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Adaptive Step Size Selection for Homotopy Methods to Solve Polynomial Equations†

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Given a C^1 path of systems of homogeneous polynomial equations f_t, $t \in [a,b]$ and an approximation x_a to a zero ζ_a of the initial system f_a, we show how to adaptively choose the step size for a Newton based homotopy method so that we approximate the lifted path (f_t, ζ_t) in the space of $(problem, solution)$ pairs. The total number of Newton iterations is bounded in terms of the length of the lifted path in the condition metric.

Keywords: Approximate zero, homotopy method, condition metric.

1. Introduction

Let us denote by $\mathscr{H}_{(d)}$ the vector space of homogeneous polynomials systems

$$f : \mathbb{C}^{n+1} \to \mathbb{C}^n,$$

$f = (f_1, \dots, f_n)$ in the variable $z = (z_0, z_1, \dots, z_n)$, with degree $(d) = (d_1, \dots, d_n)$, so that f_i has degree d_i.

Given a C^1 path of systems $t \in [a,b] \to f_t \in \mathscr{H}_{(d)}$, and a zero ζ_a of the initial system f_a, under very general conditions, the path $t \to f_t$ can be lifted to a C^1 path $t \to (f_t, \zeta_t)$ in the solution variety

$$\widehat{V} = \left\{ (f, \zeta) \in \mathscr{H}_{(d)} \times \mathbb{C}^{n+1} \setminus \{0\} \ : \ f(\zeta) = 0 \right\}.$$

If we make the additional hypothesis that $\frac{d\zeta_t}{dt}$ is (Hermitian) orthogonal to ζ_t this path is unique.

Now, given a sufficiently close approximation x_a to the zero ζ_a of the initial system f_a, predictor-corrector methods based on Newton's method may approximate the lifted path (f_t, ζ_t) by a finite number of pairs $(f_{t_i}, x_i) \in \mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$, $0 \leqslant i \leqslant k$. These algorithms

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are designed as follows: first the interval $[a,b]$ is discretized by a finite number of points $a = t_0 < t_1 < \ldots < t_k = b$, then a sequence (x_i) is constructed recursively by

$$x_0 = x_a \text{ and } x_{i+1} = N_{f_{t_{i+1}}}(x_i)$$

where N_{f_t} is the projective Newton operator associated with the system f_t. The complexity of such algorithms is measured by the size k of the subdivision (t_i). If we make a good choice for (t_i) then k is small and, for each i, x_i is an approximate zero of f_{t_i} associated with ζ_{t_i}.

The complexity of such algorithms has been related by Shub & Smale (1993) to the length l of the path (f_t) and to the condition number of the path (f_t, ζ_t): $\mu = \max_{a \leqslant t \leqslant b} \mu(f_t, \zeta_t)$. The condition number measures the size of the first order variations of the zero of a polynomial system in terms of the first order variations of the system. For $(f,\zeta) \in V$ it is given by

$$\mu(f,\zeta) = \|f\| \left\| \left(Df(\zeta)\Big|_{\zeta^\perp} \right)^{-1} \operatorname{diag}\left(\sqrt{d_i}\, \|\zeta\|^{d_i - 1} \right) \right\|$$

or ∞ when rank $Df(\zeta)\Big|_{\zeta^\perp} < n$. We extend this definition to any pair $(f,z) \in \mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$ by the same formula. Shub & Smale (1993) give the bound

$$k \leqslant C D^{3/2} l \mu^2,$$

where C is a constant, and $D = \max d_i$.

A more precise estimate is given in Shub (2009). The author proves that we can choose the step size in predictor-corrector methods so that the number of steps sufficient to approximate the lifted path is bounded in terms of the length of the lifted path in the condition metric:

$$L = \int_a^b \left(\left\| \frac{df_t}{dt} \right\|_{f_t}^2 + \left\| \frac{d\zeta_t}{dt} \right\|_{\zeta_t}^2 \right)^{1/2} \mu(f_t, \zeta_t) dt.$$

We can find such a subdivision (t_i), $0 \leqslant i \leqslant k$, with

$$k \leqslant C D^{3/2} L.$$

Its construction is given by

$$t_0 = a, \text{ and } \int_{t_i}^{t_{i+1}} \left(\left\| \frac{df_t}{dt} \right\|_{f_t} + \left\| \frac{d\zeta_t}{dt} \right\|_{\zeta_t} \right) dt = \frac{C}{D^{3/2} \mu(f_{t_i}, \zeta_{t_i})}$$

but, in this paper, some universal constants are not estimated, and no constructive algorithm is given.

In the algorithm we present below, we compute t_{i+1} from t_i so that at least one of the quantities

$$\int_{t_i}^{t_{i+1}} \left\| \frac{df_t}{dt} \right\|_{f_t} dt$$

and

$$\int_{t_i}^{t_{i+1}} \left\| \frac{d\zeta_t}{dt} \right\|_{\zeta_t} dt$$

increases of a given fraction of

$$\frac{1}{D^{3/2} \mu(f_{t_i}, \zeta_{t_i})}.$$

Moreover, to allow approximate computations, we introduce a tolerance parameter ε. The choice of this parameter depends on the considered problem and also on the round-off unit of some finite precision arithmetic. This is a difficult problem, and it is not considered here.

For an approach to a similar problem related to another algorithm, see Briquel, Cucker, Peña & Roschchina (2010).

The present algorithm reflects the geometrical structures used by Shub (2009). This structure is based on a Lipschitz-Riemannian metric defined in the solution variety V by

$$\langle .,. \rangle_{V,(f,\zeta)}\, \mu(f,\zeta)^2$$

where $\langle .,. \rangle_{V,(f,\zeta)}$ is the Riemannian metric in V inherited from the usual metric on $\mathbb{P}(\mathscr{H}_{(d)}) \times \mathbb{P}(\mathbb{C}^{n+1})$. This condition metric is studied in more details in Beltrán *et al.* (2010, TA), Beltrán & Shub (2009) and Beltrán & Shub (TA), and in Boito & Dedieu (2010).

ALGORITHM **Homotopy**

INPUT: $(f_t)_{t\in[a,b]}$, $0 \neq x_0 \in \mathbb{C}^{n+1}$, $0 < \varepsilon \leqslant 1/20$.

OUTPUT:

An integer $k \geqslant 1$,
A subdivision $a = t_0 < t_1 < \ldots < t_k = b$,
A sequence of nonzero points $x_i \in \mathbb{C}^{n+1}$, $0 \leqslant i \leqslant k$.

ALGORITHM:

$i \leftarrow 0$; $t_0 \leftarrow a$;

Repeat

Find $s \in [t_i, b]$ so that

$$\frac{20\varepsilon^2}{5D^{1/2}\mu(f_{t_i},x_i)} \leqslant \int_{t_i}^{s} \left\| \frac{df_t}{dt} \right\|_{f_t} dt \leqslant \frac{\varepsilon}{5D^{1/2}\mu(f_{t_i},x_i)}. \tag{1.1}$$

In case there is no such s, make $s = b$.

Find $s' \in [t_i, b]$ so that for all $\sigma \in [t_i, s']$,

$$\frac{20\varepsilon^2}{5D^{3/2}\mu(f_{t_i},x_i)} \leqslant \phi_{t_i,s'}(x_i) \tag{1.2}$$

$$\forall\sigma \in [t_i, s'],\ \phi_{t_i,\sigma}(x_i) \leqslant \frac{\varepsilon}{5D^{3/2}\mu(f_{t_i},x_i)} \tag{1.3}$$

where

$$\phi_{t_i,\sigma}(x_i) = \|x_i\|^{-1}\sqrt{\|X\|^2 + \|Y\|^2 - 2|\langle X, Y\rangle|},$$

with $X = Df_{t_i}(x_i)|_{x_i^\perp}^{-1} f_{t_i}(x_i)$ and $Y = Df_{t_i}(x_i)|_{x_i^\perp}^{-1} f_\sigma(x_i)$.

