} : f(\zeta) = 0, \operatorname{rank}(Df(\zeta)) = k\}$ be the affine counterpart of W^k. We also denote

$$\hat{V} = \{(f,\zeta) \in \mathcal{H}_{(d)} \times \mathbb{C}^{n+1} : f(\zeta) = 0\} = \mathcal{W} \cup \bigcup_k \hat{W}^k.$$

Note that $\hat{V}$ is a submanifold of $\mathcal{H}_{(d)} \times \mathbb{C}^{n+1}$ and $\dim \hat{V} = \dim \mathcal{H}_{(d)} + n + 1 - m$ (cf. [10, p. 194]). We use the natural notation $\hat{V}_1$, $\hat{W}_1^k$ for the linear case (i.e., when all the degrees $d_1,\dots,d_m$ are equal to 1). For example, $\hat{W}_1^k = \{(A,\zeta) \in \mathcal{M}_{m\times(n+1)}(\mathbb{C}) \times \mathbb{C}^{n+1} : A\zeta = 0, \operatorname{rank}(A) = k\}$.

Claim 1 *$\hat{W}_1^k$ is a submanifold of $\hat{V}_1$ of codimension $(m-k)(n-k)$. Let $\mathcal{R}^k$ be the set of $m \times (n+1)$ matrices of rank k, which is a submanifold of $\mathcal{M}_{m\times(n+1)}(\mathbb{C})$ of codimension $(m-k)(n+1-k)$ (this is a well-known fact; the reader may find a proof in [1]). Then, near $(A_0, \zeta_0) \in \hat{W}_1^k$, the set $\hat{W}_1^k$ is the preimage of 0 under the map $\mathcal{R}^k \times \mathbb{C}^{n+1} \longrightarrow \mathrm{Image}(A_0)$, $(A,\zeta) \mapsto \pi_{\mathrm{Image}(A_0)}(A\zeta)$, where $\pi_{\mathrm{Image}(A_0)}$ denotes orthogonal projection on the subspace $\mathrm{Image}(A_0)$. Finally, this map is a submersion, and the claim follows.*

Claim 2 *$\hat{W}^k$ is a submanifold of $\hat{V}$ of codimension $(m-k)(n-k)$. Consider the mapping $\phi_1 : \hat{V} \longrightarrow \hat{V}_1$, $(f,\zeta) \mapsto (Df(\zeta),\zeta)$, which is indeed a submersion (its normal Jacobian is known, see [6, Main Lemma]). The claim follows from Claim 1 as $\hat{W}^k = \phi_1^{-1}(\hat{W}_1^k)$.*

Claim 3 *W^k is a submanifold of V of codimension $(m-k)(n-k)$. We proceed as in [10, p. 193–194]: note that $\hat{W}^k$ contains the product of the lines through f and ζ for $(f,\zeta) \in W^k$. It follows that $\hat{W}^k$ is transversal to the product of the spheres of radius 1, $\mathbb{S}(\mathcal{H}_{(d)}) \times \mathbb{S}(\mathbb{C}^{n+1})$, and hence $\tilde{W}^k = \hat{W}^k \cap \mathbb{S}(\mathcal{H}_{(d)}) \times \mathbb{S}(\mathbb{C}^{n+1})$ is a smooth manifold. Then, the torus $S^1 \times S^1$ acts freely on $\tilde{W}^k$, and hence the quotient $W^k = \frac{\tilde{W}^k}{S^1\times S^1}$ is a manifold of the claimed codimension.*

The assertion for $\mathcal{W}$ is proved using a similar argument, formally equivalent to the one above one with $k=0$. Note that in the linear case, $\mathcal{W} = \emptyset$.

4 Proof of Proposition 2

There are two trivial implications: (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). For (2) $\Rightarrow$ (3), let $(g,\zeta) \in W$ be any pair, where $e_0 = (1,0,\dots,0)^{\mathrm{T}}$ and we have chosen any fixed representative of g. Then, we can write

$$g(z) = L(z) + h(z),$$

where $L(z)$ is of the form $\mathrm{Diag}(d_i^{1/2} z_0^{d_i-1})Cz$, C is an $m \times (n+1)$ matrix such that $Ce_0 = 0$, and $h(z) \in \mathcal{H}_{(d)}$ is such that no nonzero monomial in h contains $z_0^{d_i-1}$ or $z_0^{d_i}$. Direct computation shows that

$$\|L\| = \|C\|_F, \qquad Df(e_0) = DL(e_0) = \mathrm{Diag}\big(d_i^{1/2}\big)C,$$

which implies $C = \mathrm{Diag}(d_i^{-1/2})Dg(e_0)$. By orthogonality of the monomials in the Bombieri–Weyl inner product, we then have

$$\|g\| = \sqrt{\|L\|^2 + \|h\|^2} \geq \|L\| = \|C\|_F = \big\|\mathrm{Diag}\big(d_i^{-1/2}\big)Dg(e_0)\big\|_F.$$

Assume now that (g, e_0) satisfies (2). Then, $\|g\| = \|\mathrm{Diag}(d_i^{-1/2})Dg(e_0)\|_F$, which implies $\|h\| = 0$ and thus

$$g(z) = L(z) = \mathrm{Diag}\big(d_i^{1/2} z_0^{d_i-1}\big)Cz = \mathrm{Diag}\big(d_i^{1/2}\langle z, e_0\rangle^{d_i-1}\big)Cz.$$

Moreover, as $C = \mathrm{Diag}(d_i^{-1/2}) Dg(e_0)$, the k nonzero singular values of C are equal and $\|C\|_F = \|L\| = \|g\| = 1$ implies that each of them is equal to $1/\sqrt{k}$. Thus, we have written g as demanded by (3), by taking $B = C\sqrt{k}$. This proves (2) $\Rightarrow$ (3) in the case that $\zeta = e_0$.

For general $(f, \zeta) \in W$, let U be a unitary (orthogonal if $\mathbb{K} = \mathbb{R}$) matrix such that $U\zeta = e_0$ (equivalently, $\zeta = U * e_0$). Then, by the chain rule, $g = f \circ U^*$ satisfies:

$$(g, e_0) \in W, \qquad \mathrm{Diag}\big(d_i^{-1/2}\big) Dg(e_0) = \mathrm{Diag}\big(d_i^{-1/2}\big) Df(\zeta) U^*,$$

which using the first part implies

$$f(z) = g(Uz) = \frac{1}{\sqrt{k}} \mathrm{Diag}\big(d_i^{1/2} \langle Uz, e_0\rangle^{d_i - 1}\big) BUz = \frac{1}{\sqrt{k}} \mathrm{Diag}\big(d_i^{1/2} \langle z, \zeta\rangle^{d_i - 1}\big) BUz,$$

where B has its k nonzero singular values equal to 1. Note that this last expression is again of the form demanded in (3), as BU has the same singular values as B. This finishes the proof of Proposition 2.

Note that we have also proved an interesting inequality:

$$\|f\| \underset{(1.1)}{=} \|g\| \geq \big\|\mathrm{Diag}\big(d_i^{-1/2}\big) Dg(e_0)\big\|_F = \big\|\mathrm{Diag}\big(d_i^{-1/2}\big) Df(\zeta)\big\|_F, \tag{4.1}$$

which is valid for any $(f, \zeta) \in W$. Inequality (4.1) turns into an equality if and only if

$$f \circ U^*(z) = g(z) = \mathrm{Diag}\big(d_i^{1/2} \langle z, e_0\rangle^{d_i - 1}\big) Cz,$$

that is, if and only if

$$f(z) = \mathrm{Diag}\big(d_i^{1/2} \langle Uz, e_0\rangle^{d_i - 1}\big) CUz = \mathrm{Diag}\big(d_i^{1/2} \langle z, \zeta\rangle^{d_i - 1}\big) Ez, \tag{4.2}$$

where E is an $m \times (n+1)$ matrix and $E\zeta = 0$.

5 Proof of Proposition 3

Again we do the proof for $\mathbb{K} = \mathbb{C}$, as it is identical for $\mathbb{K} = \mathbb{R}$ substituting orthogonal groups for unitary groups. We consider the affine counterpart of $\mathcal{B}^k$ in the product of spheres, $\tilde{\mathcal{B}}^k \subseteq \tilde{W}^k$, that is, the set of pairs $(f, \zeta) \in \tilde{W}^k$ such that there is a rank k matrix B with all its singular values equal to 1 such that

$$f(z) = \frac{1}{\sqrt{k}} \mathrm{Diag}\big(d_i^{1/2} \langle z, \zeta\rangle^{d_i - 1}\big) Bz, \qquad B\zeta = 0.$$

We use the natural notation $\tilde{\mathcal{B}}_1^k$ for the linear case (i.e., when all the degrees $d_1, \ldots, d_m$ are equal to 1). That is,

$$\tilde{\mathcal{B}}_1^k = \left\{ (A, \zeta) \in \tilde{W}_1^k : A = \frac{B}{\sqrt{k}}, \text{ the } k \text{ nonzero singular values of } B \text{ are equal to } 1 \right\}.$$

Claim 1 *$\tilde{\mathcal{B}}_1^k$ is a homogeneous space of $\mathcal{U}_m \times \mathcal{U}_{n+1}$ and is a compact real submanifold of $\hat{W}_1^k$ of (real) dimension $2mk + 2nk + 2n + 1 - 3k^2$. Note that $\tilde{\mathcal{B}}_1^k$ is the orbit of (J, e_n), where $e_n = (0, \dots, 0, 1)^{\mathrm{T}} \in \mathbb{C}^{n+1}$ and $J = \frac{1}{\sqrt{k}}\left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$, under the natural action of the compact product group $\mathcal{U}_m \times \mathcal{U}_{n+1}$ defined by $((U, V), (M, \zeta)) \mapsto (UMV^*, V\zeta)$. Thus, $\tilde{\mathcal{B}}_1^k$ is a submanifold of $\hat{W}_1$ and is diffeomorphic to $\mathcal{U}_m \times \mathcal{U}_{n+1}$ modulo the isotropy group H of (J, e_n). Now,*

$$H = \left\{ \left(\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \begin{pmatrix} U_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) : U_1 \in \mathcal{U}_k, U_2 \in \mathcal{U}_{m-k}, V_2 \in \mathcal{U}_{n-k} \right\}$$

is isomorphic to $\mathcal{U}_k \times \mathcal{U}_{m-k} \times \mathcal{U}_{n-k}$, and the claim on the dimension of $\tilde{\mathcal{B}}_1^k$ follows.

Claim 2 *$\tilde{\mathcal{B}}^k$ is a compact embedded submanifold of $\tilde{W}^k$ diffeomorphic to $\tilde{\mathcal{B}}_1^k$. Now consider*

$$\phi_2 : \tilde{\mathcal{B}}_1^k \longrightarrow \hat{W}^k$$
$$(M, \zeta) \mapsto (h, \zeta), \qquad h(z) = \frac{1}{\sqrt{k}} \operatorname{Diag}\left(d_i^{1/2} \langle z, \zeta\rangle^{d_i - 1}\right) M z,$$

which is an injective immersion. As $\tilde{\mathcal{B}}_1^k$ is compact, the set $\phi_2(\tilde{\mathcal{B}}_1^k)$ is a compact embedded submanifold of $\hat{W}^k$ diffeomorphic to $\tilde{\mathcal{B}}_1^k$. Finally, note that $\tilde{\mathcal{B}}^k = \phi_2(\tilde{\mathcal{B}}_1^k)$.

Claim 3 *$\mathcal{B}^k$ is a compact real submanifold of W^k of (real) dimension $2mk + 2nk + 2n - 1 - 3k^2$, and it is diffeomorphic to $\frac{\tilde{\mathcal{B}}^k}{S^1 \times S^1}$. Just note that $\mathcal{B}^k$ is the quotient of $\tilde{\mathcal{B}}^k$ under the free action of $S^1 \times S^1$.*

Claim 4 *For $(f, \zeta) \in W$, $\widetilde{\mu}(f, \zeta) \geq k$ with equality if and only if $(f, \zeta) \in \mathcal{B}$. Write $A = \operatorname{Diag}(d_i^{-1/2}) Df(\zeta)$. Then,*

$$\widetilde{\mu}(f, \zeta)^2 = \|f\|^2 \|A^\dagger\|_F^2 \underset{(4.1)}{\geq} \|A\|_F^2 \|A^\dagger\|_F^2 \underset{(1.4)}{=} \left(\sum_{i=1}^k \sigma_i^2\right)\left(\sum_{i=1}^k \frac{1}{\sigma_i^2}\right) \geq k^2, \tag{5.1}$$

where $\sigma_1, \dots, \sigma_k$ are the nonzero singular values of A.

The last inequality in (5.1) can be easily checked by computing the partial derivatives with respect to σ_i, and checking that if they are all equal to 0 then necessarily all the σ_i have to be equal in absolute value. Assume now that $\widetilde{\mu}(f, \zeta) = k$. From (5.1), $\widetilde{\mu}(f, \zeta) = k$ implies that $\|f\| = \|A\|_F$ and all the nonzero singular values of A are equal. From Proposition 2, this implies $(f, \zeta) \in \mathcal{B}$.

Claim 5 *$\mathcal{B}$ is a smooth fiber bundle over $\mathbb{P}(\mathbb{C}^{n+1})$ with the fiber a homogeneous space of $\mathcal{U}_m \times \mathcal{U}_n$. Let $\pi_2 : \mathcal{B} \longrightarrow \mathbb{P}(\mathbb{C}^{n+1})$ be the projection on the second component. Then, the action (1.6) yields a fiber bundle structure to π_2, with fiber*

$$\mathcal{B}_\zeta = \pi_2^{-1}(\zeta) = \left\{ f \in \mathbb{P}(\mathcal{H}_{(d)}) : f(\zeta) = 0, \widetilde{\mu}(f, \zeta) = k \right\}.$$

Indeed, let $\zeta \in \mathbb{P}(\mathbb{K}^{n+1})$ and let $H_\zeta \subseteq \mathcal{U}_{n+1}$ be the stabilizer of ζ under the natural action $\mathcal{U}_{n+1} \times \mathbb{P}(\mathbb{K}^{n+1}) \to \mathbb{P}(\mathbb{K}^{n+1})$, $(U, \zeta) \mapsto U\zeta$. Let $\mathcal{N}_\zeta \subseteq \mathcal{U}_{n+1}$ be any smooth submanifold whose tangent space at the identity is the orthogonal complement of H_ζ (the particular definition of orthogonality is not important, but one may use the one induced by the Frobenius product in $\mathcal{U}_{n+1}$). For small enough open neighborhoods $\mathrm{U}_\zeta \subseteq \mathbb{P}(\mathbb{K}^{n+1})$ of ζ and $\mathrm{N}_I \subseteq \mathcal{N}_\zeta$ of I (the identity matrix), this defines a smooth diffeomorphism:

$$\begin{aligned} \varphi_\zeta : \mathrm{N}_I &\to \mathrm{U}_\zeta \\ U &\mapsto U\zeta \end{aligned}$$

Then, the fiber bundle structure defined by π_2 is locally given by the diagram

$$\begin{array}{ccc} \pi_2^{-1}(\mathrm{U}_\zeta) & \longrightarrow & U_\zeta \times \mathcal{B}_\zeta \\ \downarrow & \swarrow & \\ U_\zeta & & \end{array} \qquad\qquad \begin{array}{ccc} (f, \eta) & \longrightarrow & (\eta, f \circ ((\varphi_\zeta)^{-1}(\eta))^*) \\ \downarrow & \swarrow & \\ \eta & & \end{array}$$

From Proposition 2, the fiber $\mathcal{B}_{e_0}$ of e_0 is the set of systems $f = (f_1, \ldots, f_m) \in \mathbb{P}(\mathcal{H}_{(d)})$ such that $f_i(z) = d_i^{1/2} \sum_{j \ge 1} z_0^{d_i - 1} a_{ij} z_j$ for some matrix $A = (a_{ij}) \in \mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{C}))$ such that the k nonzero singular values of A are equal. Note that $\mathcal{U}_m \times \mathcal{U}_n$ acts transitively on that space by $(U, V, A) \mapsto UAV^*$, and hence the fiber of π_2 is diffeomorphic to $\frac{\mathcal{U}_m \times \mathcal{U}_n}{H}$, where H is the isotropy group of the element $\left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$. That is, H is the set of pairs

$$\left(\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \begin{pmatrix} \lambda U_1 & 0 \\ 0 & V_2 \end{pmatrix} \right), \quad U_1 \in \mathcal{U}_k, U_2 \in \mathcal{U}_{m-k}, V_2 \in \mathcal{U}_{n-k}, \lambda \in S^1,$$

which is diffeomorphic to $S^1 \times \mathcal{U}_k \times \mathcal{U}_{m-k} \times \mathcal{U}_{n-k}$.