In case there is no such s', make $s' = b$.

$t_{i+1} = \min(s, s')$;

Find $0 \neq x_{i+1} \in \mathbb{C}^{n+1}$ such that

$$d_R\left(N_{f_{t_{i+1}}}(x_i), x_{i+1}\right) < \frac{4\varepsilon^2}{5D^2\mu(f_{t_i},x_i)^2}; \tag{1.4}$$

$i \leftarrow i+1$;

Until $t_i = b$.

Set $k \leftarrow i$

This algorithm has the following properties:

THEOREM 1.1 Assume that $n \geqslant 2$ and $D = \max d_i \geqslant 2$. Given $0 < \varepsilon \leqslant 1/20$, a C^1 homotopy path $(f_t)_{t\in[a,b]}$ in $\mathscr{H}_{(d)}$ and an initial point $0 \neq x_0 \in \mathbb{C}^{n+1}$ satisfying

$$\frac{D^{3/2}}{2}\mu(f_a,x_0)\beta_0(f_a,x_0) < \frac{\varepsilon^2}{2},$$

where $\beta_0(f,x) = \|x\|^{-1}\|Df(x)|_{x^\perp}^{-1}f(x)\|$. then:

1. x_0 is an approximate zero of f_a with associated nonsingular zero ζ_a,
2. Let $(f_t,\zeta_t)_{t\in[a,b]}$ be a continuous lifting of $(f_t)_{t\in[a,b]}$ in the solution variety initialized at (f_a,ζ_a). If the condition length L is finite, then:

2.a. The algorithm **Homotopy** with input $(\varepsilon,(f_t),x_0)$ stops after at most

$$k = 1 + 0.65D^{3/2}\varepsilon^{-2}L$$

iterations of the main loop,

2.b. For each $i = 1\dots k$, x_i is an approximate zero of f_{t_i} with associated zero ζ_{t_i}.

REMARK 1.1

- This algorithm is robust: it is designed to allow approximate computations.
- For $\varepsilon = 1/20$, the computations in (1.1), (1.2) and (1.3) have to be exact.
- The hypothesis *"the condition length L is finite"* holds for an open dense set of the C^1 paths (C^1 topology assumed) in the solution variety V and, consequently, Theorem 1.1 holds for "almost all" inputs $(f_t)_{t\in[a,b]},x_0$.
- We reach the same complexity as in Shub (2009).

Let us now mention other approaches to the construction of *"convenient subdivisions"*.

A practical answer consists in taking $t_{i+1} = t_i + \delta t$ for an arbitrary $\delta t > 0$. If x_{i+1} fails (resp. succeeds) to be an approximate zero of $f_{t_{i+1}}$ then we take $\delta t/2$ (resp. $2\delta t$) instead of δt; see Li (2003). With such an algorithm we may jump from a lifted path $t \to (f_t,\zeta_t) \in V$ to another one $t \to (f_t,\zeta_t') \in V$. Even if, for each i, x_i is an approximate sero of f_{t_i}, we cannot certify that the sequence (f_{t_i},x_i) approximates the path (f_t,ζ_t).

In Beltrán (2011), the author presents an algorithm to construct a certified approximation of the lifted path. It requires a C^{1+Lip} path in the space of systems, and it has an additional multiplicative factor in the number of steps given by Shub (2009). This extra factor is unbounded for the class of C^1 paths considered here.

Beltrán's algorithm is studied in more detail in Beltran & Leykin (TA), which contains implementations and experimental results.

Another important problem, which is not considered here, is the choice of both the homotopy path (f_t), $a \leqslant t \leqslant b$, and the initial zero ζ_a. Classical strategies are described in Li (2003), and Sommese & Wampler (2005).

A conjectured "good choice" (see Shub & Smale , 1994) is the system

$$f_a(z_0,z_1,\dots,z_n) = \begin{cases} d_1^{\frac{1}{2}} z_0^{d_1-1} z_1 = 0, \\ \cdots \\ d_n^{\frac{1}{2}} z_0^{d_n-1} z_n = 0, \end{cases} \qquad \zeta_a = (1,0,\dots,0),$$

and a linear homotopy connecting this initial system to the target system f_b. See Shub & Smale (1994) for a precise statement. This conjecture is still unproved. In Shub & Smale (1994) an adaptive algorithm is given for linear homotopies whose number of steps is bounded by the estimate in Shub & Smale (1993). The algorithm we present here is a version of that algorithm adapted to the new context of length in the condition metric.

Beltrán & Pardo (2011) use a linear homotopy and Beltrán's strategy for the choice of the subdivision. They get, for a random choice of (f_a, ζ_a), an average running time $\mathcal{O}(N^2)$ where N is the size of the input.

Bürgisser & Cucker (TA) define an explicit algorithm, called ALH, based on the linear homotopy and a certain adaptive construction for the subdivision. They obtain the complexity

$$217\, D^{3/2} d_{\mathbb{P}}(f_a, f_b) \int_a^b \mu(f_t, \zeta_t)^2 dt$$

which is not as sharp as the estimate based on the condition length given in Shub (2009) or to our own estimate. Then, we cite the authors, *"ALH will serve as the basic routine for a number of algorithms computing zeros of polynomial systems in different contexts. In these contexts both the input system f_b and the origin (f_a, ζ_a) of the homotopy may be randomly chosen".* See this manuscript for a more detailled description.

Our feeling, based on a series of papers on the condition metric: Beltrán *et al.* (2010, TA); Beltrán & Shub (2009, TA); Boito & Dedieu (2010), is that a good choice for the homotopy path (f_t) and the initial zero ζ_a will induce a lifted path (f_t, ζ_t) close to the condition geodesic connecting (f_a, ζ_a) to (f_b, ζ_b). We are far from accomplishing this task.

Our paper is organized as follow. Section 2 recalls the geometric context and contains the main definitions. In section 3 we study the variations of the condition number $\mu(f,x)$ when we vary both the system f and the vector x. The main difficulty is to estimate universal constants which are already present in many papers (Shub & Smale (1994), Shub (2009) for example) but which are not given explicitely. Such explicit constants are necessary to design an explicit algorithm. Section 4 contains, in the same spirit, explicit material about projective alpha-theory. Section 5 is devoted to the proof of Theorem 1.1. Then in Section 6 we explain how to implement our algorithm in the case of a linear homotopy, and in Section 7 we discuss possible applications.

2. Context and definitions

2.1 *Definitions*

We begin by recalling the context. For every positive integer $l \in \mathbb{N}$, let $\mathscr{H}_l \subseteq \mathbb{C}[x_0, \ldots, x_n]$, $n \geqslant 2$, be the vector space of homogeneous polynomials of degree l. For $(d) = (d_1, \ldots, d_n) \in \mathbb{N}^n$, let $\mathscr{H}_{(d)} = \prod_{i=1}^n \mathscr{H}_{d_i}$ be the set of all systems $f = (f_1, \ldots, f_n)$ of homogeneous polynomials of respective degrees $\deg(f_i) = d_i,\ 1 \leqslant i \leqslant n$. So $f : \mathbb{C}^{n+1} \to \mathbb{C}^n$. We denote by $D := \max\{d_i : 1 \leqslant i \leqslant n\}$ the maximum of the degrees, and we suppose $D \geqslant 2$.

The *solution variety* $\widehat{V}$ is the set of points $(f, \zeta) \in \mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$ with $f(\zeta) = 0$. Since the equations are homogeneous, for all $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, $\lambda_1 f(\lambda_2 \zeta) = 0$ if and only if $f(\zeta) = 0$. So $\widehat{V}$ defines a variety $V \subset \mathbb{P}(\mathscr{H}_{(d)}) \times \mathbb{P}(\mathbb{C}^{n+1})$ where $\mathbb{P}(\mathscr{H}_{(d)})$ and $\mathbb{P}(\mathbb{C}^{n+1})$ are the projective spaces corresponding to $\mathscr{H}_{(d)}$ and $\mathbb{C}^{n+1}$ respectively; $\widehat{V}$ and V are smooth.

Most quantities we consider are defined on $\big(\mathscr{H}_{(d)} \setminus \{0\}\big) \times \big(\mathbb{C}^{n+1} \setminus \{0\}\big)$ but are constant on equivalence classes

$$\{(\lambda_1 f, \lambda_2 x) \ :\ \lambda_1,\ \lambda_2 \in \mathbb{C} \setminus \{0\}\}$$

so are defined on $\mathbb{P}(\mathscr{H}_{(d)}) \times \mathbb{P}(\mathbb{C}^{n+1})$. This product of projective spaces is the natural geometric frame for this study, but our data structure is given by pairs $(f,x) \in \big(\mathscr{H}_{(d)} \setminus \{0\}\big) \times \big(\mathbb{C}^{n+1} \setminus \{0\}\big)$. We speak interchangeably of a pair (f, ζ) in $\widehat{V}$ and its projection (f, ζ) in V.

Given a C^1 path of systems $t \in [a,b] \to f_t \in \mathbb{P}(\mathscr{H}_{(d)})$, and a zero $\zeta_a \mathbb{P}(\mathbb{C}^{n+1})$ of the initial system f_a, under very general conditions, the path $t \to f_t$ can be lifted to a unique C^1 path $t \to (f_t, \zeta_t)$ in the solution variety V.

Two important ingredients used in this paper are *projective Newton's method* introduced by Shub (1993), and the concept of *approximate zero.*

For a pair $(f,x) \in \left(\mathscr{H}_{(d)} \setminus \{0\}\right) \times \left(\mathbb{C}^{n+1} \setminus \{0\}\right)$ the projective Newton's operator N_f is defined by

$$N_f(x) = x - Df(x)|_{x^\perp}^{-1} f(x).$$

Here we assume that the restriction of the derivative $Df(x)$ to the subspace orthogonal to x

$$x^\perp = \left\{u \in \mathbb{C}^{n+1} \ : \ \langle u, x\rangle = 0\right\}$$

is invertible. It is easy to see that the line throught x is sent by N_f onto the line throught $N_f(x)$ so that N_f is in fact defined on $\mathbb{P}(\mathbb{C}^{n+1})$.

DEFINITION 2.1 We say that x is an approximate zero of f with associated zero ζ ($f(\zeta) = 0$) provided that the point $x_p = N_f(x_{p-1})$, $x_0 = x$, is defined for all $p \geqslant 1$ and

$$d_T(\zeta, x_p) \leqslant \left(\frac{1}{2}\right)^{2^p-1} d_T(\zeta, x).$$

Here d_T denotes the *tangential "distance"* as in Definition 2.2.

A well-studied class of numerical algorithms for solving polynomial systems uses homotopy (or path-following) algorithms associated with a predictor-corrector scheme. We are given a C^1 path (f_t), $a \leqslant t \leqslant b$, in the space $\mathscr{H}_{(d)}$, and a root ζ_a of f_a. Under certain genericity conditions, the path (f_t) may be lifted uniquely to a C^1 path $(f_t, \zeta_t) \in \widehat{V}$, $t \in [a,b]$, starting at the given pair (f_a, ζ_a).

Given an approximate zero x_a of f_a associated with ζ_a, our aim is to build an approximation of this path by a sequence of pairs $(f_{t_i}, x_i) \in \left(\mathscr{H}_{(d)} \setminus \{0\}\right) \times \left(\mathbb{C}^{n+1} \setminus \{0\}\right)$, $1 \leqslant i \leqslant k$ where, $a = t_0 < t_1 < \ldots < t_k = b$ is a subdivision of the interval $[a,b]$, and where x_i is an approximate zero of f_{t_i} associated with ζ_{t_i}. To simplify our notations we let $f_{t_i} = f_i$.

The construction of the suddivision (t_i) is given in the "Algorithm Homotopy". The construction of the sequence (x_i) uses the predictor-corrector scheme based on projective Newton's method, studied for the first time in Shub & Smale (1993). This sequence is defined recursively by

$$x_{i+1} = N_{f_{i+1}}(x_i).$$

In fact, to allow computation errors, we choose x_{i+1} in a suitable neighborhood of $N_{f_{i+1}}(x_i)$. Theorem 1.1 proves that for each $i = 1 \cdots k$, x_i is an approximate zero of f_{t_i} associated with ζ_{t_i} and it gives an estimate for the integer k in terms of the maximum degree D, and the condition length of the path (f_t, ζ_t), $a \leqslant t \leqslant b$.

DEFINITION 2.2 As it is clear from the context, systems and vectors are supposed nonzero.

1. $\mathscr{H}_{(d)}$ is endowed with the unitarily invariant inner product (see Blum *et. al.* (1998) section 12.1)

$$\langle f, g\rangle = \sum_{i=1}^{n} \sum_{|\alpha|=d_i} \frac{\alpha_0! \ldots \alpha_n!}{d_i!} f_{i,\alpha} \overline{g_{i,\alpha}}$$

where $\alpha = (\alpha_0, \ldots, \alpha_n)$, $|\alpha| = \alpha_0 + \cdots + \alpha_n$,

$$f_i(x_0, \ldots, x_n) = \sum_{|\alpha|=d_i} f_{i,\alpha} x_0^{\alpha_0} \ldots x_n^{\alpha_n},$$

and

$$g_i(x_0, \ldots, x_n) = \sum_{|\alpha|=d_i} g_{i,\alpha} x_0^{\alpha_0} \ldots x_n^{\alpha_n}.$$

2. $d_R(x,y)$ is the Riemannian distance in $\mathbb{P}_n(\mathbb{C})$.

3. $d_P(x,y) = \sin d_R(x,y) = \min_{0 \neq \lambda \in \mathbb{C}} \frac{\|x - \lambda y\|}{\|x\|}$ is the projective distance.

4. $d_T(x,y) = \tan d_R(x,y)$ is the tangential "distance". It does not satisfy the triangle inequality.

5. One has $d_P(x,y) \leqslant d_R(x,y) \leqslant d_T(x,y)$.

6. When rank $Df(x) = n$, we denote by θ_x the angle between x and ker $Df(x)$. $\theta_x = 0$ when $f(x) = 0$.

7. $\theta_{L,M}$ denotes the angle between the complex lines L and M so that $\theta_x = \theta_{x,\ker Df(x)}$.

8. $\psi(u) = 1 - 4u + 2u^2$ decreases from 1 to 0 on the interval $[0,(2-\sqrt{2})/2]$.