Claim 6 *If $m = k \le n$ then $\mathcal{U}_{n+1}$ acts transitively on $\mathcal{B}$. Let $(f, \zeta) \in \mathcal{B}$ and choose norm 1 representatives such that*

$$f(z) = \frac{1}{\sqrt{k}} \mathrm{Diag}\big(d_i^{1/2} \langle z, \zeta \rangle^{d_i - 1}\big) U \begin{pmatrix} I_m & 0 \end{pmatrix} V^* z, \tag{5.2}$$

for $U \in \mathcal{U}_m$, $V \in \mathcal{U}_{n+1}$ such that $(\, I_m \; 0 \,) V^* \zeta = 0$. We can choose V such that $V^* \zeta = e_n$. Then, note that

$$U \begin{pmatrix} I_m & 0 \end{pmatrix} V^* = \begin{pmatrix} I_m & 0 \end{pmatrix} R^*, \quad R^* = \begin{pmatrix} U & 0 \\ 0 & I_{n+1-m} \end{pmatrix} V^*,$$

so that $(f, \zeta) = (g \circ R^*, Re_n)$, where

$$g(z) = \frac{1}{\sqrt{k}} \mathrm{Diag}\big(d_i^{1/2} \langle z, e_n \rangle^{d_i - 1}\big) \begin{pmatrix} I_m & 0 \end{pmatrix} z.$$

Thus, $\mathcal{B}$ is the orbit of g and $\mathcal{U}_{n+1}$ acts transitively on $\mathcal{B}$.

Claim 7 *If all the d_i's are equal, $\mathcal{U}_m \times \mathcal{U}_{n+1}$ acts transitively on $\mathcal{B}$. This is clear from point* (3) *in Proposition* 2 *and from* (1.7).

6 The Frobenius Condition Number and Its Gradient Flow

6.1 Gradient of $\widetilde{\mu}$

In this section we compute lower and upper bounds for the norm of the gradient of $\widetilde{\mu}$.

Proposition 7 *Let $\mathcal{R}_k$ be the manifold of $m \times (n+1)$ rank k matrices of Frobenius norm equal to some $c > 0$. Let $A \in \mathcal{R}_k$, $\dot{A} \in T_A\mathcal{R}_k$. Then, $\psi(A) = \|A^\dagger\|_F$ defined on $\mathcal{R}_k$ is a smooth function, and*

$$D\psi(A)(\dot{A}) = -\frac{\mathbb{R}e\langle A^\dagger, A^\dagger \dot{A} A^\dagger\rangle_F}{\|A^\dagger\|_F}.$$

Proof Smoothness of ψ follows from that of †. This last is a well-known folk result, but we do not find an appropriate reference, so we include a short proof for completeness. Recall from (1.3) that

$$A^\dagger = (A|_{\mathrm{Ker}(A)^\perp})^{-1} \circ \pi_{\mathrm{Image}(A)},$$

where $\mathrm{Ker}(A)^\perp$ is the orthogonal complement of the kernel of $A \in \mathcal{R}^k$ and $\mathrm{Image}(A)$ is its image. Now, $\mathrm{Ker}(A)^\perp = \mathrm{Image}(A^*)$, so as far as their dimension is fixed (i.e., as far as A does not leave $\mathcal{R}_k$), all these subspaces move smoothly with A and so does $A^\dagger$.

Now we compute $D\psi$ using implicit differentiation. Recall the following well-known identities, all of which easily follow from (1.4):

$$AA^\dagger A = A, \qquad A^\dagger A A^\dagger = A^\dagger, \qquad (A^\dagger)^* = (A^\dagger)^* A^\dagger A = AA^\dagger (A^\dagger)^*. \tag{6.1}$$

Let P be any matrix of the same size of $A^\dagger$. From identities (6.1) and the elementary properties of $\langle\cdot,\cdot\rangle_F$, we have

$$\begin{aligned}\langle A^\dagger, A^\dagger A P A A^\dagger\rangle_F &= \langle I_m, A^\dagger A P A A^\dagger (A^\dagger)^*\rangle_F = \langle I_m, A^\dagger A P (A^\dagger)^*\rangle_F = \langle A^\dagger, A^\dagger A P\rangle_F \\ &= \langle I_{n+1}, (A^\dagger)^* A^\dagger A P\rangle_F = \langle I_{n+1}, (A^\dagger)^* P\rangle_F = \langle A^\dagger, P\rangle_F,\end{aligned}$$

that is,

$$\langle A^\dagger, P\rangle_F = \langle A^\dagger, A^\dagger A P A A^\dagger\rangle_F. \tag{6.2}$$

Let $\dot{A}$ be tangent to $\mathcal{R}_k$ at A, and let B denote the derivative of the Moore–Penrose inverse at A in the direction of $\dot{A}$. From the first equality in (6.1),

$$\dot{A}A^\dagger A + ABA + AA^\dagger \dot{A} = \dot{A}, \tag{6.3}$$

and hence

$$\langle A^\dagger, B\rangle_F = \langle A^\dagger, A^\dagger ABAA^\dagger\rangle_F = \langle A^\dagger, A^\dagger(\dot{A} - \dot{A}A^\dagger A - AA^\dagger \dot{A})A^\dagger\rangle_F$$
$$= \langle A^\dagger, A^\dagger \dot{A}A^\dagger - A^\dagger \dot{A}A^\dagger AA^\dagger - A^\dagger AA^\dagger \dot{A}A^\dagger\rangle_F = -\langle A^\dagger, A^\dagger \dot{A}A^\dagger\rangle_F.$$

From the chain rule applied to the composition of $A \mapsto A^\dagger = M$, $M \mapsto \|M\|_F^2 = t$, and $t \mapsto \sqrt{t}$ we then get

$$D\psi(A)(\dot{A}) = \frac{\mathbb{R}e\langle A^\dagger, B\rangle_F}{\|A^\dagger\|_F} = -\frac{\mathbb{R}e\langle A^\dagger, A^\dagger \dot{A}A^\dagger\rangle_F}{\|A^\dagger\|_F}. \qquad \square$$

Lemma 1 *For any $k \times k$ diagonal matrix P, the following inequality holds*:

$$\frac{\|P^3\|_F^2 - \|P\|_F^4}{\|P\|_F^2} \geq \frac{\|P\|_F^4}{k^2}\left(1 - \frac{k^2}{\|P\|_F^2}\right).$$

Proof Note that the inequality of the lemma is equivalent to

$$\|P^3\|_F^2 - \|P\|_F^4 \geq \frac{\|P\|_F^6}{k^2} - \|P\|_F^4,$$

so it suffices to prove that $\|P\|_F^6 \leq k^2\|P^3\|_F^2$. Now, recall the inequality

$$(\sigma_1^2 + \cdots + \sigma_k^2)^3 \leq k^2(\sigma_1^6 + \cdots + \sigma_k^6),$$

which holds for every choice of real numbers $\sigma_1, \ldots, \sigma_k$. Let $\sigma_i = |\lambda_i|$ where $P = \mathrm{Diag}(\lambda_i)$ in this last formula to get $\|P\|_F^6 \leq k^2\|P^3\|_F^2$. $\square$

Proof of Proposition 4 First note that $\widetilde{\mu}$ is a smooth function on W, because $\|Df(\zeta)^\dagger \,\mathrm{Diag}(\|\zeta\|^{d_i-1}d_i^{1/2})\|$ moves smoothly with f, ζ, for † is a smooth function in $\mathcal{R}_k$, as claimed in Proposition 7.

Now we prove the bounds on the norm of the gradient of $\widetilde{\mu}$. Let $(f, \zeta) \in W$ and let $(f_t, \zeta_t) \subseteq W$ be a C^1 curve with tangent vector $(\dot{f}, \dot{\zeta})$ at $(f, \zeta) = (f_0, \zeta_0)$. Assume that we have chosen representatives f_t, ζ_t such that $\|f_t\| = \|\zeta_t\| = 1$ and $\dot{f}_t \in f_t^\perp$, $\dot{\zeta}_t \in \zeta_t^\perp$ (that is, (f_t, ζ_t) is a horizontal lift to the product of spheres). Then,

$$\widetilde{\mu}(f_t, \zeta_t) = \|A_t^\dagger\|_F, \qquad A_t = \mathrm{Diag}(d_i^{-1/2})Df_t(\zeta_t) \in \mathcal{M}_{m\times(n+1)}(\mathbb{K}).$$

Thus, from Proposition 7,

$$D\widetilde{\mu}(f, \zeta)(\dot{f}, \dot{\zeta}) = -\frac{\mathbb{R}e\langle A^\dagger, A^\dagger(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}A_t)A^\dagger\rangle_F}{\|A^\dagger\|_F},$$

where

$$A = A_0 = \mathrm{Diag}(d_i^{-1/2})Df(\zeta).$$

Now, note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A_t = B + C_{\dot\zeta}, \qquad B = \mathrm{Diag}\big(d_i^{-1/2}\big) D\dot f(\zeta),$$
$$C_{\dot\zeta} = \mathrm{Diag}\big(d_i^{-1/2}\big) D^2 f(\zeta)(\dot\zeta, \cdot),$$

where $C_{\dot\zeta}$ is seen as an $m \times (n+1)$ matrix satisfying $C_{\dot\zeta} z = D^2 f(\zeta)(\dot\zeta, z)$ for $z \in \mathbb{K}^{n+1}$.

We thus have

$$\begin{aligned}\big\| D\widetilde{\mu}(f, e_0)(\dot f, \dot\zeta)\big\| &= \left\| \frac{\mathbb{R}e\langle (A^\dagger)^* A^\dagger (A^\dagger)^*, B + C_{\dot\zeta}\rangle_F}{\|A^\dagger\|_F} \right\| \\ &\le \big\|A^\dagger\big\|_F^2 \big(\|B\|_F + \|C_{\dot\zeta}\|_F\big) \\ &= \widetilde{\mu}(f,\zeta)^2 \big(\|B\|_F + \|C_{\dot\zeta}\|_F\big).\end{aligned}$$

From the higher derivative estimate (1.2) we have

$$\big\|d_i^{-1/2} D\dot f_i(\zeta)\big\| \le d_i^{1/2} \|\dot f_i\|, \quad i = 1, \dots, m,$$

which implies

$$\|B\|_F = \sqrt{\sum_{i=1}^m \big\|d_i^{-1/2} D\dot f_i(\zeta)\big\|^2} \le d^{1/2} \sqrt{\sum_{i=1}^m \|\dot f_i\|^2} = d^{1/2} \|\dot f\|.$$

Again from (1.2) we have

$$\big\|d_i^{-1/2} D^2 f_i(\zeta)(\dot\zeta, \cdot)\big\| \le d_i^{1/2}(d_i - 1)\|f_i\|\|\dot\zeta\|, \quad i = 1, \dots, m,$$

which implies

$$\begin{aligned}\|C_{\dot\zeta}\|_F &= \sqrt{\sum_{i=1}^m \big\|d_i^{-1/2} D^2 \dot f_i(\zeta)(\dot\zeta, \cdot)\big\|^2} \le d^{1/2}(d-1)\|\dot\zeta\| \sqrt{\sum_{i=1}^m \|f_i\|^2} \\ &= d^{1/2}(d-1)\|\dot\zeta\|\|f\| = d^{1/2}(d-1)\|\dot\zeta\|.\end{aligned}$$

We have thus proved that

$$\begin{aligned}\|B\|_F + \|C_{\dot\zeta}\|_F &\le d^{1/2}\|\dot f\| + d^{1/2}(d-1)\|\dot\zeta\| \\ &\le \big(d + d(d-1)^2\big)^{1/2} \big\|(\dot f, \dot\zeta)\big\| \le d^{3/2} \big\|(\dot f, \dot\zeta)\big\|.\end{aligned}$$

The upper bound for the norm of the gradient follows immediately:

$$\big\|\mathrm{grad}\, \widetilde{\mu}(f,\zeta)\big\| = \sup \frac{\|D\widetilde{\mu}(f,\zeta)(\dot f, \dot\zeta)\|}{\|(\dot f, \dot\zeta)\|} \le d^{3/2} \widetilde{\mu}(f,\zeta)^2.$$

Now we prove the lower bound. We have seen in (1.5) that $\widetilde{\mu}$ is invariant under unitary transformations. Thus, we may assume that $\zeta = e_0 = (1, 0, \dots, 0)^{\mathrm{T}}$. Let $\dot{\zeta} = 0$ and $\dot{h}$ be defined as

$$\dot{h}(z) = \dot{h}(z_0, \dots, z_n) = \mathrm{Diag}\big(d_i^{1/2} z_0^{d_i - 1}\big)\big(A^\dagger\big)^* A^\dagger \big(A^\dagger\big)^* z,$$

where $A = \mathrm{Diag}(d_i^{-1/2}) Df(e_0)$ satisfies $Ae_0 = 0$. So, $\mathrm{Diag}(d_i^{-1/2}) D\dot{h}(e_0) = (A^\dagger)^* A^\dagger (A^\dagger)^*$. We claim that $\dot{h}(e_0) = 0$. Indeed, let $A = UDV^*$ be a singular value decomposition of A, where $D = \mathrm{Diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)$ is an $m \times (n+1)$ matrix. Then, $Ae_0 = 0$ implies $V^* e_0 \in \mathrm{Span}(e_{k+1}, \dots, e_n)$. Thus,

$$\big(A^\dagger\big)^* e_0 = \big(V D^\dagger U^*\big)^* e_0 = U\big(D^\dagger\big)^* V^* e_0 = 0,$$

and $\dot{h}(e_0) = 0$.

Let $P = \mathrm{Diag}(\sigma_1^{-1}, \dots, \sigma_k^{-1})$, which implies $\|A\|_F = \|P^{-1}\|_F$ and $\|A^\dagger\|_F = \|P\|_F$. Then, let $\dot{f}$ be the projection of h onto the orthogonal complement of f, that is,

$$\dot{f} = \dot{h} - \langle \dot{h}, f \rangle f.$$

Note that $\dot{f}(e_0) = 0$ and from [10, Lemma 17, Chap. 12] (or just from straightforward computation),

$$\begin{aligned}
&\|\dot{h}\|^2 = \big\|\big(A^\dagger\big)^* A^\dagger \big(A^\dagger\big)^*\big\|_F^2 = \big\|P^3\big\|_F^2,\\
&\langle \dot{h}, f \rangle = \big\langle \big(A^\dagger\big)^* A^\dagger \big(A^\dagger\big)^*, A\big\rangle = \big\langle A^\dagger, A^\dagger A A^\dagger\big\rangle = \big\langle A^\dagger, A^\dagger\big\rangle = \|P\|_F^2,\\
&\mathrm{Diag}\big(d_i^{-1/2}\big) D\dot{f}(e_0) = \big(A^\dagger\big)^* A^\dagger \big(A^\dagger\big)^* - \|P\|_F^2 A.
\end{aligned}$$

Hence,

$$\|\dot{f}\|^2 = \|\dot{h}\|^2 - \big|\langle \dot{h}, f\rangle\big|^2 = \big\|P^3\big\|_F^2 - \|P\|_F^4, \tag{6.4}$$

and

$$\begin{aligned}
D\widetilde{\mu}(f, e_0)(\dot{f}, 0) &= -\frac{\mathbb{R}e\langle (A^\dagger)^* A^\dagger (A^\dagger)^*, (A^\dagger)^* A^\dagger (A^\dagger)^* - \|P\|_F^2 A\rangle_F}{\|A^\dagger\|_F}\\
&= -\frac{\|P^3\|_F^2 - \|P\|_F^4}{\|P\|_F}.
\end{aligned}$$

Note also that

$$1 = \|f\| \underset{(4.1)}{\geq} \|A\|_F = \big\|P^{-1}\big\|_F,$$

which as in (5.1) implies

$$\|P\|_F^2 \geq \frac{k^2}{\|P^{-1}\|_F^2} \geq k^2.$$

Thus, from Lemma 1,

$$\frac{\|P^3\|_F^2 - \|P\|_F^4}{\|P\|_F} \geq \frac{\|P\|_F^2}{k}\sqrt{1 - \frac{k^2}{\|P\|_F^2}} \geq 0,$$

and

$$\left|D\widetilde{\mu}(f, e_0)(\dot{f}, 0)\right| = \frac{\|P^3\|_F^2 - \|P\|_F^4}{\|P\|_F}. \tag{6.5}$$

We have then proved

$$\begin{aligned}\left\|\operatorname{grad}\widetilde{\mu}(f, e_0)\right\| &\geq \frac{|D\widetilde{\mu}(f, e_0)(\dot{f}, 0)|}{\|\dot{f}\|} \underset{(6.4),(6.5)}{=} \left(\frac{\|P^3\|_F^2 - \|P\|_F^4}{\|P\|_F^2}\right)^{1/2} \\ &\underset{\text{Lemma 1}}{\geq} \frac{\|P\|_F^2}{k}\sqrt{1 - \frac{k^2}{\|P\|_F^2}} = \frac{\widetilde{\mu}(f, e_0)^2}{k}\sqrt{1 - \frac{k^2}{\widetilde{\mu}(f, e_0)^2}},\end{aligned}$$

as desired. □

As for the last claim of the proposition, if $(f, \zeta) \in W \setminus \mathcal{B}$, then from Proposition 3 $\mu(f, \zeta) > k$, and from Proposition 4 the gradient of $\widetilde{\mu}$ does not vanish at (f, ζ) and thus (f, ζ) is not a critical point. On the other hand, if $(f, \zeta) \in \mathcal{B}$ then $\widetilde{\mu}$ attains its global minimum value at (f, ζ) and hence (f, ζ) is a critical point of $\widetilde{\mu}$.