9. The norm of a linear (resp. multi-linear) operator is always the operator norm.

10. The condition number

$$\mu(f,x) = \|f\| \left\| Df(x)|_{x^\perp}^{-1} \mathrm{diag}\left(\sqrt{d_i}\|x\|^{d_i-1}\right)\right\|,$$

is also denoted μ_{proj} in Shub & Smale (1993), and μ_{norm} in Blum *et. al.* (1998).

11. $D = \max d_i$. We suppose $D \geqslant 2$.

12. $u = D^{3/2}\mu(f,x)d_R(x,y)/2$.

13. $v = D^{1/2}\mu(f,x)\min_{|\lambda|=1}\left\|\frac{f}{\|f\|} - \lambda\frac{g}{\|g\|}\right\|$ where it is assumed that $f \neq 0$ and $g \neq 0$. The quantity expressed as a norm is $2\sin d_R(f,g)/2 \leqslant d_R(f,g)$.

14. $\beta_0(f,x) = \|x\|^{-1}\|Df(x)|_{x^\perp}^{-1} f(x)\|$.

15. $\gamma_0(f,x) = \|x\| \max_{k\geqslant 2}\|Df(x)|_{x^\perp}^{-1}\frac{D^k f(x)}{k!}\|^{1/(k-1)}$.

16. $\alpha_0(f,x) = \beta_0(f,x)\gamma_0(f,x)$.

17. $\alpha_0 = (13-3\sqrt{17})/4 = .15767\ldots$

18. $\delta(f,x) = \|x\|^{-1}\|Df(x)|_{x^\perp}^{-1}\mathrm{diag}(d_i)f(x)\|$.

19. Let $X = Df_t(x)|_{x^\perp}^{-1} f_t(x)$ and $Y = Df_t(x)|_{x^\perp}^{-1} f_s(x)$. Then define

$$\begin{aligned}\phi_{t,s}(x) &= \min_{|\lambda|=1}\|x\|^{-1}\|X-\lambda Y\| = \|x\|^{-1}\sqrt{\|X\|^2+\|Y\|^2-2\max_{|\lambda|=1}\mathrm{Re}(\langle X,Y\rangle)}\\ &= \|x\|^{-1}\sqrt{\|X\|^2+\|Y\|^2-2|\langle X,Y\rangle|}.\end{aligned}$$

20. Let a C^1 path $a \leqslant t \leqslant b \to f_t \in \mathscr{H}_{(d)} \setminus \{0\}$ be given. We denote by

$$\dot{f}_t = \frac{df_t}{dt}$$

the derivative of the path with respect to t, and, for any $g \in \mathscr{H}_{(d)}$,

$$\|g\|_{f_t} = \left\|\Pi_{f_t^\perp} g\right\| / \|f_t\|$$

the norm of the projection of g onto the subspace orthogonal to f_t divided by the norm of f_t. The length of (f_t) in $\mathbb{P}(\mathscr{H}_{(d)})$ is given by

$$l(b) = \int_a^b \left\|\dot{f}_t\right\|_{f_t} dt.$$

When $\|f_t\| = 1$ and $\langle f_t, \dot{f}_t\rangle = 0$ for each t we have

$$l(b) = \int_a^b \left\|\dot{f}_t\right\| dt.$$

21. The condition length of the path $(f_t, x_t) \in \left(\mathscr{H}_{(d)} \setminus \{0\}\right) \times \left(\mathbb{C}^{n+1} \setminus \{0\}\right)$, $a \leqslant t \leqslant b$, is

$$L(b) = \int_a^b \left(\|\dot{f}_t\|_{f_t}^2 + \|\dot{x}_t\|_{x_t}^2\right)^{1/2} \mu(f_t, x_t) dt,$$

where $\dot{x}_t$ is the derivative of the path x_t with respect to t, and where the norm $\|\dot{x}_t\|_{x_t}$ is defined as in the previous item with $\left\|\dot{f}_t\right\|_{f_t}$. When $\|f_t\| = \|x_t\| = 1$ and $\langle f_t, \dot{f}_t\rangle = \langle x_t, \dot{x}_t\rangle = 0$ for each t we have

$$L(b) = \int_a^b \left(\left\|\dot{f}_t\right\|^2 + \|\dot{x}_t\|^2\right)^{1/2} \mu(f_t, x_t) dt.$$

We will also use some invariants related with non homogeneous polynomial systems. For $(d) = (d_1, \ldots, d_n) \in \mathbb{N}^n$, let $\mathscr{P}_{(d)} = \prod_{i=1}^n \mathscr{P}_{d_i}$ be the set of polynomial systems $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ in the variables $X = (X_1, \ldots, X_n)$, of respective degrees $\deg(F_i) \leqslant d_i$, $1 \leqslant i \leqslant n$. The homogeneous counterpart of F is the system $f = (f_1, \ldots, f_n) \in \mathscr{H}_{(d)}$ defined by

$$f_i(x_0, x_1, \ldots, x_n) = x_0^{d_i} F_i\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

The norm on $\mathscr{P}_{(d)}$ is defined by $\|F\| = \|f\|$. We also let:

22. $\beta(F, X) = \|DF(X)^{-1}F(X)\|$.
23. $\gamma(F, X) = \max_{k \geqslant 2} \|DF(X)^{-1} \frac{D^k F(X)}{k!}\|^{1/(k-1)}$.
24. $\alpha(F, X) = \beta(F, X)\gamma(F, X)$.

3. Variation of the condition number

A necessary ingredient for the proof of Theorem 1.1 is the following theorem which gives the variations of the condition number when both the system and the point vary:

THEOREM 3.1 Let two nonzero systems $f, g \in \mathscr{H}_{(d)}$, and two nonzero vectors $x, y \in \mathbb{C}^{n+1}$ be given such that rank $Df(x)|_{x^\perp} = n$, $u \leqslant 1/20$, and $v \leqslant 1/20$. Then rank $Dg(y)|_{y^\perp} = n$, and

$$(1 - 3.805u - v)\mu(g, y) \leqslant \mu(f, x) \leqslant (1 + 3.504u + v)\mu(g, y).$$

COROLLARY 3.1 Let $0 < \varepsilon \leqslant 1/4$, two nonzero systems $f, g \in \mathscr{H}_{(d)}$, and two nonzero vectors $x, y \in \mathbb{C}^{n+1}$ be given such that $u \leqslant \varepsilon/5$, $v \leqslant \varepsilon/5$, and rank $Df(x)|_{x^\perp} = n$. One has rank $Dg(y)|_{y^\perp} = n$, and

$$(1 - \varepsilon)\mu(g, y) \leqslant \mu(f, x) \leqslant (1 + \varepsilon)\mu(g, y).$$

The proof of these results is obtained from the following series of lemmas.

LEMMA 3.1 Let $f \in \mathscr{H}_{(d)}$ and $x \in \mathbb{C}^{n+1}$. We have

1. For any $i = 1 \ldots n$, $\|f_i(x)\| \leqslant \|f_i\| \, \|x\|^{d_i}$, so that $\|f(x)\| \leqslant 1$ when f and x are normalized.
2. For any $i = 1 \ldots n$, $\|Df_i(x)\| \leqslant d_i \|f_i\| \, \|x\|^{d_i - 1}$, so that $\|Df(x)\| \leqslant D$ when f and x are normalized.
3. $\gamma_0(f, x) \leqslant \frac{D^{3/2}}{2} \mu(f, x)$.

Proof. These inequalities come from Proposition 1, and Theorem 2, p. 267, in Blum *et. al.* (1998). □

LEMMA 3.2 $1 \leqslant \sqrt{n} = \min_{f,x} \mu(f,x)$.