6.2 Hessian of $\widetilde{\mu}$

The (Riemannian) Hessian (see [23, Chap. 1]) of a C^2 function $\mathrm{f} : U \to \mathbb{R}$ defined in an open set U of a Riemannian manifold M can be defined as

$$\operatorname{Hess}(\mathrm{f})(p)(w, w) = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{f}\bigl(\gamma(t)\bigr), \tag{6.6}$$

where $\gamma(t)$ is a geodesic in M such that $\gamma(0) = p$, $\dot{\gamma}(0) = w$. If p is a critical point of f, then any curve $\gamma(t)$ such that $\gamma(0) = p$, $\dot{\gamma}(0) = w$ can be used to compute the Hessian with the formula above. That is, we do not need γ to be a geodesic.

It is not easy to compute the Hessian of $\widetilde{\mu}$, but we can prove the following.

Proposition 8 *For $p = (f, \zeta) \in \mathcal{B}$ and $w \in T_pW$, $\operatorname{Hess}(\widetilde{\mu})(p)(w, w)$ is positive unless $w \in T_p\mathcal{B}$.*

The proof uses several technical results.

Lemma 2 *Let $(f, e_0) \in \mathcal{B}$. Assume some representative of f has been chosen and let $h \in \mathcal{H}_{(d)}$ be such that $h(e_0) = 0$. Then,*

$$\langle f, h\rangle = \bigl\langle \operatorname{Diag}\bigl(d_i^{-1/2}\bigr) Df(e_0), \operatorname{Diag}\bigl(d_i^{-1/2}\bigr) Dh(e_0)\bigr\rangle.$$

Proof From Proposition 2, f is of the form

$$f(z)=\mathrm{Diag}\big(d_i^{1/2}z_0^{d_i-1}\big)Bz,$$

for some matrix B such that $Be_0=0$. Then, $Df(e_0)=\mathrm{Diag}(d_i^{1/2})B$. On the other hand, we can write

$$h(z)=\mathrm{Diag}\big(d_i^{1/2}z_0^{d_i-1}\big)Cz+p(z),$$

where $Ce_0=0$ and no monomial in $p(z)$ contains $z_0^{d_i-1}$. Again, we have $Dh(e_0)=\mathrm{Diag}(d_i^{1/2})C$. From the definition of Bombieri–Weyl's product,

$$\langle f,h\rangle=\sum_{i=1}^{m}\sum_{j=0}^{n}\binom{d_i}{d_i-1}^{-1}d_iB_{ij}\overline{C_{ij}}=\sum_{i=1}^{m}\sum_{j=0}^{n}B_{ij}\overline{C_{ij}}=\langle B,C\rangle,$$

which yields the desired equality. □

Lemma 3 *Let $\mathbb{E}$ be a finite dimensional real (complex) vector space endowed with an inner (Hermitian) product $\langle\cdot,\cdot\rangle_{\mathbb{E}}$ and its associated norm $\|\cdot\|_{\mathbb{E}}$. Let $t\mapsto v_t\subseteq\mathbb{E}$ be a curve, twice differentiable at $t=0$, and assume that $v=v_0\neq 0$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\|v_t\|_{\mathbb{E}}=\frac{\mathbb{R}e\langle v,\dot v\rangle_{\mathbb{E}}}{\|v\|_{\mathbb{E}}}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\|v_t\|_{\mathbb{E}}=\frac{\|\dot v\|_{\mathbb{E}}^2+\mathbb{R}e\langle v,\ddot v\rangle_{\mathbb{E}}}{\|v\|_{\mathbb{E}}}-\frac{(\mathbb{R}e\langle v,\dot v\rangle_{\mathbb{E}})^2}{\|v\|_{\mathbb{E}^3}}.$$

Proof Note that

$$\begin{aligned}\|v_t\|_{\mathbb{E}}^2=\langle v_t,v_t\rangle_{\mathbb{E}}&=\Big\langle v+t\dot v+\frac{t^2}{2}\ddot v+\mathrm{o}\big(t^2\big),v+t\dot v+\frac{t^2}{2}\ddot v+\mathrm{o}\big(t^2\big)\Big\rangle_{\mathbb{E}}\\&=\langle v,v\rangle_{\mathbb{E}}+2t\mathbb{R}e\langle v,\dot v\rangle_{\mathbb{E}}+t^2\big(\|\dot v\|_{\mathbb{E}}^2+\mathbb{R}e\langle v,\ddot v\rangle_{\mathbb{E}}\big)+\mathrm{o}\big(t^2\big).\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\|v_t\|_{\mathbb{E}}^2=2\mathbb{R}e\langle v,\dot v\rangle_{\mathbb{E}},\qquad\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\|v_t\|_{\mathbb{E}}^2=2\big(\|\dot v\|_{\mathbb{E}}^2+\mathbb{R}e\langle v,\ddot v\rangle_{\mathbb{E}}\big).$$

Now, for a generic twice differentiable nonzero function $\alpha(t)$, we have

$$\big(\alpha^2\big)'=2\alpha\alpha',\qquad\big(\alpha^2\big)''=2\big(\alpha'\big)^2+2\alpha\alpha'',$$

which implies $\alpha'=(\alpha^2)'/(2\alpha)$ and

$$\alpha''=\frac{(\alpha^2)''-2(\alpha')^2}{2\alpha}.$$

Applied to our case $\alpha = \|v_t\|_{\mathbb{E}}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \|v_t\|_{\mathbb{E}} = \frac{2\mathbb{R}e\langle v, \dot{v}\rangle_{\mathbb{E}}}{2\|v_t\|_{\mathbb{E}}} = \frac{\mathbb{R}e\langle v, \dot{v}\rangle_{\mathbb{E}}}{\|v\|_{\mathbb{E}}},$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \|v_t\|_{\mathbb{E}} = \frac{2(\|\dot{v}\|_{\mathbb{E}}^2 + \mathbb{R}e\langle v, \ddot{v}\rangle_{\mathbb{E}}) - 2(\frac{\mathbb{R}e\langle v, \dot{v}\rangle_{\mathbb{E}}}{\|v\|_{\mathbb{E}}})^2}{2\|v\|_{\mathbb{E}}}.$$

The lemma follows. □

The next result, which is of independent interest, follows the ideas behind Proposition 7. Recall that, for any fixed $c > 0$, $\mathcal{R}_k$ is the manifold of $m \times (n+1)$ rank k matrices of Frobenius norm equal to c, and $\psi : \mathcal{R}_k \to \mathbb{R}$ is defined as $\psi(A) = \|A^\dagger\|$. Its proof (a long and technical computation) is delayed to Appendix A.

Proposition 9 *Let $A \in \mathcal{R}^k$ be such that its k nonzero singular values are equal. Then, A is a critical point of ψ, and*

$$\frac{c^3}{k} D^2\psi(A)(\dot{A}, \dot{A}) = \frac{2k}{c^2}\mathbb{R}e\big\langle A, \dot{A}A^*\dot{A}\big\rangle_F - \frac{k}{c^2}\big\|\dot{A}A^*\big\|_F^2 - \frac{k}{c^2}\big\|A^*\dot{A}\big\|_F^2 + \|\dot{A}\|_F^2 + \frac{3k^2}{c^4}\big\|A^*\dot{A}A^*\big\|_F^2.$$

Corollary 4 *With the notation of Proposition 7, let $c = \sqrt{k}$ and let $A \in \mathcal{R}_k$ be such that all its nonzero singular values are equal* (*which implies that they all must be equal to* 1), *and assume that there exists $\zeta \in \mathbb{P}(\mathbb{K}^{n+1})$ such that $A\zeta = 0$. Let*

$$T = \big\{\dot{A} \in T_A\mathcal{R}_k : D^2\psi(A)(\dot{A}, \dot{A}) = 0, \langle A, \dot{A}\rangle_F = 0, \dot{A}\zeta = 0\big\}.$$

Then, T is a real vector space and has real dimension

- $2kn + 2km - 3k^2 - 1$ *if* $\mathbb{K} = \mathbb{C}$,
- $kn + km - \frac{3}{2}k^2 - \frac{1}{2}k$ *if* $\mathbb{K} = \mathbb{R}$.

Proof First, assume that

$$A = J = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

is such that the first k entries of its main diagonal are 1 and the rest of the entries are 0. So, the zero ζ is in the subspace spanned by the last $n+1-k$ vectors of the standard basis. Recall the well-known fact (see for example [1]) that $\mathcal{R}_k$ is a submanifold of $\mathcal{M}_{m\times(n+1)}(\mathbb{K})$ of dimension $m(n+1) - (m-k)(n+1-k) - 1 = km + kn + k - k^2 - 1$, and its tangent space at J is the set of $\dot{A}$ of the form

$$\dot{A} = \begin{pmatrix} \dot{A}_1 & \dot{A}_2 \\ \dot{A}_3 & 0 \end{pmatrix}, \qquad \mathbb{R}e\big(\mathrm{trace}(\dot{A}_1)\big) = 0,$$

where the blocks are compatible with those of J. Note that the last hypothesis on $\dot{A}_1$ comes from the fact that $\mathcal{R}_k$ is considered inside a sphere in the space of matrices.

From the expression for $D^2\psi(I)(\dot{A},\dot{A})$ of Proposition 9, we have

$$D^2\psi(J)(\dot{A},\dot{A})=0 \Leftrightarrow 2\mathbb{R}e\langle J,\dot{A}J^*\dot{A}\rangle_F = \|\dot{A}J^*\|_F^2+\|J^*\dot{A}\|_F^2-\|\dot{A}\|_F^2-3\|J^*\dot{A}J^*\|_F^2.$$

Now,

$$\begin{aligned}\mathbb{R}e\langle J,\dot{A}J^*\dot{A}\rangle_F &= \mathbb{R}e\big(\mathrm{trace}\big(\dot{A}_1^2\big)\big)=\mathbb{R}e\langle \dot{A}_1,\dot{A}_1^*\rangle,\\ \|\dot{A}J^*\|_F^2 &= \|\dot{A}_1\|_F^2+\|\dot{A}_3\|_F^2,\\ \|J^*\dot{A}\|_F^2 &= \|\dot{A}_1\|_F^2+\|\dot{A}_2\|_F^2,\\ \|\dot{A}\|_F^2 &= \|\dot{A}_1\|_F^2+\|\dot{A}_2\|_F^2+\|\dot{A}_3\|_F^2,\\ 3\|J^*\dot{A}J^*\|_F^2 &= 3\|\dot{A}_1\|_F^2.\end{aligned}$$

We conclude that

$$\begin{aligned}D^2\psi(J)(\dot{A},\dot{A})=0 &\Leftrightarrow \mathbb{R}e\langle \dot{A}_1,\dot{A}_1^*\rangle = -\|\dot{A}_1\|_F^2 = -\mathbb{R}e\langle \dot{A}_1,\dot{A}_1\rangle\\ &\Leftrightarrow \mathbb{R}e\langle \dot{A}_1,\dot{A}_1+\dot{A}_1^*\rangle_F=0.\end{aligned}$$

Now, for any matrices B,C of size $m\times(n+1)$ we have $\langle B,C\rangle=\mathrm{trace}(BC^*)$ and thus,

$$\mathbb{R}e\langle B,C\rangle=\mathbb{R}e\big(\mathrm{trace}\big(BC^*\big)\big)=\mathbb{R}e\big(\mathrm{trace}\big(C^*B\big)\big)=\mathbb{R}e\langle C^*,B^*\rangle=\mathbb{R}e\langle B^*,C^*\rangle.$$

In particular,

$$\mathbb{R}e\langle \dot{A}_1,\dot{A}_1+\dot{A}_1^*\rangle_F=0 \Rightarrow \mathbb{R}e\langle \dot{A}_1^*,\dot{A}_1+\dot{A}_1^*\rangle_F=0 \Rightarrow \mathbb{R}e\langle \dot{A}_1+\dot{A}_1^*,\dot{A}_1+\dot{A}_1^*\rangle_F=0,$$

and hence $\dot{A}_1+\dot{A}_1^*=0$. We conclude that

$$T=\left\{\dot{A}=\begin{pmatrix}\dot{A}_1 & \dot{A}_2\\ \dot{A}_3 & 0\end{pmatrix}: \dot{A}_1 \text{ is skew-symmetric and } \mathrm{trace}(\dot{A}_1)=0,\ \dot{A}\zeta=0\right\},$$

and the corollary follows in this case (note that in particular we have proved that T is a real vector space).

Now, for general A, write a singular value decomposition $A=UJV^*$ with U and V unitary (orthogonal if $\mathbb{K}=\mathbb{R}$), and note that ψ is invariant under multiplication by U and V. That is, $\psi(P)=\psi(U^*PV)$ for any matrix P. Hence,

$$D^2\psi(A)(\dot{A},\dot{A})=D^2\psi(J)\big(U^*\dot{A}V,U^*\dot{A}V\big).$$

Hence, the dimension of T is the same as that for the case $A=J$. □

Recall that we have chosen $p=(f,\zeta)\in\mathcal{B}$. Assume that representatives such that $\|f\|=\sqrt{k}$, $\|\zeta\|=1$ are chosen. From Proposition 2, this implies that $\|\operatorname{Diag}(d_i^{-1/2})Df(\zeta)\|_F=\sqrt{k}$.