Proof. Let given a matrix $A \in \mathbb{C}^{n\times m}$, $m \geqslant n$, and $A = U\Sigma V^*$ a singular value decomposition with

$$\Sigma = \mathrm{diag}(\sigma_1(A) \geqslant \ldots \geqslant \sigma_n(A)) \in \mathbb{C}^{n\times m},$$

$\sigma_i(A) \geqslant 0$, $U \in \mathbb{U}_n$, $V \in \mathbb{U}_m$ unitary matrices. We define

$$\kappa(A) = \sigma_n(A)^{-1}\|A\|_F = \sigma_n(A)^{-1}\sqrt{\sigma_1(A)^2 + \ldots + \sigma_n(A)^2}$$

when $\sigma_n(A) > 0$, and ∞ otherwise. We see easily that

$$\min_A \kappa(A) = \sqrt{n},$$

and this minimum is obtained when $\sigma_i(A) = 1$, $1 \leqslant i \leqslant n$.

For the case of polynomial systems we have

$$\mu(f,x) = \|f\|\|\left(\mathrm{diag}(d_i^{-1/2})Df(x)|_{x^\perp}\right)^{-1}\| = \|f\|\sigma_n\left(\mathrm{diag}(d_i^{-1/2})Df(x)|_{x^\perp}\right)^{-1}.$$

Using the unitary invariance of the norm in $\mathscr{H}_{(d)}$ (Blum *et. al.*, 1998, Sec. 12.1) , that is $\|f\circ U\| = \|f\|$ for any $U \in \mathbb{U}_{n+1}$, considering a U such that $Ue_0 = x$ with $e_0 = (1,0,\ldots,0)^T \in \mathbb{C}^{n+1}$, we see that

$$\|\mathrm{diag}(d_i^{-1/2})Df(x)|_{x^\perp}\|_F \leqslant \|f\|.$$

Thus, our previous estimate shows that $\mu(f,x) \geqslant \sqrt{n}$.

To prove the equality $\sqrt{n} = \min_{f,x}\mu(f,x)$ we use the unitary invariance of the condition number

$$\mu(f,x) = \mu(f\circ U, U^*x)$$

for any $U \in \mathbb{U}_{n+1}$, and the equality $\sqrt{n} = \mu(f,e_0)$ when $f_i(z) = \sqrt{d_i}z_0^{d_i-1}z_i$, $1 \leqslant i \leqslant n$, and $e_0 = (1,0,\ldots,0)^T \in \mathbb{C}^{n+1}$. □

LEMMA 3.3 For any f and x one has $d_R(x,y) \leqslant u/\sqrt{2n} \leqslant u/2$, and $d_R(x,y)\delta(f,x) \leqslant u$.

Proof. We suppose that both system f and vectors x,y are normalized. We have by Lemma 3.2

$$u = \frac{D^{3/2}}{2}\mu(f,x)d_R(x,y) \geqslant \frac{2^{3/2}}{2}\sqrt{n}d_R(x,y).$$

For the second inequality we have

$$\begin{aligned} d_R(x,y)\delta(f,x) &\leqslant d_R(x,y)\|Df(x)|_{x^\perp}^{-1}\mathrm{diag}(d_i^{1/2})\|\|\mathrm{diag}(d_i^{1/2})\|\|f(x)\| \\ &\leqslant d_R(x,y)\mu(f,x)D^{1/2}\|f(x)\| \\ &\leqslant d_R(x,y)\mu(f,x)D^{3/2}/2 \\ &= u \end{aligned}$$

thanks to the inequalities $2 \leqslant D$, $\|f_i(x)\| \leqslant \|f_i\|\|x\|^{d_i}$ (Lemma 3.1), and the hypothesis $\|f\| = \|x\| = 1$. □

LEMMA 3.4 When rank $Df(x)|_{x^\perp} = n$ we have

$$\|Df(x)|_{x^\perp}^{-1}Df(x)|_{y^\perp}\| \leqslant 1 + d_P(x,y)\tan\theta_x.$$

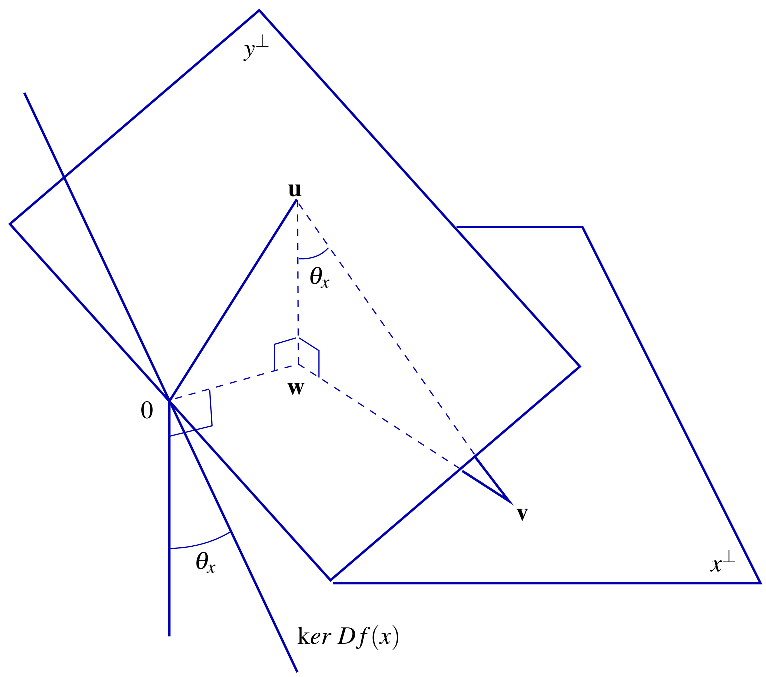

FIG. 1.

Proof. Take $\mathbf{u} \in y^\perp$ and define $\mathbf{v} = Df(x)|_{x^\perp}^{-1} Df(x)|_{y^\perp}\mathbf{u}$ so that

$$\mathbf{v} = (\mathbf{u} + \ker Df(x)) \cap x^\perp.$$

$\mathbf{v} \in x^\perp$ is the projection of $\mathbf{u} \in y^\perp$ along $\ker Df(x)$. Let us denote by $\mathbf{w}$ the orthogonal projection of $\mathbf{u}$ onto $x^\perp$. See Figure 1. We have $\|\mathbf{w}\| = \|\mathbf{u}\| \cos\theta_{\mathbf{u},\mathbf{w}}$ and

$$\|\mathbf{w} - \mathbf{v}\| = \|\mathbf{w} - \mathbf{u}\| \tan\theta_x = \|\mathbf{u}\| \sin\theta_{\mathbf{u},\mathbf{w}} \tan\theta_x,$$

so that

$$\|\mathbf{v}\| \leqslant \|\mathbf{u}\|(\cos\theta_{\mathbf{u},\mathbf{w}} + \sin\theta_{\mathbf{u},\mathbf{w}}\tan\theta_x) \leqslant \|\mathbf{u}\|(1 + \sin\theta_{x,y}\tan\theta_x)$$

because $\theta_{x^\perp,y^\perp} = \theta_{x,y} = \max_{\substack{\|\mathbf{u}\|=1 \\ \mathbf{u}\in y^\perp}} \theta_{\mathbf{u},\mathbf{w}}$. (See Definition 2.2(7)) □

LEMMA 3.5 Assume that $\|x\| = \|y\| = 1$. When $\operatorname{rank} Df(x)|_{x^\perp} = n$ and $u < 1$ one has

$$\|Df(x)|_{x^\perp}^{-1} Df(y)|_{y^\perp}\| \leqslant \frac{1}{(1-u)^2} + d_P(x,y)\tan\theta_x.$$

Proof. Assume without loss of generality that y is scaled such that $|\langle x-y, x\rangle|$ is minimal under the constraint that $\|y\| = 1$. In particular, this implies that $\|x - y\| \leqslant d_R(x,y)$.