Lemma 4 *Let $w=h+v\in T_pW$, where $v\in T_p\mathcal{B}$. Then,*

$$\operatorname{Hess}(\widetilde{\mu})(p)(w,w)=\operatorname{Hess}(\widetilde{\mu})(p)(h,h).$$

Proof As $\widetilde{\mu}|_{\mathcal{B}}\equiv k$ is constant and $\mathcal{B}$ is a set of critical points of $\widetilde{\mu}$, $v\in T_p\mathcal{B}$ implies $\operatorname{Hess}(\widetilde{\mu})(p)(v,v)=0$. Now, the Hessian is a bilinear form and as p is a minimum of $\widetilde{\mu}$ it is positive semidefinite. Thus, $\operatorname{Hess}(\widetilde{\mu})(p)(v,v)=0$ implies $\operatorname{Hess}(\widetilde{\mu})(p)(h,v)=0$ for every h and the lemma follows. □

Let $V_p\subseteq T_pW$ be the set of tangent vectors $(\dot{f},0)$ such that $\operatorname{Hess}(\widetilde{\mu})(f,\zeta)(v,v)=0$, and let

$$H_p=\left\{(\dot{h},0):\left\langle(\dot{h},0),(\dot{f},0)\right\rangle=0\ \forall(\dot{f},0)\in V_p\right\}\subseteq T_pW. \tag{6.7}$$

Note that the real codimension of H_p in T_pW equals

$$\operatorname{codim}_{\mathbb{R}}(H_p)=\dim_{\mathbb{R}}\left(\mathbb{P}\left(\mathbb{K}^{n+1}\right)\right)+\dim_{\mathbb{R}}(V_p).$$

Lemma 5 *The real dimension of V_p equals*

- $2kn+2km-3k^2-1$ *if* $\mathbb{K}=\mathbb{C}$,
- $kn+km-\frac{3}{2}k^2-\frac{1}{2}k$ *if* $\mathbb{K}=\mathbb{R}$.

Proof By the unitary invariance of (1.1) and (1.5), we may assume that $\zeta=e_0$. Let $(f_t,e_0)\subseteq W$ be a $\mathcal{C}^1$ curve, $p=(f_0,e_0)=(f,e_0)$, with tangent vector $v=(\dot{f}_t,0)$. Let $A_t=\operatorname{Diag}(d_i^{-1/2})Df_t(e_0)$, so $\dot{A}_t=\operatorname{Diag}(d_i^{-1/2})D\dot{f}_t(e_0)$ and $\ddot{A}_t=\operatorname{Diag}(d_i^{-1/2})D\ddot{f}_t(e_0)$. Choose representatives of f_t such that $\|A_t\|_F^2=k,\forall t$. In particular, all the nonzero singular values of A_0 are equal to 1, and

$$0=\sqrt{k}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\|A_t\|\underset{\text{Lemma 3}}{=}\|\dot{A}_0\|_F^2+\mathbb{R}e\langle A_0,\ddot{A}_0\rangle_F.$$

Then, we have

$$\sqrt{k}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\|f_t\|\underset{\text{Lemma 3}}{=}\mathbb{R}e\langle f,\dot{f}_0\rangle\underset{\text{Lemma 2}}{=}\mathbb{R}e\langle A_0,\dot{A}_0\rangle_F=0, \tag{6.8}$$

$$\sqrt{k}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\|f_t\|\underset{\text{Lemma 3}}{=}\|\dot{f}\|^2+\mathbb{R}e\langle f,\ddot{f}_0\rangle\underset{(4.1)}{\geq}\|\dot{A}_0\|_F^2+\mathbb{R}e\langle A_0,\ddot{A}_0\rangle_F=0,$$

the inequality being an equality if $\|\dot{f}_0\|=\|\dot{A}_0\|_F$, that is, as in Sect. 4, if $\dot{f}$ has the form

$$\dot{f}(z)=\operatorname{Diag}\left(d_i^{-1/2}z_0^{d_i-1}\right)\dot{A}z,\qquad Ae_0=0.$$

Recall that if α and β are two differentiable functions of one variable we have $(\alpha\beta)'' = \alpha''\beta + 2\alpha'\beta' + \alpha\beta''$. Hence, using (6.8) we get

$$\begin{aligned}\mathrm{Hess}(\widetilde{\mu})(f, e_0)(v, v) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\big(\widetilde{\mu}(f_t, e_0)\big) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\big(\|f_t\|\,\|A_t^\dagger\|_F\big)\\ &= \sqrt{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\big(\|f_t\|\big) + \sqrt{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\big(\|A_t^\dagger\|_F\big)\\ &\geq \sqrt{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\big(\|A_t^\dagger\|_F\big) = D^2\psi(A)(\dot{A}_0, \dot{A}_0),\end{aligned}$$

in the notation of Corollary 4.

We have proved that

$$V_p = \big\{(\dot{f}, 0) : \dot{f}z = \mathrm{Diag}\big(d_i^{1/2} z_0^{d_i - 1}\big)\dot{A}z,\ \dot{A} \in T\big\},$$

where T is as in Corollary 4 (with $\zeta = 0$). In particular, the dimension of V_p is equal to that of T, and the lemma then follows from Corollary 4. □

Proof of Proposition 8 From the definition of H_p, we have that $H_p \cap T_p\mathcal{B} = \{0\}$. Moreover, from Lemma 5 and Proposition 3, the codimension of H_p in T_pW is equal to the dimension of $\mathcal{B}$ (both for $\mathbb{K} = \mathbb{R}, \mathbb{C}$).

We conclude that H_p and $T_p\mathcal{B}$ are complementary subspaces of T_pW. Let $w = h + v$, where $v \in T_p\mathcal{B}$ and $h \in H_p$. Then, from Lemma 4 and the definition of H_p,

$$\mathrm{Hess}(\widetilde{\mu})(p)(w, w) = \mathrm{Hess}(\widetilde{\mu})(p)(h, h) > 0,$$

unless $h = 0$ (i.e., unless $w \in T_p\mathcal{B}$). □

7 Distances and Integral Curves of the Gradient Flow

In this section the symbol X denotes the smooth vector field in $W \setminus \mathcal{B}$ defined by

$$X(f, \zeta) = \frac{\mathrm{grad}\,\widetilde{\mu}(f, \zeta)}{\|\mathrm{grad}\,\widetilde{\mu}(f, \zeta)\|^2}. \tag{7.1}$$

Given $p \in W$, the integral curve $\alpha_p(t)$ is a curve which satisfies $\alpha_p(0) = p$, $\mathrm{d}/\mathrm{d}t(\alpha_p(t)) = X(\alpha_p(t))$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mu}\big(\alpha_p(t)\big) = D\widetilde{\mu}\big(\alpha_p(t)\big)\frac{\mathrm{d}}{\mathrm{d}t}\alpha_p(t) = \frac{\langle \mathrm{grad}\,\widetilde{\mu}(\alpha_p(t)), \mathrm{grad}\,\widetilde{\mu}(\alpha_p(t))\rangle}{\|\mathrm{grad}\,\widetilde{\mu}(\alpha_p(t))\|^2} = 1.$$

Hence, for $p = (f_0, \zeta_0) \in W \setminus \mathcal{B}$, the integral curve $\alpha_p(t)$ is defined for $t \in (k - \widetilde{\mu}(p), \infty)$ and

$$\widetilde{\mu}\big(\alpha_p(t)\big) = \widetilde{\mu}(p) + t.$$

Lemma 6

$$\int \frac{k}{s\sqrt{s^2-k^2}}\,\mathrm{d}s = \arccos\frac{k}{s}, \qquad \int \frac{k}{\sqrt{s^2-k^2}}\,\mathrm{d}s = k\ln\bigl(s+\sqrt{s^2-k^2}\bigr).$$

Proposition 10 *Let $\alpha : [a,b] \longrightarrow W$ be a piecewise $\mathcal{C}^1$ curve, and let* $\mathrm{Length}_F(\alpha)$ *be the length of α in the Fubini–Study metric. Then,*

$$\mathrm{Length}_F(\alpha) \geq \frac{1}{d^{3/2}}\left|\frac{1}{\widetilde{\mu}(\alpha(a))} - \frac{1}{\widetilde{\mu}(\alpha(b))}\right|.$$

If α is an integral curve of X, then

$$\mathrm{Length}_F(\alpha) \leq \arccos\frac{k}{\widetilde{\mu}(\alpha(b))} - \arccos\frac{k}{\widetilde{\mu}(\alpha(a))}.$$

Moreover, for any $p=(f,\zeta)\in W\setminus\mathcal{B}$, both $\lim_{t\mapsto\infty}\alpha_p(t)\in\Sigma'$, Σ' the set of ill-posed inputs, see (2.1), and $\lim_{t\mapsto k-\widetilde{\mu}(p)}\alpha_p(t)\in\mathcal{B}$ exist.

Proof For the lower bound, let $\tau:[a,b]\to\mathbb{R}$ be defined as $\tau(s)=\widetilde{\mu}(\alpha(s))$. Note that τ is piecewise C^1 and

$$\tau'(s) = D\widetilde{\mu}\bigl(\alpha(s)\bigr)\dot{\alpha}(s) \leq \bigl\|\mathrm{grad}\mu\bigl(\alpha(s)\bigr)\bigr\|\bigl\|\dot{\alpha}(s)\bigr\| \underset{\text{Prop. 4}}{\leq} d^{3/2}\widetilde{\mu}\bigl(\alpha(s)\bigr)^2\bigl\|\dot{\alpha}(s)\bigr\|.$$

Thus,

$$\mathrm{Length}_F(\alpha) = \int_a^b \bigl\|\dot{\alpha}(s)\bigr\|\,\mathrm{d}s = \int_a^b \frac{d^{3/2}\widetilde{\mu}(\alpha(s))\|\dot{\alpha}(s)\|}{d^{3/2}\widetilde{\mu}(\alpha(s))^2}\,\mathrm{d}s \geq \int_a^b \frac{\tau'(s)}{d^{3/2}\widetilde{\mu}(\alpha(s))^2}\,\mathrm{d}s,$$

and the change of variables $t=\tau(s)$ then yields

$$\begin{aligned}\mathrm{Length}_F(\alpha) \geq \int_{\tau(a)}^{\tau(b)} \frac{\mathrm{d}t}{d^{3/2}t^2} &= \frac{1}{d^{3/2}}\left[\frac{-1}{t}\right]_{t=\tau(a)=\widetilde{\mu}(\alpha(a))}^{t=\tau(b)=\widetilde{\mu}(\alpha(b))} \\ &= \frac{1}{d^{3/2}}\left(\frac{1}{\widetilde{\mu}(\alpha(a))} - \frac{1}{\widetilde{\mu}(\alpha(b))}\right).\end{aligned}$$

Reversing the order of a,b gives the claimed lower bound.

For the upper bound, note that if α is an integral curve for X,

$$\mathrm{Length}_F(\alpha) = \int_a^b \bigl\|\dot{\alpha}(s)\bigr\|\,\mathrm{d}s = \int_a^b \frac{1}{\|\mathrm{grad}\,\widetilde{\mu}(\alpha(s))\|}\,\mathrm{d}s,$$

and from Proposition 4,

$$\mathrm{Length}_F(\alpha) \leq \int_a^b \frac{k}{\widetilde{\mu}(\alpha(s))^2\sqrt{1-\frac{k^2}{\widetilde{\mu}(\alpha(s))^2}}}\,\mathrm{d}s$$

$$= \int_a^b \frac{k}{(\widetilde{\mu}(\alpha(a)) + s - a)^2 \sqrt{1 - \frac{k^2}{(\widetilde{\mu}(\alpha(a))+s-a)^2}}} \, \mathrm{d}s$$

$$= \int_{\widetilde{\mu}(\alpha(a))}^{\widetilde{\mu}(\alpha(b))} \frac{k}{s\sqrt{s^2 - k^2}} \, \mathrm{d}s$$

and the upper bound of the proposition follows from Lemma 6.

For the last claim of the lemma, let $\{t_j\}_{j\in\mathbb{N}}$ be an increasing sequence converging to ∞ as $j \mapsto \infty$. Then, for $j_1 \geq j_2$,

$$d\big(\alpha_p(t_{j_1}), \alpha_p(t_{j_2})\big) \leq L(t_{j_1}, t_{j_2}) \leq \arccos \frac{k}{\widetilde{\mu}(\alpha_p(t_{j_2}))} - \arccos \frac{k}{\widetilde{\mu}(\alpha_p(t_{j_1}))}$$
$$\leq \frac{\pi}{2} - \arccos \frac{k}{\widetilde{\mu}(\alpha_p(t_{j_1}))}.$$

As this last term tends to 0 as j_1 tends to infinity, we have that $\alpha_p(t_j)$ is a Cauchy sequence in the compact manifold V, which implies that it has a limit. As we have chosen any subsequence, we conclude that $\alpha_p(t)$ has a limit $(h, \eta) \in V$ as t goes to ∞. Note that $\lim_{t\mapsto\infty} \widetilde{\mu}(\alpha_p(t)) = \infty$, which by continuity of $\widetilde{\mu}$ implies that $(h, \eta) \in V \setminus W$ (because if $(h, \eta) \in W$ then $\widetilde{\mu}(h, \eta)$ would be a real number).

Let $\alpha_p(t) = (f_t, \zeta_t)$. Then,

$$\mathrm{Diag}\big(d_i^{1/2} Df_t(\zeta_t)\big) \mapsto \mathrm{Diag}\big(d_i^{1/2} Dh(\eta)\big) \quad \text{as } t \mapsto \infty.$$

Thus, $Dh(\eta)$ is the limit of rank k matrices and thus has rank at most $k - 1$, that is, $(h, \eta) \in \Sigma'$.

Similarly, if $\{t_j\}_{j\in\mathbb{N}} \in (k - \widetilde{\mu}(p), 0)$ is a decreasing sequence converging to $k - \widetilde{\mu}(p)$ and $j_1 \geq j_2$, from the definition of integral curve we have

$$d\big(\alpha_p(t_{j_2}), \alpha_p(t_{j_1})\big) = d\big(\alpha_{\alpha_p(t_{j_2})}(0), \alpha_{\alpha_p(t_{j_2})}(t_{j_1} - t_{j_2})\big)$$
$$\leq \arccos \frac{k}{\widetilde{\mu}(\alpha_p(t_{j_1}))} - \arccos \frac{k}{\widetilde{\mu}(\alpha_p(t_{j_2}))}$$
$$= \arccos \frac{k}{\widetilde{\mu}(p) + t_{j_1}} - \arccos \frac{k}{\widetilde{\mu}(p) + t_{j_2}}.$$

Because arccos and $x \mapsto 1/x$ are locally Lipschitz in our range, this last is at most $K|t_{j_1} - t_{j_2}|$ for some $k \geq 0$ and hence $\alpha_p(t_j)$ is a Cauchy sequence. As t_j is any sequence, we conclude that $\alpha_p(t)$ converges to some (h, η) as t approaches $k - \widetilde{\mu}(p)$. Moreover,

$$\widetilde{\mu}(h, \eta) = \lim_{t\mapsto k-\widetilde{\mu}(p)} \widetilde{\mu}\big(\alpha_p(t)\big) = \lim_{t\mapsto k-\widetilde{\mu}(p)} \widetilde{\mu}(p) - t = k,$$

which by Proposition 3 implies $(h, \eta) \in \mathcal{B}$. The proposition is now proved. □

Corollary 5 (See Proposition 5, item 3) *For any two pairs* $(f,\zeta),(h,\eta)\in W$,

$$\left|\frac{1}{\widetilde{\mu}(f,\zeta)}-\frac{1}{\widetilde{\mu}(h,\eta)}\right|\le d\big((f,\zeta),(h,\eta)\big)d^{3/2}.$$

In particular, if $\varepsilon=d((f,\zeta),(h,\eta))d^{3/2}\widetilde{\mu}(f,\zeta)$ *satisfies* $\varepsilon<1$, *then*

$$\frac{\widetilde{\mu}(f,\zeta)}{1+\varepsilon}\le\widetilde{\mu}(h,\eta)\le\frac{\widetilde{\mu}(f,\zeta)}{1-\varepsilon}.$$

Proof The first part is immediate using the lower bound for the length of a curve given by Proposition 10. For the second part, note that

$$\widetilde{\mu}(f,\zeta)\left(\frac{1}{\widetilde{\mu}(f,\zeta)}-\frac{1}{\widetilde{\mu}(h,\eta)}\right)\le\widetilde{\mu}(f,\zeta)\left|\frac{1}{\widetilde{\mu}(f,\zeta)}-\frac{1}{\widetilde{\mu}(h,\eta)}\right|\le\varepsilon.$$

Namely,

$$1-\frac{\widetilde{\mu}(f,\zeta)}{\widetilde{\mu}(h,\eta)}\le\varepsilon,$$

which implies $\widetilde{\mu}(h,\eta)\le\widetilde{\mu}(f,\zeta)/(1-\varepsilon)$. The other inequality is obtained in the same way, using:

$$\widetilde{\mu}(f,\zeta)\left(-\frac{1}{\widetilde{\mu}(f,\zeta)}+\frac{1}{\widetilde{\mu}(h,\eta)}\right)\le\widetilde{\mu}(f,\zeta)\left|\frac{1}{\widetilde{\mu}(f,\zeta)}-\frac{1}{\widetilde{\mu}(h,\eta)}\right|.$$ □

Proposition 11 *Let* $\alpha:[a,b]\longrightarrow W$ *be a curve, and let* $\mathrm{Length}_{F,\kappa}(a,b)$ *be the length of* α *in the Frobenius condition metric. Then,*

$$\mathrm{Length}_{F,\kappa}(a,b)\ge\frac{1}{d^{3/2}}\left|\ln\frac{\widetilde{\mu}(\alpha(b))}{\widetilde{\mu}(\alpha(a))}\right|.$$

Moreover, if α *is an integral curve of* X, *then*

$$\mathrm{Length}_{F,\kappa}(a,b)\le k\ln\frac{\widetilde{\mu}(\alpha(b))+\sqrt{\widetilde{\mu}(\alpha(b))^2-k^2}}{\widetilde{\mu}(\alpha(a))+\sqrt{\widetilde{\mu}(\alpha(a))^2-k^2}}.$$

Proof The proof is the same as that of Proposition 10, but now we use the Frobenius condition metric, so we must bound

$$\mathrm{Length}_{F,\kappa}(a,b)=\int_a^b\widetilde{\mu}\big(\alpha(s)\big)\big\|\dot{\alpha}(t)\big\|\,\mathrm{d}s.$$

That is, for any curve α,

$$\mathrm{Length}_{F,\kappa}(a,b)\ge\int_{\widetilde{\mu}(\alpha(a))}^{\widetilde{\mu}(\alpha(b))}\frac{1}{d^{3/2}t}\,\mathrm{d}t=\frac{1}{d^{3/2}}\ln\frac{\widetilde{\mu}(\alpha(b))}{\widetilde{\mu}(\alpha(a))}.$$

Changing the order of a and b gives the first claim of the proposition. For the second claim, if α is an integral curve of X, as in Proposition 10 we get

$$\mathrm{Length}_{F,\kappa}(a,b) \le \int_{\widetilde{\mu}(\alpha(a))}^{\widetilde{\mu}(\alpha(b))} \frac{k}{\sqrt{s^2-k^2}}\,\mathrm{d}s,$$

which is computed using Lemma 6. □

Corollary 6 (See Proposition 5, item 2) *Let $(f,\zeta)\in W=W^k$ and let $d((f,\zeta),\Sigma')$ be the Fubini–Study distance from (f,ζ) to the set of ill-posed problems. Then,*

$$\frac{1}{d^{3/2}\widetilde{\mu}(f,\zeta)} \le d\big((f,\zeta),\Sigma'\big) \le \arcsin\frac{k}{\widetilde{\mu}(f,\zeta)} \le \frac{\pi}{2}\frac{k}{\widetilde{\mu}(f,\zeta)}.$$

Moreover, let $d((f,\zeta),\mathcal{B})$ be the Fubini–Study distance from (f,ζ) to the set $\mathcal{B}$. Then,

$$\frac{1}{d^{3/2}}\left(\frac{1}{k}-\frac{1}{\widetilde{\mu}(f,\zeta)}\right) \le d\big((f,\zeta),\mathcal{B}\big) \le \arccos\frac{k}{\widetilde{\mu}(f,\zeta)}.$$

Proof Let $p=(f,\zeta)$ and let $\alpha_p(t)$ be the associated integral curve of X. Let $(h,\eta)\in V$ be the limit as t goes to ∞ of $\alpha_p(t)$, which exists from Proposition 10 and which is in Σ'. Then,

$$\begin{aligned} d\big((f,\zeta),\Sigma'\big) \quad &\le \quad d\big((f,\zeta),(h,\zeta)\big) = \lim_{t\mapsto\infty} d\big((f,\zeta),\alpha_p(t)\big) \\ &\underset{\text{Prop. 10}}{\le} \frac{\pi}{2} - \arccos\frac{k}{\widetilde{\mu}(f,\zeta)} = \arcsin\frac{k}{\widetilde{\mu}(f,\zeta)}. \end{aligned}$$

On the other hand, let $\alpha(t)\subseteq W$, $t\in[a,b)$ be any curve such that $\alpha(0)=(f,\zeta)$ and $\alpha(t)$ tends to Σ' as $t\mapsto b$. Then, $\widetilde{\mu}(\alpha(t))$ tends to ∞ as $t\mapsto b$ and, hence, for any $r>\widetilde{\mu}(f,\zeta)$ there is some t_0 such that $\widetilde{\mu}(\alpha(t_0))=r$. Thus,

$$\int_a^b \big\|\dot{\alpha}(t)\big\|\,\mathrm{d}t \ge \int_a^{t_0} \big\|\dot{\alpha}(t)\big\|\,\mathrm{d}t \underset{\text{Prop. 10}}{\ge} \frac{1}{d^{3/2}}\left(\frac{1}{\widetilde{\mu}(f,\zeta)}-\frac{1}{r}\right).$$

As this is valid for any $r>\widetilde{\mu}(f,\zeta)$, we conclude that the length of any curve joining (f,ζ) to Σ' is at least $(d^{3/2}\widetilde{\mu}(f,\zeta))^{-1}$, as desired. The bounds on the distance to $\mathcal{B}$ are computed in a similar way: let $(h,\eta)=\lim_{t\mapsto k-\widetilde{\mu}(f,\zeta)}\alpha_p(t)\in\mathcal{B}$, which exists from Proposition 10. Then, the curve

$$\alpha(t)=\begin{cases} \alpha_p(t), & k-\widetilde{\mu}(f,\zeta)<t\le 0,\\ (h,\eta), & t=k-\widetilde{\mu}(f,\zeta)\end{cases}$$

is a C^1 curve with extremes (f,ζ) and (h,η). For any $t_0>k-\widetilde{\mu}(f,\zeta)$, $\mathrm{Image}(\alpha(t))=\mathrm{Image}(\beta(s))$, where

$$\beta(s)=\alpha_{\alpha(t_0)}(s), \quad s\in[0,t_0].$$

Thus, from Proposition 10, the length of α is at most

$$\arccos\frac{k}{\widetilde{\mu}(\beta(0))}-\arccos\frac{k}{\beta(t_0)}=\arccos\frac{k}{\widetilde{\mu}(f,\zeta)}-\arccos\frac{k}{\widetilde{\mu}(f,\zeta)-t_0}.$$

As this is valid for any $t_0>k-\widetilde{\mu}(f,\zeta)$, we conclude that the length of α is at most

$$\arccos\frac{k}{\widetilde{\mu}(f,\zeta)}-\arccos\frac{k}{\widetilde{\mu}(f,\zeta)-(k-\widetilde{\mu}(f,\zeta))}=\arccos\frac{k}{\widetilde{\mu}(f,\zeta)}.$$

Hence,

$$d\big((f,\zeta),\mathcal{B}\big)\le d\big((f,\zeta),(h,\eta)\big)\le\arccos\frac{k}{\widetilde{\mu}(f,\zeta)}.$$

On the other hand, let $\alpha:[a,b]\to W$ be a C^1 curve such that $\alpha(a)\in\mathcal{B}$ and $\alpha(b)=(f,\zeta)$. From Proposition 10, the length of α is at least

$$\frac{1}{d^{3/2}}\left(\frac{1}{k}-\frac{1}{\widetilde{\mu}(f,\zeta)}\right).$$

This is thus a lower bound for the distance from (f,ζ) to $\mathcal{B}$. □

Corollary 7 (See Proposition 5, item 1) *Let $(f,\zeta)\in W$ and let $d_{F,\kappa}((f,\zeta),\mathcal{B})$ be the distance in the Frobenius condition metric from (f,ζ) to the set $\mathcal{B}$. Then,*

$$\frac{1}{d^{3/2}}\ln\frac{\widetilde{\mu}(f,\zeta)}{k}\le d_{F,\kappa}\big((f,\zeta),\mathcal{B}\big)\le k\ln\frac{\widetilde{\mu}(f,\zeta)+\sqrt{\widetilde{\mu}(f,\zeta)^2-k^2}}{k}.$$

Proof The proof mimics that of Corollary 6, but using Proposition 11 instead of Proposition 10 for computing the bounds. □

Corollary 8 *Let $\alpha:[a,b]\to\subseteq W$ be a $\mathcal{C}^1$ embedded curve with extremes $\alpha(a)=(f,\zeta)$ and $\alpha(b)=(h,\eta)$. Let $MAX=\max_{(f,\zeta)\in\alpha}\widetilde{\mu}(f,\zeta)$ and $MIN=\min_{(f,\zeta)\in\alpha}\widetilde{\mu}(f,\zeta)$, and assume that*

$$\widetilde{\mu}(f,\zeta)\le\widetilde{\mu}(h,\eta). \tag{7.2}$$

Then,

$$\frac{MAX}{MIN}\le\sqrt{\frac{\widetilde{\mu}(h,\eta)}{\widetilde{\mu}(f,\zeta)}}\,e^{\frac{d^{3/2}\,\mathrm{Length}_{F,\kappa}(\alpha)}{2}},$$

where $\mathrm{Length}_{F,\kappa}(\alpha)$ *is the length of α in the Frobenius condition metric.*

Proof Let

$$t_0=\inf\big\{t\in[a,b]:\widetilde{\mu}\big(\alpha(t)\big)=MIN\big\},\qquad t_1=\inf\big\{t\in[a,b]:\widetilde{\mu}\big(\alpha(t)\big)=MAX\big\}.$$

Assume first that $t_0 \le t_1$. Then, considering the subintervals $[a,t_0]$, $[t_0,t_1]$, and $[t_1,b]$ we have

$$\begin{aligned}\mathrm{Length}_{F,\kappa}(\alpha) &\underset{\text{Prop. 11}}{\ge} \frac{1}{d^{3/2}}\left(\ln\frac{\mu(\alpha(a))}{MIN}+\ln\frac{MAX}{MIN}+\ln\frac{MAX}{\widetilde{\mu}(\alpha(b))}\right)\\ &= \frac{1}{d^{3/2}}\ln\frac{\widetilde{\mu}(\alpha(a))MAX^2}{\widetilde{\mu}(\alpha(b))MIN^2},\end{aligned}$$

and hence

$$\frac{MAX^2}{MIN^2}\le\frac{\widetilde{\mu}(h,\eta)}{\widetilde{\mu}(f,\zeta)}\mathrm{e}^{d^{3/2}\,\mathrm{Length}_{F,\kappa}(\alpha)}.$$

On the other hand, if $t_1 \le t_0$ then considering the subintervals $[a,t_1]$, $[t_1,t_0]$, and $[t_0,b]$ we get

$$\mathrm{Length}_{F,\kappa}(\alpha) \underset{\text{Prop. 11}}{\ge} \frac{1}{d^{3/2}}\left(\ln\frac{MAX}{\mu(\alpha(a))}+\ln\frac{MAX}{MIN}+\ln\frac{\widetilde{\mu}(\alpha(b))}{MAX}\right),$$

which now implies

$$\frac{MAX^2}{MIN^2}\le\frac{\widetilde{\mu}(f,\zeta)}{\widetilde{\mu}(h,\eta)}\mathrm{e}^{d^{3/2}\,\mathrm{Length}_{F,\kappa}(\alpha)}\underset{(7.2)}{\le}\frac{\widetilde{\mu}(h,\eta)}{\widetilde{\mu}(f,\zeta)}\mathrm{e}^{d^{3/2}\mathrm{Length}_{F,\kappa}(\alpha)}.$$

In any of the two cases, the corollary follows. □

Corollary 9 (See Proposition 5, item 4) *Let $k=m\le n$, $(f,\zeta)\in W$ and let $h\in\mathbb{P}(\mathcal{H}_{(d)})$ be a system such that*

$$\varepsilon=2d(f,h)\widetilde{\mu}(f,\zeta)^2 d^{3/2}\sqrt{1+\frac{1}{m^2}}<1.$$

Then, ζ can be continued to a zero η of h and

$$\frac{\widetilde{\mu}(f,\zeta)}{\sqrt{1+\varepsilon}}\le\widetilde{\mu}(h,\eta)\le\frac{\widetilde{\mu}(f,\zeta)}{\sqrt{1-\varepsilon}}.$$

Proof Let $\tilde{W}$ be as in the proof of Proposition 1; i.e., it is the set of pairs $(h,\eta)\in\mathbb{S}\times\mathbb{S}(\mathbb{K}^{n+1})$ such that $h(\eta)=0$ and $Dh(\eta)$ has maximal rank. Note that $\tilde{W}$ is the preimage of 0 under the regular mapping $\mathbb{S}\times\mathbb{S}(\mathbb{K}^{n+1})\to\mathbb{K}^n$ sending (h,η) to $h(\eta)$. Thus, the tangent space to $\tilde{W}$ is

$$T_{(h,\eta)}\tilde{W}=\left\{(\dot{h},\dot{\eta})\in\mathcal{H}_{(d)}\times\mathbb{K}^{n+1}:\mathbb{R}e\langle h,\dot{h}\rangle=0,\mathbb{R}e\langle\eta,\dot{\eta}\rangle=0,\dot{h}(\eta)+Dh(\eta)\dot{\eta}=0\right\}.$$

It follows that the projection on the first coordinate $W\overset{\pi_1}{\to}\mathbb{S}$ is a submersion, for given $\dot{h}\in T_h\mathbb{S}$ the vector $(\dot{h},Dh(\eta)^\dagger\dot{h}(\eta))$ is in $T_{(h,\eta)}W$ and projects to $\dot{h}$. Given

$(f,\zeta),(h,\eta)\in W$ as in the corollary, let $f_t\in\mathbb{S}$ be a horizontal lift to the sphere of the geodesic from f to h. We consider f_t parametrized by arc length and extended to be defined for $t\in[0,d(f,h)+\delta]$ (some $\delta>0$.) Thus, f_t is a smooth embedded injective curve in $\mathbb{S}$ such that $f_0=f$, $\|\dot f_t\|=1$, $f_{d(f,h)}=h$. Let

$$t_0=\sup\big\{t\in\big[0,d(f,h)+\delta\big]:f_s\in\pi_1(W)\text{ for every }s\in[0,t]\big\}>0.$$

The preimage $\pi_1^{-1}(\{f_t:0\le t<t_0\})$ is a submanifold of W, and we can define on that submanifold a vector field

$$X(f_t,\eta)=\big(\dot f_t,Df_t(\eta)^\dagger\dot f_t(\eta)\big).$$

Consider the integral curve $\alpha(t)=(f_t,\zeta_t)$ of X, and note that $\alpha(0)=(f,\zeta)$, $\alpha(t)\subseteq W$ and thus $f_t(\zeta_t)=0$. In particular, the zero ζ of f can be continued to a zero ζ_t of f_t, and

$$\begin{aligned}\|\dot\zeta_t\| &= \big\|Df_t(\zeta_t)^\dagger\dot f_t(\zeta_t)\big\|\le\big\|Df_t(\zeta_t)^\dagger\big\|_F\big\|\dot f_t(\zeta_t)\big\|\\ &\underset{(1.2)\text{ and }\|\zeta_t\|=1}{\le}\big\|Df_t(\zeta_t)^\dagger\big\|_F\|\dot f_t\|\le\widetilde\mu(f_t,\zeta_t)\|\dot f_t\|.\end{aligned}\tag{7.3}$$

Note that from Proposition 4, if $t\in(0,s)$,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\widetilde\mu\big(\alpha(t)\big)\right| &= \big|D\widetilde\mu(f_t,\zeta_t)\dot\alpha(t)\big|\le d^{3/2}\widetilde\mu\big(\alpha(t)\big)^2\big\|(\dot f_t,\dot\zeta_t)\big\|\\ &\underset{(7.3)}{\le}d^{3/2}\widetilde\mu\big(\alpha(t)\big)^2\sqrt{1+\widetilde\mu\big(\alpha(t)\big)^2}\|\dot f_t\|\underset{\widetilde\mu\ge k=m}{\le}d^{3/2}\sqrt{1+\frac{1}{m^2}}\widetilde\mu\big(\alpha(t)\big)^3.\end{aligned}\tag{7.4}$$