Since $Df(y) = Df(x) + \sum_{k\geqslant 1} \frac{D^{k+1}f(x)}{k!}(y-x)^k$ we get

$$Df(x)|_{x^\perp}^{-1} Df(y)|_{y^\perp} = Df(x)|_{x^\perp}^{-1} Df(x)|_{y^\perp} + \sum_{k\geqslant 1}(k+1)Df(x)|_{x^\perp}^{-1}\frac{D^{k+1}f(x)}{(k+1)!}(y-x)^k|_{y^\perp}.$$

Then, we use $\|Df(x)|_{x^\perp}^{-1}\frac{D^{k+1}f(x)}{(k+1)!}\| \leqslant \gamma_0(f,x)^k$, Lemma 3.4 and Lemma 3.1 to obtain

$$\begin{aligned}\|Df(x)|_{x^\perp}^{-1} Df(y)|_{y^\perp}\| &\leqslant 1 + d_P(x,y)\tan\theta_x + \sum_{k\geqslant 1}(k+1)\gamma_0(f,x)^k\|x-y\|^k \\ &\leqslant \frac{1}{(1-u)^2} + d_P(x,y)\tan\theta_x\end{aligned}$$

□

LEMMA 3.6 When rank $Df(x)|_{x^\perp} = n$ then

$$\tan\theta_x = \delta(f,x) \leqslant D^{1/2}\mu(f,x).$$

Proof. We assume that x and f are normalized. Let $y = Df(x)|_{x^\perp}^{-1} Df(x)x$ be the projection of x onto $x^\perp$ along $\ker Df(x)$ so that $\|y\| = \tan\theta_x$. By Euler's Theorem for homogeneous functions, one also has

$$y = Df(x)|_{x^\perp}^{-1}\operatorname{diag}(d_i) f(x)$$

thus,

$$\begin{aligned}\tan\theta_x &= \delta(f,x)\\ &\leqslant \|Df(x)|_{x^\perp}^{-1}\operatorname{diag}(d_i^{1/2})\|\|\operatorname{diag}(d_i^{1/2})f(x)\|\end{aligned}$$

Using Lemma 3.1 we obtain

$$\tan\theta_x \leqslant D^{1/2}\mu(f,x).$$

□

LEMMA 3.7 When rank $Df(x)|_{x^\perp} = n$ one has

$$\|Df(x)|_{x^\perp}^{-1} Df(x)|_{y^\perp}\| \leqslant 1 + \delta(f,x) d_P(x,y).$$

Moreover, when $\|x\| = \|y\|$ and $u < 1$, we have

$$\|Df(x)|_{x^\perp}^{-1} Df(y)|_{y^\perp}\| \leqslant \frac{1}{(1-u)^2} + \delta(f,x) d_P(x,y).$$

Proof. The first assertion comes from lemmas 3.4 and 3.6. The second assertion is a consequence of lemmas 3.5 and 3.6. □

LEMMA 3.8 Assume that $\|x\| = \|y\| = 1$ and $\|f\| = 1$. When rank $Df(x)|_{x^\perp} = n$, and $u < (2-\sqrt{2})/2$, then rank $Df(y)|_{x^\perp} = n$ and

$$\|Df(y)|_{x^\perp}^{-1} Df(x)|_{x^\perp}\| \leqslant \frac{(1-u)^2}{\psi(u)}.$$

Moreover, if $u \leqslant 1/19$,

$$\|Df(y)|_{y^\perp}^{-1} Df(x)|_{x^\perp}\| \leqslant (1 + d_P(x,y)\delta(f,y))\frac{(1-u)^2}{\psi(u)} \leqslant 1 + 3.805u.$$

Proof. Assume without loss of generality that y is scaled such that $|\langle x-y, x\rangle|$ is minimal under the constraint that $\|y\| = 1$.

We have

$$Df(y) = Df(x) + \sum_{k\geqslant 1} \frac{D^{k+1}f(x)}{k!}(y-x)^k$$

so that

$$Df(x)|_{x^\perp}^{-1}(Df(y) - Df(x)) = \sum_{k\geqslant 1} Df(x)|_{x^\perp}^{-1}\frac{D^{k+1}f(x)}{k!}(y-x)^k,$$

and, like in the proof of Lemma 3.5,

$$\|Df(x)|_{x^\perp}^{-1}(Df(y) - Df(x))\| \leqslant \sum_{k\geqslant 1}(k+1)u^k = \frac{2u-u^2}{(1-u)^2} < 1$$

(because $u < (2-\sqrt{2})/2$). Now we apply the perturbation formula for operators X with $\|X\| < 1$:

$$\|(I-X)^{-1}\| \leqslant \frac{1}{1-\|X\|}.$$

Setting $X = -Df(x)|_{x^\perp}^{-1}(Df(y)-Df(x))|_{x^\perp}$, we obtain that

$$I_{x^\perp} - X = Df(x)|_{x^\perp}^{-1} Df(y)|_{x^\perp}$$

is invertible (i.e. $Df(y)|_{x^\perp}$ is invertible), and

$$\|Df(y)|_{x^\perp}^{-1} Df(x)|_{x^\perp}\| \leqslant \frac{1}{1-\frac{2u-u^2}{(1-u)^2}} = \frac{(1-u)^2}{\psi(u)}.$$

We will prove the second statement in two steps. First, prove it under the assumption that $Df(y)|_{y^\perp}$ is invertible. Then, we remove this assumption.

The first step goes as follows. Combining the first statement with Lemma 3.4 we obtain:

$$\begin{aligned}
\|Df(y)|_{y^\perp}^{-1} Df(x)|_{x^\perp}\| &\leqslant \|Df(y)|_{y^\perp}^{-1} Df(y)|_{x^\perp}\| \|Df(y)|_{x^\perp}^{-1} Df(x)|_{x^\perp}\| \\
&\leqslant (1+d_P(x,y)\tan\theta_y)\frac{(1-u)^2}{\psi(u)} \\
&= (1+d_P(x,y)\delta(f,y))\frac{(1-u)^2}{\psi(u)}
\end{aligned}$$

A bound for $\delta(f,y)$ is

$$\begin{aligned}
\delta(f,y) &= \left\| Df(y)_{y^\perp}^{-1} \operatorname{diag}(d_i) f(y) \right\| \\
&\leqslant \|Df(y)|_{y^\perp}^{-1} Df(x)|_{x^\perp}\| \left\| Df(x)_{x^\perp}^{-1} \operatorname{diag}(d_i) f(y) \right\| \\
&\leqslant \|Df(y)|_{y^\perp}^{-1} Df(x)|_{x^\perp}\| \mu(f,x)\sqrt{D}
\end{aligned}$$

using $|f(y)| \leqslant 1$ since $\|f\| = 1$ and $\|y\| = 1$.