The standard proof for Gronwall's inequality can be repeated here:

$$d^{3/2}\sqrt{1+\frac{1}{m^2}}\ge\frac{1}{\widetilde\mu(\alpha(t))^3}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde\mu\big(\alpha(t)\big)=-\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\widetilde\mu(\alpha(t))^2}\right).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\widetilde\mu(\alpha(t))^2}\right)\ge-2d^{3/2}\sqrt{1+\frac{1}{m^2}},$$

and

$$\frac{1}{\widetilde\mu(\alpha(t))^2}\ge\frac{1}{\widetilde\mu(f,\zeta)^2}-2td^{3/2}\sqrt{1+\frac{1}{m^2}}.$$

Equivalently,

$$\widetilde\mu\big(\alpha(t)\big)^2\le\frac{\widetilde\mu(f,\zeta)^2}{1-2t\widetilde\mu(f,\zeta)^2d^{3/2}\sqrt{1+\frac{1}{m^2}}}.$$

In particular, as far as

$$t < \frac{1}{2\widetilde{\mu}(f,\zeta)^2 d^{3/2}\sqrt{1+\frac{1}{m^2}}},$$

we have that $(f_t,\zeta_t) \in W$ and thus $f_t \in \pi_1(W)$. The hypothesis of the corollary is that $d(f,h)$ is smaller than this quantity, so we conclude that $t_0 > d(f,h)$. Moreover, $\eta = \zeta_{d(f,g)}$ is a zero of h and

$$\widetilde{\mu}(h,\eta)^2 \le \frac{\widetilde{\mu}(f,\zeta)^2}{1 - 2d(f,h)\widetilde{\mu}(f,\zeta)^2 d^{3/2}\sqrt{1+\frac{1}{m^2}}} = \frac{\widetilde{\mu}(f,\zeta)^2}{1-\varepsilon}.$$

A similar argument changing $\mathrm{d}/\mathrm{d}t(\widetilde{\mu}(\alpha))$ to $-\mathrm{d}/\mathrm{d}t(\widetilde{\mu}(\alpha))$ in (7.4) proves that

$$\widetilde{\mu}(h,\eta)^2 \ge \frac{\widetilde{\mu}(f,\zeta)^2}{1+\varepsilon},$$

as claimed. □

7.1 Proof of Proposition 6

From Proposition 10, $\psi(p) = \lim_{t\mapsto\infty}\alpha_p(t)$ exists for every $p \in W\setminus\mathcal{B}$. It remains to see that it is a continuous function. Let $p \in W\setminus\mathcal{B}$. By continuity of the solution curves of a vector field with respect to the initial conditions, for any $t_0 > 0$ the mapping defined by $\psi_{t_0}(q) = \alpha_q(t_0)$ is continuous. Thus, given $\varepsilon > 0$ there exists $\delta_{t_0} > 0$ such that $d(p,q) < \delta$ implies $d(\alpha_p(t_0),\alpha_q(t_0)) < \varepsilon$. Now,

$$\widetilde{\mu}\big(\alpha_p(t_0)\big) = \widetilde{\mu}(p) + t_0, \qquad \widetilde{\mu}\big(\alpha_q(t_0)\big) = \widetilde{\mu}(q) + t_0.$$

Let

$$\mathrm{Length}_F(\alpha_{\alpha_p(t_0)})$$

be the length (in the Fubini–Study metric) of the curve $\alpha_{\alpha_p(t_0)}(s)$ for $s \in [0,\infty]$. From Proposition 10, we have

$$\begin{aligned}\mathrm{Length}_F(\alpha_{\alpha_p(t_0)}) &\le \lim_{s\mapsto\infty}\left(\arccos\frac{k}{\widetilde{\mu}(p)+t_0+s} - \arccos\frac{k}{\widetilde{\mu}(p)+t_0}\right)\\ &= \frac{\pi}{2} - \arccos\frac{k}{\widetilde{\mu}(p)+t_0} = \arcsin\frac{k}{\widetilde{\mu}(p)+t_0}.\end{aligned}$$

In particular,

$$d\big(\alpha_p(t_0),\psi(p)\big) = d\Big(\alpha_p(t_0), \lim_{s\mapsto\infty}\alpha_{\alpha_p(t_0)}(s)\Big) \le \arcsin\frac{k}{\widetilde{\mu}(p)+t_0} \le \frac{k}{t_0},$$

and similarly

$$d\big(\alpha_q(t_0),\psi(q)\big) \le \arcsin\frac{k}{\widetilde{\mu}(q)+t_0} \le \frac{k}{t_0}.$$

Hence,

$$d\big(\psi(p),\psi(q)\big)\le d\big(\psi(p),\alpha_p(t_0)\big)+d\big(\alpha_p(t_0),\alpha_q(t_0)\big)+d\big(\alpha_q(t_0),\psi(q)\big)$$
$$\le \frac{2k}{t_0}+d\big(\alpha_p(t_0),\alpha_q(t_0)\big).$$

Now, let $\varepsilon>0$ and let $t_0>2k\varepsilon^{-1}$. Then, for a point $q\in W\setminus\mathcal{B}$ such that $d(p,q)\le\delta_{t_0}$ we have

$$d\big(\psi(p),\psi(q)\big)\le\frac{2k}{2k\varepsilon^{-1}}+\varepsilon=3\varepsilon.$$

The continuity of ψ is now proved.

8 Topology of $\mathcal{B}$ in the Case that $k=m\le n$

Theorem 3 describes the topology of W in quite an explicit way, as it suffices to study the topology of $\mathcal{B}$. The homotopy, homology, and cohomology groups of W coincide with those of $\mathcal{B}$, which may be easier to compute. For example, if $\mathbb{K}=\mathbb{R}$ and $n=1$ then $\mathcal{B}$ is connected and diffeomorphic to the unit circle S^1. We compute some of these groups for the particular case that $k=m\le n$. These results are summarized in the following table.

	$\mathbb{K}=\mathbb{R}$ $m=n>1$ n and $d_1+\cdots+d_n-1$ even	$\mathbb{K}=\mathbb{R}$ $m=n>1$ other cases	$\mathbb{K}=\mathbb{C}$ $m<n$	$\mathbb{K}=\mathbb{C}$ $m=n$
$\pi_0(\mathcal{B})$	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\pi_1(\mathcal{B})$	8 elements	4 elements	$\{0\}$	$\mathbb{Z}/a\mathbb{Z}$
$\pi_2(\mathcal{B})$	$\{0\}$	$\{0\}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$
$\pi_k(\mathcal{B}), k\ge 3$	$\pi_k(\mathcal{SO}_{n+1})$	$\pi_k(\mathcal{SO}_{n+1})$	?	$\pi_k(\mathcal{SU}_{n+1})$

where $a=\gcd(n,d_1+\cdots+d_n-1)$ and $\mathbb{Z}/a\mathbb{Z}$ is the finite cyclic group of a elements.

For other values of m,n,k, similar results may be achieved using the techniques of this section. Recall that a sequence of homomorphisms

$$\cdots\longrightarrow A_{n+1}\xrightarrow{\alpha_{n+1}}A_n\xrightarrow{\alpha_n}A_{n-1}\longrightarrow\cdots,$$

is said to be *exact* if $\mathrm{Ker}(\alpha_n)=\mathrm{Image}(\alpha_{n+1})$ for each n. Given a fiber bundle $F\to E\xrightarrow{p}B$ (F the fiber, B the base space), where B is path connected and $b_0\in B$, $p_0\in p^{-1}(b_0)$, there exists a long exact sequence (see for example Theorem 4.41 and Proposition 4.48 in [17]):

$$\cdots\to\pi_n(F,p_0)\to\pi_n(E,p_0)\xrightarrow{p_*}\pi_n(B,b_0)\to\pi_{n-1}(F,p_0)$$
$$\to\cdots\to\pi_0(E,p_0)\to 0,\qquad(8.1)$$

where $\pi_n(X, x_0)$ is the n-th homotopy group of a space X with base point x_0, and p_* is the homomorphism induced by p (see [17, pp. 340–342]). In the case that the path-connected components of B are pairwise homeomorphic, and the same property holds for E, F, the homotopy groups do not depend on the base points, and hence they are just denoted by $\pi_n(X)$. We now write the last terms of the sequence (8.1) in this last case, assuming also that $\mathcal{B}$ is path connected:

$$\pi_2(F) \to \pi_2(E) \overset{p_*}{\to} \pi_2(B) \to \pi_1(F) \to \pi_1(E) \overset{p_*}{\to} \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to 0. \tag{8.2}$$

This long exact sequence together with Proposition 3 will be our main tool in the study of the topology of $\mathcal{B}$.

We now recall some well-known facts about homotopy groups of spheres and projective spaces, that the reader may find useful in subsequent computations. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. The lower homotopy groups of S^n are known[2] (see [17, Cor. 4.9 and 4.25]):

$$\pi_k\big(S^n\big) = 0, \quad k < n, \qquad \pi_n\big(S^n\big) = \mathbb{Z}.$$

The fiber bundle $S^1 \to S^{2n+1} \to \mathbb{P}(\mathbb{C}^{n+1})$ and (8.1) then imply

$$\pi_k\big(\mathbb{P}\big(\mathbb{C}^{n+1}\big)\big) = \pi_k\big(S^{2n+1}\big), \quad k > 2,$$
$$\pi_2\big(\mathbb{P}\big(\mathbb{C}^{n+1}\big)\big) = \mathbb{Z}, \qquad \pi_1\big(\mathbb{P}\big(\mathbb{C}^{n+1}\big)\big) = 0.$$

Similarly, from the fiber bundle $\{-1, 1\} \to S^n \to \mathbb{P}(\mathbb{R}^{n+1})$ one can see that (for $n > 1$):

$$\pi_k\big(\mathbb{P}\big(\mathbb{R}^{n+1}\big)\big) = \pi_k\big(S^n\big), \quad k > 1, \qquad \pi_1\big(\mathbb{P}\big(\mathbb{R}^{n+1}\big)\big) = \mathbb{Z}/2\mathbb{Z}.$$

We will also use the following equalities (see, for example, [19, Chap. 7, Sect. 12]):

$$\pi_1(\mathcal{U}_{n+1}) = \mathbb{Z}, \qquad \pi_2(\mathcal{SU}_n) = \pi_1(\mathcal{SU}_n) = \{0\}, \quad n \geq 1. \tag{8.3}$$

The topological structure of W seems surprisingly simple. For example, the table says that if $\mathbb{K} = \mathbb{C}$, $m = n = k$, and all the degrees $d_1, \ldots, d_n$ are equal, then W is a simply connected manifold. The topology of V can also be studied from the fiber bundle given by the projection on the second coordinate $V \to \mathbb{P}(\mathbb{K}^{n+1})$, whose fiber is a projective space. The long exact sequence (8.2) yields for example that V is simply connected if $\mathbb{K} = \mathbb{C}$. If $\mathbb{K} = \mathbb{R}$ and $n \geq 3$ (or $n = 2, d \geq 2$) we have the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(V) \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

and thus $\pi_1(V)$ has exactly 4 elements.

[2] The higher homotopy groups of spheres are, on the other hand, one of the most important open problems in algebraic topology.

Now we prove the results claimed in the table above. From Proposition 3, $\mathcal{B}$ is the orbit of (g, e_0) under the action of $\mathcal{G}_{n+1}$, where g is the (normalized) homogenization of the identity, i.e.,

$$g = (g_1, \dots, g_m) \in \mathbb{P}(\mathcal{H}_{(d)}), \quad g_i = d_i^{1/2} X_0^{d_i - 1} X_i, \; i = 1 \dots m. \tag{8.4}$$

Thus, $\mathcal{B}$ is diffeomorphic to $\mathcal{G}_{n+1}/H_1$, where

$$H_1 = \left\{ \begin{pmatrix} \rho & & \\ & \mathrm{Diag}(\lambda \rho^{1-d_i}) & \\ & & Q \end{pmatrix}, \; \rho, \lambda \in \mathcal{G}_1, \, Q \in \mathcal{G}_{n-m} \right\}$$

is the stabilizer of (g, e_0).

8.1 Case $\mathbb{K} = \mathbb{C}$

Note that the stabilizer H_1 is a connected set, and $\pi_1(\mathcal{U}_{n+1}) = \mathbb{Z}$. The long exact sequence (8.2) then shows that $\pi_1(\mathcal{B})$ is the homomorphic image of $\mathbb{Z}$ and hence it is a cyclic group.

Clearly, $\mathcal{B}$ is also the orbit of (g, e_0) under the action of the special unitary group $\mathcal{SU}_{n+1}$ on W. Let H_2 be the stabilizer of this action, so that $\mathcal{B}$ is diffeomorphic to $\frac{\mathcal{SU}_{n+1}}{H_2}$. Moreover, H_2 is the set of matrices

$$\begin{pmatrix} \rho & & \\ & \mathrm{Diag}(\lambda \rho^{1-d_i}) & \\ & & Q \end{pmatrix}, \quad \rho, \lambda \in S^1, \; Q \in \mathcal{U}_{n-m}, \; \rho^{m+1-d_1-\cdots-d_m} \lambda^m \det(Q) = 1. \tag{8.5}$$

Lemma 7 *Let $m < n$. Then, $\mathcal{B}$ is simply connected and $\pi_2(\mathcal{B}) = \mathbb{Z} \oplus \mathbb{Z}$.*

Proof Note that if $m < n$, we can define a surjection $H_2 \longrightarrow \mathbb{T}^2 = S^1 \times S^1$ by sending an element of H_2 in the form (8.5) to (ρ, λ). This defines on H_2 a structure of fiber bundle with fiber homeomorphic to $\mathcal{SU}_{n-m}$. Then, (8.2) reads

$$\pi_2(\mathcal{SU}_{n-m}) \to \pi_2(H_2) \to \pi_2(\mathbb{T}^2) \to \pi_1(\mathcal{SU}_{n-m}) \to \pi_1(H_2) \to \pi_1(\mathbb{T}^2) \to \{0\}.$$

That is, using (8.3),

$$\{0\} \to \pi_2(H_2) \to \{0\} \to \{0\} \to \pi_1(H_2) \to \mathbb{Z} \oplus \mathbb{Z} \to \{0\}.$$

We conclude that $\pi_2(H_2) = \{0\}$ and $\pi_1(H_2) = \mathbb{Z} \oplus \mathbb{Z}$. So, the long exact sequence for the fiber bundle $H_2 \to \mathcal{SU}_{n+1} \to \mathcal{B}$,

$$\pi_2(H_2) \to \pi_2(\mathcal{SU}_{n+1}) \to \pi_2(\mathcal{B}) \to \pi_1(H_2) \to \pi_1(\mathcal{SU}_{n+1}) \to \pi_1(\mathcal{B}) \to \{0\},$$

again using (8.3) reads

$$\{0\} \to \{0\} \to \pi_2(\mathcal{B}) \to \mathbb{Z} \oplus \mathbb{Z} \to \{0\} \to \pi_1(\mathcal{B}) \to \{0\},$$

and the lemma follows. □

Lemma 8 *Let $m = n$. Then, H_2 is a disjoint union of $a = gcd(n, d_1 + \cdots + d_n - 1)$ diffeomorphic copies of S^1.*

Proof Let $b = n + 1 - d_1 - \cdots - d_n$, so that $a = gcd(n, b)$, and let $G_a \subseteq S^1$ be the group of a-th roots of 1. Note that H_2 is isomorphic to $G_{b,n} = \{(\rho, \lambda) \in S^1 \times S^1 : \rho^b \lambda^n = 1\} \subseteq \mathbb{T}^2$. First, assume that $a = 1$. Then, $G_{b,n}$ can be parametrized by $\{(t^n, t^{-b}) : t \in S^1\}$ and is a torus knot that goes around $\mathbb{T}^2$, b times in one direction and n times in the other direction (see [21, pp. 13–14]). This proves that $G_{b,n}$ is a circle, thus diffeomorphic to S^1. Now, for the general case, consider the Lie epimorphism

$$\begin{aligned} \varphi : G_{b,n} &\longrightarrow G_a \\ (\rho, \lambda) &\mapsto \rho^{b/a} \lambda^{n/a}. \end{aligned}$$

Then, the a cosets $\{\varphi^{-1}(r) : r \in G_a\}$ are disjoint and pairwise diffeomorphic. Moreover,

$$\varphi^{-1}(1) = \big\{(\rho, \lambda) \in G_{b,n} : \rho^{b/a} \lambda^{n/a} = 1\big\} = \big\{(\rho, \lambda) \in S^1 \times S^1 : \rho^{b/a} \lambda^{n/a} = 1\big\},$$

is diffeomorphic to S^1 by the first part of the proof. The lemma follows. □

Proposition 12 *Let $m = n$ and $a = gcd(n, d_1 + \cdots + d_n - 1)$. Then,*

$$\begin{aligned} &\pi_k(\mathcal{B}) \equiv \pi_k(\mathcal{SU}_{n+1}) \quad \textit{for } k \geq 3 \\ &\pi_2(\mathcal{B}) \equiv \mathbb{Z} \\ &\pi_1(\mathcal{B}) \equiv \mathbb{Z}/a\mathbb{Z}, \end{aligned}$$

where $\mathbb{Z}/a\mathbb{Z}$ is the finite cyclic group of order a.

Proof The long exact sequence (8.1) for the fibration $H_2 \to \mathcal{SU}_{n+1} \to \mathcal{B}$ reads

$$\cdots \longrightarrow \pi_k(H_2) \longrightarrow \pi_k(\mathcal{SU}_{n+1}) \xrightarrow{\pi_*} \pi_k(\mathcal{B}) \longrightarrow \pi_{k-1}(H_2) \longrightarrow \cdots$$

Now, from Lemma 8, for $k \geq 2$ we have that $\pi_k(H_2) = \{0\}$ and we conclude that

$$\pi_k(\mathcal{B}) \equiv \pi_k(\mathcal{SU}_{n+1}), \quad \forall k \geq 3,$$

while the last terms of the long exact sequence read

$$\begin{aligned} \pi_2(\mathcal{SU}_{n+1}) &\xrightarrow{\pi_*} \pi_2(\mathcal{B}) \to \pi_1(H_2) \to \pi_1(\mathcal{SU}_{n+1}) \\ &\xrightarrow{\pi_*} \pi_1(\mathcal{B}) \to \pi_0(H_2) \to \pi_0(\mathcal{SU}_{n+1}), \end{aligned}$$

that is,

$$\{0\} \xrightarrow{\pi_*} \pi_2(\mathcal{B}) \longrightarrow \mathbb{Z} \longrightarrow \{0\} \xrightarrow{\pi_*} \pi_1(\mathcal{B}) \longrightarrow \pi_0(H_2) \longrightarrow \{0\}.$$

Note that, although H_2 is not connected if $a > 1$, we do not need to write the base points here because all the connected components of H_2 are pairwise diffeomorphic by Lemma 8. We conclude that $\pi_2(\mathcal{B}) \equiv \mathbb{Z}$ and $\pi_1(\mathcal{B})$ has finite cardinal a. As stated in the beginning of this section, $\pi_1(\mathcal{B})$ is cyclic, so it equals $\mathbb{Z}/a\mathbb{Z}$. □

8.2 Case $\mathbb{K} = \mathbb{R}$

Now we introduce the analogous lemmas on the topology of $\mathcal{B}$ for the real case, and we include a proof when necessary. The stabilizer H_1 of (g, e_0) under the action of the orthogonal group on W is now the set of matrices

$$\begin{pmatrix} \rho & & \\ & \mathrm{Diag}(\lambda\rho^{1-d_i}) & \\ & & Q, \end{pmatrix}$$

where $\rho, \lambda \in \mathbb{S}^0 = \{-1, 1\}$ and $Q \in \mathcal{O}_{n-m}$. In particular, H_1 is diffeomorphic to $\mathbb{S}^0 \times \mathbb{S}^0 \times \mathcal{O}_{n-m}$. Let g be as in (8.4).

Lemma 9 *Let $H_2 \subseteq \mathcal{SO}_{n+1}$ be the stabilizer of (g, e_0) under the action of $\mathcal{SO}_{n+1}$ on W. If $m < n$ or one of $n, d_1 + \cdots + d_n - 1$ is odd, then $\mathcal{B}$ equals the orbit of (g, e_0) under the action of $\mathcal{SO}_{n+1}$ on W and hence $\mathcal{B}$ is diffeomorphic to $\mathcal{SO}_{n+1}/H_2$. If $m = n$ and both $n, d_1 + \cdots + d_n - 1$ are even, $\mathcal{B}$ has two diffeomorphic connected components, and each of them is diffeomorphic to $\mathcal{SO}_{n+1}/H_2$.*

Proof Let $(f, \zeta) \in \mathcal{B}$. Consider the action of the special orthogonal group $\mathcal{SO}_{n+1}$ on W. From Proposition 3 we can write $(f, \zeta) = (g \circ U^*, Ue_0)$, where $U \in \mathcal{O}_{n+1}$. Let $V = U\,\mathrm{Diag}(\rho, \rho^{1-d_1}\lambda, \ldots, \rho^{1-d_m}\lambda, Q)$, where $\rho, \lambda \in \mathbb{S}^0$ and $Q \in \mathcal{O}_{n-m}$ are generic. Note that $\mathrm{Diag}(\rho, \rho^{1-d_1}\lambda, \ldots, \rho^{1-d_m}\lambda, Q)^* \in H_1$, and hence

$$g \circ \mathrm{Diag}\big(\rho, \rho^{1-d_1}\lambda, \ldots, \rho^{1-d_m}\lambda, Q\big)^* = g.$$

Thus, $(g \circ V^*, Ve_0) = (f, \zeta)$, and $V \in \mathcal{SO}_{n+1}$ when

$$\rho^{1+m-d_1-\cdots-d_m}\lambda^m \det(Q) = \frac{1}{\det(U)} \in \{-1, 1\}.$$

If $m < n$ or one of $n, d_1 + \cdots + d_n - 1$ is odd, we can choose ρ, λ, Q that satisfy this equation. Hence, if $m < n$ or one of $n, d_1 + \cdots + d_n - 1$ is odd, we have that $\mathcal{B}$ equals the orbit of (g, e_0) under the action of $\mathcal{SO}_{n+1}$ on W. Otherwise, $\mathcal{B}$ is diffeomorphic to $\mathcal{O}_{n+1}/H_1$, where H_1 is the discrete subgroup of matrices of the form $\mathrm{Diag}(\rho, \rho^{1-d_1}\lambda, \ldots, \rho^{1-d_n}\lambda)$. Now, in this last case $\mathcal{O}_{n+1}/H_1$ has two diffeomorphic connected components because $H_1 \subseteq \mathcal{SO}_{n+1}$. This finishes the proof. □

Recall (cf., for example, [19, Chap. 7, Sect. 12]) that

$$\pi_0(\mathcal{SO}_r) = \{0\}, \qquad \pi_2(\mathcal{SO}_r) = 0, \quad \forall r \geq 1,$$
$$\pi_1(\mathcal{SO}_1) = \{0\}, \qquad \pi_1(\mathcal{SO}_2) = \mathbb{Z}, \qquad \pi_1(\mathcal{SO}_r) = \mathbb{Z}_2, \quad \forall r \geq 3,$$

and note that, if $m = n$,

- $\sharp\pi_0(H_2) = 2$ except if both $n, d_1 + \cdots + d_n - 1$ are even, and
- in that case, $\sharp\pi_0(H_2) = 4$.

Lemma 9 and the long exact sequence (8.1) applied to the fibration $H_2 \to \mathcal{SO}_{n+1} \to \mathcal{B}$ yield some information about the homotopy groups of $\mathcal{B}$ in the different cases:

- Let $m = n = 1$. Then, $\mathcal{B}$ is diffeomorphic to S^1 (this follows directly from Lemma 9, for $\mathcal{SO}_2$ is diffeomorphic to S^1).
- Let $m = n > 1$. Then, $\pi_k(\mathcal{B}) = \pi_k(\mathcal{SO}_{n+1})$, for every $k \geq 2$, and the last terms of (8.2) read

  $$0 = \pi_1(H_2) \to \mathbb{Z}_2 \to \pi_1(\mathcal{B}) \to \pi_0(H_2) \to 0 \to 0,$$

  which implies $\sharp(\pi_1(\mathcal{B})) = 2\sharp(\pi_0(H_2))$.

All the claims in the table at the beginning of this section are now proved.

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Appendix A: Proof of Proposition 9

Fix some smooth curve A_t, $A_0 = A$, $\dot{A}_0 = \dot{A}$. Let $\ddot{A}_0 = \ddot{A}$ (one may assume that A_t is a geodesic in $\mathcal{R}_k$, but it is not important), and assume that A has all its nonzero singular values equal to $ck^{-1/2}$, which implies that

$$A^\dagger = \frac{k}{c^2} A^*. \tag{A.1}$$

Thus,

$$\begin{aligned} (A^\dagger)^* A^\dagger (A^\dagger)^* &= \frac{k^2}{c^4} A A^\dagger A = \frac{k^2}{c^4} A, \\ A A^* A &= \frac{c^2}{k} A A^\dagger A = \frac{c^2}{k} A, \\ A^* A A^* &= \frac{c^4}{k^2} A^\dagger A A^\dagger = \frac{c^4}{k^2} A^\dagger = \frac{c^2}{k} A^* \end{aligned} \tag{A.2}$$

and for any matrix P of the same size of A,

$$\langle A^\dagger, A^\dagger P A^\dagger \rangle_F = \langle (A^\dagger)^* A^\dagger (A^\dagger)^*, P \rangle_F \underset{(A.2)}{=} \frac{k^2}{c^4} \langle A, P \rangle_F.$$

In particular,

$$\mathbb{R}e\langle A^\dagger, A^\dagger \dot{A} A^\dagger \rangle_F = \frac{k^2}{c^4} \mathbb{R}e\langle A, \dot{A} \rangle_F = 0,$$

for every $\dot{A} \in T_A\mathcal{R}_k$, because $\mathcal{R}_k$ is contained in the sphere (for the Frobenius norm) of radius c. From Proposition 7, this implies that A is a critical point of ψ in $\mathcal{R}_k$.

Let B, C be the first and second derivatives at 0 of the Moore–Penrose inverse along A_t.

From (6.6),

$$D^2\psi(A)(\dot{A},\dot{A}) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \|A_t^\dagger\|_F = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \left\| A^\dagger + tB + \frac{t^2}{2}C + \mathrm{o}(t^2)\right\|_F \underset{\text{Lemma 3}}{=} \frac{\mathbb{R}e\langle A^\dagger, C\rangle_F + \|B\|_F^2}{\|A^\dagger\|_F}. \tag{A.3}$$

We compute each of these terms. From (6.3) (which holds for every A_t) we obtain by differentiating w.r.t. t at $t=0$,

$$ACA = \ddot{A} - \ddot{A}A^\dagger A - AA^\dagger\ddot{A} - 2\dot{A}BA - 2AB\dot{A} - 2\dot{A}A^\dagger\dot{A}. \tag{A.4}$$

From (6.2) and (6.1) and using the fact that

$$A^\dagger\big(\ddot{A} - \ddot{A}A^\dagger A - AA^\dagger\ddot{A}\big)A^\dagger = -A^\dagger\ddot{A}A^\dagger,$$

we conclude the following:

$$\begin{aligned}
\langle A^\dagger, C\rangle_F &\underset{(6.2)}{=} \langle A^\dagger, A^\dagger ACAA^\dagger\rangle_F \\
&\underset{(A.4)}{=} -\langle A^\dagger, A^\dagger\ddot{A}A^\dagger\rangle_F - \langle A^\dagger, A^\dagger\big(2\dot{A}BA + 2AB\dot{A} + 2\dot{A}A^\dagger\dot{A}\big)A^\dagger\rangle_F \\
&= -\langle (A^\dagger)^* A^\dagger (A^\dagger)^*, \ddot{A}\rangle_F \\
&\quad - \langle (A^\dagger)^* A^\dagger (A^\dagger)^*, 2\dot{A}BA + 2AB\dot{A} + 2\dot{A}A^\dagger\dot{A}\rangle_F \\
&\underset{(A.2)}{=} \frac{k^2}{c^4}\big(-\langle A, \ddot{A}\rangle_F - 2\langle A, \dot{A}BA + AB\dot{A} + \dot{A}A^\dagger\dot{A}\rangle_F\big).
\end{aligned}$$

Moreover, A_t is contained in the sphere of Frobenius norm c and hence $\mathbb{R}e\langle A, \ddot{A}\rangle_F = -\|\dot{A}\|_F^2$. We have thus proved:

$$\mathbb{R}e\langle A^\dagger, C\rangle_F = \frac{k^2}{c^4}\big(\|\dot{A}\|_F^2 - 2\mathbb{R}e\langle A, \dot{A}BA + AB\dot{A} + \dot{A}A^\dagger\dot{A}\rangle_F\big). \tag{A.5}$$

On the other hand, it easily follows from (1.4) that for all t we have $A_t^* = A_t^\dagger A_t A_t^*$ and $A_t^* = A_t^* A_t A_t^\dagger$, so

$$\begin{aligned}
\dot{A}^* &= BAA^* + A^\dagger\dot{A}A^* + A^\dagger A\dot{A}^*, \\
\dot{A}^* &= \dot{A}^*AA^\dagger + A^*\dot{A}A^\dagger + A^*AB.
\end{aligned}$$

That is,

$$\begin{aligned}
BAA^* &= \dot{A}^* - A^\dagger\dot{A}A^* - A^\dagger A\dot{A}^*, \\
A^*AB &= \dot{A}^* - \dot{A}^*AA^\dagger - A^*\dot{A}A^\dagger.
\end{aligned} \tag{A.6}$$

The first equality in (A.6) implies

$$\langle A, \dot{A}BA\rangle_F = \langle I, \dot{A}BAA^*\rangle_F = \|\dot{A}\|_F^2 - \langle A, \dot{A}A^\dagger\dot{A}\rangle_F - \langle\dot{A}, \dot{A}A^\dagger A\rangle_F. \tag{A.7}$$

Similarly, the second equality in (A.6) implies

$$\langle A, AB\dot{A}\rangle_F = \|\dot{A}\|_F^2 - \langle A, \dot{A}A^\dagger\dot{A}\rangle_F - \langle\dot{A}, AA^\dagger\dot{A}\rangle_F. \tag{A.8}$$

Thus,

$$\begin{aligned}
&\langle A, \dot{A}BA + AB\dot{A} + \dot{A}A^\dagger\dot{A}\rangle_F \\
&\quad = 2\|\dot{A}\|_F^2 - \langle A, \dot{A}A^\dagger\dot{A}\rangle_F - \langle\dot{A}, AA^\dagger\dot{A}\rangle_F - \langle\dot{A}, \dot{A}A^\dagger A\rangle_F \\
&\quad \underset{(A.1)}{=} 2\|\dot{A}\|_F^2 - \frac{k}{c^2}\langle A, \dot{A}A^*\dot{A}\rangle_F - \frac{k}{c^2}\langle\dot{A}, AA^*\dot{A}\rangle_F - \frac{k}{c^2}\langle\dot{A}, \dot{A}A^*A\rangle_F \\
&\quad = 2\|\dot{A}\|_F^2 - \frac{k}{c^2}\langle A, \dot{A}A^*\dot{A}\rangle_F - \frac{k}{c^2}\langle A^*\dot{A}, A^*\dot{A}\rangle_F - \frac{k}{c^2}\langle\dot{A}A^*, \dot{A}A^*\rangle_F \\
&\quad = 2\|\dot{A}\|_F^2 - \frac{k}{c^2}\langle A, \dot{A}A^*\dot{A}\rangle_F - \frac{k}{c^2}\|A^*\dot{A}\|_F^2 - \frac{k}{c^2}\|\dot{A}A^*\|_F^2.
\end{aligned} \tag{A.9}$$

We put together (A.5) and (A.9) to conclude:

$$\mathbb{R}e\langle A^\dagger, C\rangle_F = \frac{2k^3}{c^6}\left(\mathbb{R}e\langle A, \dot{A}A^*\dot{A}\rangle_F + \|\dot{A}A^*\|_F^2 + \|A^*\dot{A}\|_F^2\right) - \frac{3k^2}{c^4}\|\dot{A}\|_F^2. \tag{A.10}$$