Combining both inequations and setting $M = \|Df(y)|_{y^\perp}^{-1} Df(x)|_{x^\perp}\|$, we obtain:

$$M \leqslant (1+Mu)\frac{(1-u)^2}{\psi(u)}$$

that simplifies to

$$M \leqslant \frac{(1-u)^2}{\psi(u)-u(1-u)^2} = \frac{(1-u)^2}{1-5u+4u^2-u^3} \leqslant 1+3.805u.$$

The last bound follows from the fact that the numerator and the denominator have alternating signs, so the Taylor expansion at zero of the fraction has terms of the same sign (positive). Hence,

$$\frac{\frac{(1-u)^2}{1-5u+4u^2-u^3} - 1}{u}$$

is an increasing function. In particular, for $u = 1/19$, this is smaller than 3.805.

Now, we must prove that $Df(y)|_{y^\perp}$ is invertible. Let $(x_t)_{t\in[0,d_R(x,y)]}$ denote a minimizing geodesic (arc of great circle) between x and y.

Let W be the subset of all $t \in [0, d_R(x,y)]$ so that $Df(x_t)|_{x_t}$ is invertible. It is an open set, and $0 \in W$.

We claim that W is a closed set. Indeed, let $s \in \overline{W}$. Then there is a sequence of $t_i \in W$ with $t_i \to s$. We know from the second statement (restricted) that

$$\|Df(x_{t_i})|_{x_{t_i}^\perp}^{-1} Df(x)_{x^\perp}\| \leqslant 1 + 3.805u.$$

Hence, for $\tau = t_i$,

$$h(\tau) = \|Df(x_\tau)|_{x_\tau^\perp}^{-1} Df(x)_{x^\perp}\|_F^2 \leqslant n(1+3.805u)^2$$

The function $h(\tau)$ is a rational function of a real parameter τ, so its domain is an open set and contains s. By continuity, $h(s) \leqslant n(1+3.805u)^2$. Thus, $Df(x_s)_{x_s^\perp}$ is invertible, and $s \in W$. As W is a non-empty open and closed subset of an interval, $W = [0, d_R(x,y)]$ and $Df(y)|_{y^\perp}$ must be invertible. □

LEMMA 3.9 Suppose that rank $Df(x)|_{x^\perp} = n$, $u < 1$ and $\mu(f,y)$ is finite. Then

$$\mu(f,x) \leqslant \mu(f,y)\left(\frac{1}{(1-u)^2} + \delta(f,x) d_P(x,y)\right).$$

Proof. Suppose that $\|x\| = \|y\| = \|f\| = 1$. We can bound

$$\mu(f,x) = \|Df(x)|_{x^\perp}^{-1} Df(y)|_{y^\perp} Df(y)|_{y^\perp}^{-1} \mathrm{diag}(\sqrt{d_i})\|$$

and we conclude with Lemma 3.7. □

LEMMA 3.10 When rank $Df(x)|_{x^\perp} = n$, and $u < 1/19$ we have

$$\mu(f,y) \leqslant (1+3.805u)\mu(f,x).$$

Proof. Suppose that $\|x\| = \|y\| = \|f\| = 1$. We have

$$\mu(f,y) = \|Df(y)|_{y^\perp}^{-1} Df(x)|_{x^\perp} Df(x)|_{x^\perp}^{-1} \mathrm{diag}(\sqrt{d_i})\| \leqslant (1+3.805u)\mu(f,x)$$

by Lemma 3.8. □

LEMMA 3.11 Assume that $\|f\| = \|g\| = 1$. Suppose that rank $Df(x)|_{x^\perp} = n$, and $v < 1$. Then rank $Dg(x)|_{x^\perp} = n$, and

$$\|Df(x)|_{x^\perp}^{-1} Dg(x)|_{x^\perp}\| \leqslant 1 + v,$$

$$\|Dg(x)|_{x^\perp}^{-1} Df(x)|_{x^\perp}\| \leqslant \frac{1}{1-v},$$

$$(1-v)\mu(g,x) \leqslant \mu(f,x) \leqslant (1+v)\mu(g,x).$$

Proof. Suppose that $\|x\| = 1$. Also, assume without loss of generality that g is scaled so that $v = D^{1/2}\mu(f,x)\|f-g\|$.

One has $Df(x)|_{x^\perp}^{-1} Dg(x)|_{x^\perp} = I_{x^\perp} - (I_{x^\perp} - \text{ idem})$ and

$$I_{x^\perp} - \text{ idem} = Df(x)|_{x^\perp}^{-1} \mathrm{diag}(d_i^{1/2}) \mathrm{diag}(d_i^{-1/2}) D(f-g)(x)|_{x^\perp}$$

which norm is bounded by (using Lemma 3.1)

$$\mu(f,x) D^{1/2} \|f-g\| \leqslant v < 1.$$

This proves the first inequality. Thus $Df(x)|_{x^\perp}^{-1}Dg(x)|_{x^\perp}$ is invertible and the norm of its inverse is bounded by $1/(1-v)$ (Neumann's Perturbation Theorem). This gives

$$\mu(g,x) \leqslant \|Dg(x)|_{x^\perp}^{-1}Df(x)|_{x^\perp}\|\|Df(x)|_{x^\perp}^{-1}\mathrm{diag}((d_i^{1/2})\| \leqslant \frac{\mu(f,x)}{1-v}.$$

The last inequality is obtained via

$$\mu(f,x) \leqslant \|Df(x)|_{x^\perp}^{-1}Dg(x)|_{x^\perp}\|\|Dg(x)|_{x^\perp}^{-1}\mathrm{diag}((d_i^{1/2})\| \leqslant (1+v)\mu(g,x).$$

□

LEMMA 3.12 Let $\|f\| = \|g\| = 1$. Let $u_g = D^{3/2}\mu(g,x)d_R(x,y)/2$. Suppose that rank $Df(x)|_{x^\perp} = n$, and $v < 1$. Then

$$(1-v)u_g \leqslant u \leqslant (1+v)u_g,$$

and

$$(1-v)\delta(g,x) - v \leqslant \delta(f,x) \leqslant (1+v)\delta(g,x) + v.$$

Proof. As before, suppose that $\|x\| = 1$. and assume without loss of generality that g is scaled so that $v = D^{1/2}\mu(f,x)\|f-g\|$.

The first double inequality is a consequence of Lemma 3.11. For the second one, one has:

$$\delta(g,x) = \|Dg(x)|_{x^\perp}^{-1}Df(x)|_{x^\perp}Df(x)|_{x^\perp}^{-1}\mathrm{diag}(d_i)f(x) + Dg(x)|_{x^\perp}^{-1}\mathrm{diag}(d_i)(g(x)-f(x))\|.$$

By Lemma 3.11$\|Dg(x)|_{x^\perp}^{-1}Df(x)|_{x^\perp}\| \leqslant 1/(1-v)$ so that

$$\delta(g,x) \leqslant \frac{\delta(f,x)}{1-v} + \mu(g,x)D^{1/2}\|f-g\|.$$

Again by Lemma 3.11 $\mu(g,x) \leqslant \mu(f,x)/(1-v)$ so that

$$\delta(g,x) \leqslant \frac{\delta(f,x)}{1-v} + \frac{\mu(f,x)D^{1/2}\|f-g\|}{1-v} \leqslant \frac{\delta(f,x)+v}{1-v}.$$