The same formulas used to obtain equation (A.10) help to compute $\|B\|_F^2$, starting with one of the equalities in (6.1):

$$A_t^\dagger A_t A_t^\dagger = A_t^\dagger \quad \text{which implies} \quad BAA^\dagger + A^\dagger\dot{A}A^\dagger + A^\dagger AB = B.$$

Thus, we have

$$\begin{aligned}
\|B\|_F^2 &= \left\|BAA^\dagger + A^\dagger\dot{A}A^\dagger + A^\dagger AB\right\|_F^2 \\
&\underset{(A.1)}{=} \left\|\frac{k}{c^2}BAA^* + \frac{k^2}{c^4}A^*\dot{A}A^* + \frac{k}{c^2}A^*AB\right\|_F^2.
\end{aligned}$$

Note that from (A.6) we have:

$$\begin{aligned}
BAA^* + A^*AB &= 2\dot{A}^* - A^\dagger\dot{A}A^* - A^\dagger A\dot{A}^* - \dot{A}^*AA^\dagger - A^*\dot{A}A^\dagger \\
&\underset{(A.1)}{=} 2\dot{A}^* - 2\frac{k}{c^2}A^*\dot{A}A^* - \frac{k}{c^2}A^*A\dot{A}^* - \frac{k}{c^2}\dot{A}^*AA^*.
\end{aligned} \tag{A.11}$$

We have then proved that

$$\|B\|_F^2 = \frac{k^4}{c^8}\left\|\frac{2c^2}{k}\dot{A}^* - \dot{A}^*AA^* - A^*A\dot{A}^* - A^*\dot{A}A^*\right\|_F^2.$$

By expanding the terms in the squared norm we get:

$$\begin{aligned}\|B\|_F^2 &= \frac{4k^2}{c^4}\|\dot{A}\|_F^2 + \frac{k^4}{c^8}\|\dot{A}^*AA^*\|_F^2 + \frac{k^4}{c^8}\|A^*A\dot{A}^*\|_F^2 + \frac{k^4}{c^8}\|A^*\dot{A}A^*\|_F^2 \\ &\quad - \frac{4k^3}{c^6}\langle\dot{A}^*, \dot{A}^*AA^*\rangle_F - \frac{4k^3}{c^6}\langle\dot{A}^*, A^*A\dot{A}^*\rangle_F - \frac{4k^3}{c^6}\langle\dot{A}^*, A^*\dot{A}A^*\rangle_F \\ &\quad + 2\frac{k^4}{c^8}\langle\dot{A}^*AA^*, A^*A\dot{A}^*\rangle_F + 2\frac{k^4}{c^8}\langle\dot{A}^*AA^*, A^*\dot{A}A^*\rangle_F \\ &\quad + 2\frac{k^4}{c^8}\langle A^*A\dot{A}^*, A^*\dot{A}A^*\rangle_F. \end{aligned} \tag{A.12}$$

We can rewrite many of the terms in this last equality:

$$\begin{aligned} \|\dot{A}^*AA^*\|^2 &= \langle\dot{A}^*AA^*, \dot{A}^*AA^*\rangle_F = \langle\dot{A}^*A, \dot{A}^*AA^*A\rangle_F \underset{(A.2)}{=} \frac{c^2}{k}\|\dot{A}^*A\|_F^2, \\ \|A^*A\dot{A}^*\|^2 &= \langle A^*A\dot{A}^*, A^*A\dot{A}^*\rangle_F = \langle A\dot{A}^*, AA^*A\dot{A}^*\rangle_F \underset{(A.2)}{=} \frac{c^2}{k}\|A\dot{A}^*\|_F^2, \\ \langle\dot{A}^*, \dot{A}^*AA^*\rangle_F &= \langle\dot{A}^*A, \dot{A}^*A\rangle_F = \|\dot{A}^*A\|_F^2, \\ \langle\dot{A}^*, A^*A\dot{A}^*\rangle_F &= \langle A\dot{A}^*, A\dot{A}^*\rangle_F = \|A\dot{A}^*\|_F^2, \\ \langle\dot{A}^*AA^*, A^*A\dot{A}^*\rangle_F &= \langle A\dot{A}^*A, A\dot{A}^*A\rangle_F = \|A\dot{A}^*A\|_F^2 = \|A^*\dot{A}A^*\|_F^2, \\ \langle\dot{A}^*AA^*, A^*\dot{A}A^*\rangle_F &= \langle\dot{A}^*, A^*\dot{A}A^*AA^*\rangle_F \underset{(A.2)}{=} \frac{c^2}{k}\langle\dot{A}^*, A^*\dot{A}A^*\rangle_F, \\ \langle A^*A\dot{A}^*, A^*\dot{A}A^*\rangle_F &= \langle\dot{A}^*, A^*AA^*\dot{A}A^*\rangle_F \underset{(A.2)}{=} \frac{c^2}{k}\langle\dot{A}^*, A^*\dot{A}A^*\rangle_F. \end{aligned}$$

Substituting all these formulas in (A.12) we get

$$\|B\|_F^2 = \frac{4k^2}{c^4}\|\dot{A}\|_F^2 - \frac{3k^3}{c^6}\|A^*\dot{A}\|_F^2 - \frac{3k^3}{c^6}\|\dot{A}A^*\|_F^2 + \frac{3k^4}{c^8}\|A^*\dot{A}A^*\|_F^2. \tag{A.13}$$

We can then add (A.10) and (A.13) to get

$$\begin{aligned}\mathbb{R}e\langle A^\dagger, C\rangle_F + \|B\|_F^2 &= \frac{k^3}{c^6}\big(2\mathbb{R}e\langle A, \dot{A}A^*\dot{A}\rangle_F - \|A^*\dot{A}\|_F^2 - \|\dot{A}A^*\|_F^2\big) \\ &\quad + \frac{k^2}{c^4}\|\dot{A}\|_F^2 + \frac{3k^4}{c^8}\|A^*\dot{A}A^*\|_F^2. \end{aligned} \tag{A.14}$$

Finally, from (A.3), this last formula, and using $\|A^\dagger\| = k/c$, we get

$$D^2\psi(A)(\dot{A},\dot{A}) = \frac{k^2}{c^5}\big(2\mathbb{R}e\langle A, \dot{A}A^*\dot{A}\rangle_F - \|A^*\dot{A}\|_F^2 - \|\dot{A}A^*\|_F^2\big) + \frac{k}{c^3}\|\dot{A}\|_F^2 + \frac{3k^3}{c^7}\|A^*\dot{A}A^*\|_F^2.$$

The proposition follows immediately from this last claim.

Appendix B: Proof of Theorem 3

The proof of Theorem 3 uses dynamical systems theory, more specifically, the theory of invariant manifolds. The reader may find background in [18].

Let M be a finite dimensional manifold and $\mathcal{B} \subseteq M$ be a compact submanifold of M. Let TM be the tangent bundle to M and $T_{\mathcal{B}}M$ the restriction of TM to $\mathcal{B}$. Let $X : M \longrightarrow TM$ be a smooth vector field which vanishes on $\mathcal{B}$. The derivative of X at $b \in \mathcal{B}$ is then a well-defined map $T_bX : T_bM \longrightarrow T_bM$ that can be computed in any coordinate system. We write $T_{\mathcal{B}}X : T_{\mathcal{B}}M \longrightarrow T_{\mathcal{B}}M$ for the global version.

For all $b \in \mathcal{B}$ and for every eigenvector $v \in T_bW \setminus T_b\mathcal{B}$, we now assume that the real part of the corresponding eigenvalue is not zero. Let $N^s \subseteq T_{\mathcal{B}}M$ (resp. $N^u \subseteq T_{\mathcal{B}}M$) be the vector subbundle such that N_b^s is the generalized eigenspace of the eigenvalues of T_bX with negative (resp. positive) real part. Using local coordinates, it is easy to see that N^s, N^u, and $N = N^s \oplus N^u$ are smooth vector subbundles of $T_{\mathcal{B}}M$, invariant under $T_{\mathcal{B}}X$. We write

$$T^sX = T_{\mathcal{B}}X|_{N^s} : N^s \longrightarrow N^s,$$
$$T^uX = T_{\mathcal{B}}X|_{N^u} : N^u \longrightarrow N^u.$$

We define a flow $\psi_t : N \longrightarrow N$ by $\psi_t(b, v) = (b, \exp(tT_bX)(v))$, where $\exp(\cdot)$ is the usual exponential of a linear map. Then, ψ_t leaves invariant N^u and N^s and we can choose metrics so that $\psi_t^s = \psi_t|_{N^s}$ contracts vector lengths and $\psi_t^u = \psi_t|_{N^u}$ expands vector lengths (note that this choice of metric may disagree with any previously imposed Riemannian structure on M. In particular, if X is defined as a gradient flow, the new metric might be different from the one defining X).

Recall that ϕ_t is the solution flow of the vector field $V(x) = -\operatorname{grad}\widetilde{\mu}(x)$ on W. For simplicity, we assume that ϕ_t is defined for all t. The manifold $\mathcal{B}$ is normally hyperbolic (see [18, Sect. 1]) for the flow ϕ_t. Here we restrict the application of the theory of normally hyperbolic manifolds to the case that the manifold $\mathcal{B}$ consists of fixed points. Much of what we say applies with greater generality.

We consider the stable and unstable manifolds of $\mathcal{B}$, $W^s(\mathcal{B}) = \{x \in M : \phi_t(x) \mapsto \mathcal{B}, t \mapsto \infty\}$, and $W^u(\mathcal{B}) = \{x \in M : \phi_t(x) \mapsto \mathcal{B}, t \mapsto -\infty\}$. Recall that for $b \in \mathcal{B}$ we also have $W^s(b) = \{x \in M : \phi_t(x) \mapsto b, t \mapsto \infty\}$ and $W^u(b) = \{x \in M : \phi_t(x) \mapsto b, t \mapsto -\infty\}$. Finally, two flows ϕ_t^1 and ϕ_t^2 are (locally) topologically conjugate if there is a (local) homeomorphism h such that $h\phi_t^1 = \phi_t^2 h$ on the domain of definition of h, when defined.

We state the following result for stable manifolds; it is identical for unstable manifolds (changing s to u). Note that we are in the situation of [18, Sect. 1], with the extra assumption that the invariant manifold $\mathcal{B}$ consists of fixed points of the flow. This extra assumption will imply smoothness (instead of the usual continuity) of the foliation W^s.

Theorem 4 *In the conditions above*:

(1) $W^s(\mathcal{B})$ *is a* $\mathcal{C}^\infty$ *manifold tangent to* N^s *at* $\mathcal{B}$.
(2) $W^s(\mathcal{B})$ *is* $\mathcal{C}^\infty$ *foliated by the* $W^s(b)$ $(b \in \mathcal{B})$, *which are tangent to* N^s_b *at* $b \in \mathcal{B}$.
(3) N^s *is* $\mathcal{C}^\infty$ *diffeomorphic to* $W^s(\mathcal{B})$ *by a diffeomorphism* σ *which is the identity on* $\mathcal{B}$ *and which maps the fibration of* N^s *as a vector bundle over* $\mathcal{B}$ *to the foliation of* $W^s(\mathcal{B})$ *by the* $W^s(b)$ *leaves. The derivative* $D\sigma|_{N^s_b}$ *is the identity for all* $b \in \mathcal{B}$.
(4) ψ^s_t *is topologically conjugate to* $\phi_t|_{W^s(\mathcal{B})}$ *by a* $\mathcal{C}^0$ *conjugacy* h^s *which is the identity on* $\mathcal{B}$.
(5) ϕ_t *and* ψ_t *are topologically conjugate on a neighborhood of* $\mathcal{B}$.

Proof

(1) See [18, Theorem 4.1].
(2) By the overflowing property on $W^s(\mathcal{B})$ locally near $\mathcal{B}$ and the fact that $\phi_t|_{\mathcal{B}} = \mathrm{Id}_{\mathcal{B}}$, by the $\mathcal{C}^r$-section theorem [28, Theorem 5.18] applied to the graph transform whose fixed section is the tangent bundle to the manifolds $W^s(b)$, for every $r > 0$ there is a neighborhood U^s_r of $\mathcal{B}$ such that the $W^s(b)$ foliation of $W^s(\mathcal{B})$ is $\mathcal{C}^r$ near $\mathcal{B}$. Saturating by the flow shows that the global foliations are $\mathcal{C}^r$ for all $r > 0$ and hence $\mathcal{C}^\infty$.
(3) To see that σ exists locally near $\mathcal{B}$, consider the manifold $\{(x, b) : x \in W^s(b)\} \subseteq W^s(\mathcal{B}) \times \mathcal{B}$ which is the graph of a smooth function and hence smooth. Then consider the smooth mapping $\varphi : x \mapsto (x, b) \mapsto \pi_{N^s_b} \exp_b^{-1}(x)$ with exp the exponential map in M and $\pi_{N^s_b}$ the orthogonal projection. Clearly, the derivative of φ at b is nonsingular; hence, locally near every $b \in \mathcal{B}$, φ is a smooth diffeomorphism. By the compactness of $\mathcal{B}$, we conclude that φ is a smooth diffeomorphism locally near $\mathcal{B}$. Take $\sigma = \varphi^{-1}$.

    Now we globalize σ. Put a metric on N^s so that the orbits of ψ^s_t are transversal to $\mathbb{S}^s_b(r)$ for all $r < r_0$, where $\mathbb{S}^s_b(r)$ is the sphere of radius r in N^s_b, and the same is true for $\sigma^{-1}(\phi_t)$. Here, r_0 is small enough such that σ is defined. Now choose $0 < a_1 < a_2 < r_0$ and alter the vector field defining the ψ^s_t flow to a $\mathcal{C}^\infty$ vector field such that it remains transversal to the spheres for $r < r_0$, and the new flow $\tilde{\psi}^s_t$ and ϕ_t are σ-conjugate on the set $V^{a_2}_{a_1} = \{(b, v) \in N^s : a_1 < \|v\| < a_2\}$. Finally, define

$$\begin{aligned} \tilde{\sigma} : N^s &\longrightarrow W^s(\mathcal{B}), \\ (b, v) &\mapsto \sigma(b, v) \quad \text{if } \|v\| < a_2, \\ (b, v) &\mapsto \phi_{-t}\sigma\tilde{\psi}_t(b, v) \quad \text{if } \|v\| \geq a_2, \end{aligned}$$

    where t in this last formula is any positive number such that $\tilde{\psi}_t(v) \in V^{a_2}_{a_1}$. Note that as $\tilde{\psi}_t$ and ϕ_t are σ-conjugate on $\sigma(V^{a_2}_{a_1})$, $\tilde{\sigma}$ is well defined, smooth and indeed a diffeomorphism. Moreover, $\tilde{\sigma} = \sigma$ on a neighborhood of $\mathcal{B}$ and satisfies the claimed properties.

(4) The local version is proven in [22]. To get the global version from a local conjugacy h_{loc} define $h(x) = \psi_t h_{\text{loc}} \phi_t^{-1}(x)$ for any t such that $\phi_t^{-1}(x)$ is in the domain of definition of h_{loc}.
(5) Same as (4). □

Remark 3 By a result of Hartman [16] (see also [20]), for any fixed $b \in \mathcal{B}$, $\phi_t|_{W^s(b)}$ is $\mathcal{C}^1$ conjugate to $\psi_t^s|_{N_b^s}$. We don't know how such a $\mathcal{C}^1$ conjugacy may be made global so as to be $\mathcal{C}^1$ in b. In the context of equivariant gradient flow as we are considering, the proof in [20] might be adaptable.

B.1 Proof of Theorem 3

Note that $\widetilde{\mu}$ is proper and bounded from below. Thus $W^s(\mathcal{B}) = W$. The derivative at $\mathcal{B}$ satisfies $-D\,\text{grad}\,\widetilde{\mu} = -D^2\widetilde{\mu}$, which from Proposition 8 is symmetric negative definite on the normal bundle to $\mathcal{B}$ and its eigenvalues are pointwise real and negative. Hence we can apply Theorem 4, from which Theorem 3 follows.

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