Similarly $\delta(f,x) =$

$$\|Df(x)|_{x^\perp}^{-1}Dg(x)|_{x^\perp}Dg(x)|_{x^\perp}^{-1}\mathrm{diag}(d_i)g(x) + Df(x)|_{x^\perp}^{-1}\mathrm{diag}(d_i)(f(x)-g(x))\| \leqslant$$

$$\|Df(x)|_{x^\perp}^{-1}Dg(x)|_{x^\perp}\|\delta(g,x) + v.$$

Lemma 3.11 shows that $\|Df(x)|_{x^\perp}^{-1}Dg(x)|_{x^\perp}\| \leqslant 1+v$ so that

$$\delta(f,x) \leqslant (1+v)\delta(g,x) + v.$$

□

LEMMA 3.13 When rank $Df(x)|_{x^\perp} = n$, $u, v \leqslant 1/20$, then

$$\mu(g,y)\frac{(1-v)}{1+3.805\frac{u}{1-v}} \leqslant \mu(f,x) \leqslant (1+v)\left(\frac{1}{(1-\frac{u}{1-v})^2} + \frac{u}{1-v}\right)\mu(g,y).$$

Proof. As before, let $u_g = D^{3/2}\mu(g,x)d_R(x,y)/2$. Lemma 3.12 allows us to bound $u_g \leqslant u/(1-v) \leqslant 1/19$. So we may apply Lemma 3.10 to g instead of f so that

$$\mu(g,y) \leqslant (1+3.805u_g)\mu(g,x)$$

Then, we bound $\mu(g,x)$ by $\mu(f,x)/(1-v)$ (Lemma 3.11) and obtain the first inequality. In particular, $\mu(g,y)$ is finite.

To prove the second one we apply Lemma 3.11, and Lemma 3.9 to obtain

$$\mu(f,x) \leqslant (1+v)\mu(g,x) \leqslant (1+v)\left(\frac{1}{(1-u_g)^2} + \delta(g,x)d_P(x,y)\right)\mu(g,y).$$

By Lemma 3.3 and Lemma 3.12 we have

$$\delta(g,x)d_P(x,y) \leqslant u_g \leqslant \frac{u}{1-v}$$

and we are done. □

3.1 *Proof of Theorem 3.1*

To prove the inequalities $(1-3.805-v)\mu(g,y) \leqslant \mu(f,x) \leqslant (1+3.504u+v)\mu(g,y)$ we use Lemma 3.13 which gives

$$B(u,v)\mu(g,y) \leqslant \mu(f,x) \leqslant A(u,v)\mu(g,y)$$

with

$$A(u,v) = (1+v)\left(\frac{1}{(1-\frac{u}{1-v})^2} + \frac{u}{1-v}\right),\ B(u,v) = \frac{(1-v)}{1+3.805\frac{u}{1-v}}.$$

$$A(u,v) = 1+3u+v+u\frac{6v^3+9uv^2-12v^2+4u^2v-12uv+6v-2u^2+3u}{(1-v)(1-u-v)^2}$$

and the last parenthesis is less than 0.504 when $u,v < 1/20$.

The function $B(u,v)-(1-3.805u-v)$ is increasing in u and v, and vanishes at the origin. □

3.2 *Proof of Corollary 3.1*

Since $u,\ v \leqslant \varepsilon/5$ and $\varepsilon \leqslant 1/4$ we get $u,v \leqslant 1/20$. Thus we can apply Theorem 3.1 which gives

$$(1-\varepsilon)\mu(g,y) \leqslant (1-3.805u-v)\mu(g,y) \leqslant \mu(f,x) \leqslant (1+3.504u+v)\mu(g,y) \leqslant (1+\varepsilon)\mu(g,y).$$

□

4. Alpha theory in projective spaces

THEOREM 4.1 Let $0 < \alpha \leqslant \alpha_0 = (13-3\sqrt{17})/4 = 0.15767\ldots$ Let $f \in \mathscr{H}_{(d)}$ and $x \in \mathbb{C}^{n+1}$ both nonzero. If

$$\frac{D^{3/2}}{2}\mu(f,x)\beta_0(f,x) \leqslant \alpha,$$

then there is a zero $\zeta \in \mathbb{C}^{n+1}$ of f satisfying: $d_T(x,\zeta) \leqslant \sigma(\alpha)\beta_0(f,x)$ with

$$\sigma(\alpha) = \frac{1}{4} + \frac{1-\sqrt{(1+\alpha)^2-8\alpha}}{4\alpha}.$$

Furthermore, if $y = N_f(x)$, then $d_R(y,\zeta) \leqslant (\sigma(\alpha)-1)\beta(f,x)$. Moreover, when

$$\frac{D^{3/2}}{2}\mu(f,x)\beta_0(f,x) \leqslant \alpha \leqslant 0.049,$$

then x is an approximate zero of f corresponding to ζ, and so does y.

Proof. The proof below follows the lines of Shub & Smale (1994). We suppose that f and x are normalized ($\|f\| = \|x\| = 1$). We consider the non-homogeneous polynomial system, defined for a variable $X \in x^{\perp}$ by

$$F(X) = f(x+X).$$

Let us denote by N_F the usual Newton operator:

$$N_F(X) = X - DF(X)^{-1}F(X).$$

Then $DF(X) = Df(x+X)|_{x^{\perp}}$. In particular, $DF(0) = Df(x)|_{x^{\perp}}$ and we have,

$$\begin{array}{rcl} y = N_f(x) & = & \lambda(x+N_F(0)), \lambda \in \mathbb{C}\setminus\{0\} \\ \beta_0(f,x) & = & \beta(F,0) \\ \gamma_0(f,x) & \geqslant & \gamma(F,0) \\ \alpha_0(f,x) & \geqslant & \alpha(F,0). \end{array}$$

Since, by Lemma 3.1, $\alpha_0(f,x) \leqslant \beta_0(f,x)\mu(f,x)D^{3/2}/2$, by Theorem 1, p. 462, in Shub & Smale (1993), 0 is an approximate zero of F and hence (Definition 1 ibid.) the sequence $(X_k)_{k\geqslant 0}$ defined recursively by $X_{k+1} = N_F(X_k)$, $X_0 = 0$, converges quadratically to a zero Z of F. Namely,

$$\|X_{k+1} - X_k\| \leqslant 2^{-2^k+1}\|X_1 - X_0\|.$$

Moreover, by the same theorem,

$$\|Z - X_0\| \leqslant \frac{1+\alpha(F,0) - \sqrt{(1+\alpha(F,0))^2 - 8\alpha(F,0)}}{4\gamma(F,0)}.$$

Thus, for $\zeta = (x+Z)/\|x+Z\|$, we can bound

$$d_T(\zeta,x) \leqslant \|Z - X_0\| \leqslant \sigma(\alpha_0(f,x))\beta(F,0) \leqslant \sigma(\alpha)\beta_0(f,x).$$

Again by Theorem 1, p.462, of Shub & Smale (1993),

$$\|Z - X_1\| \leqslant \frac{1-3\alpha(F,0) - \sqrt{(1+\alpha(F,0))^2 - 8\alpha(F,0)}}{4\gamma(F,0)}$$

which implies that

$$d_R(\zeta,y) \leqslant (\sigma(\alpha)-1)\beta_0(f,x).$$

Let us prove that x is an approximate zero. This will follow directly from Blum *et. al.* (1998), Chap. 14, Theorem 1. In order to apply this Theorem, we have to check its hypothesis

$$d_T(\zeta,x)\gamma_0(f,\zeta) \leqslant \frac{3-\sqrt{7}}{2}.$$

Using Lemma 3.1 this is obtained from

$$d_T(\zeta,x)\frac{D^{3/2}}{2}\mu(f,\zeta) \leqslant \frac{3-\sqrt{7}}{2}. \tag{4.1}$$

We notice that $u = \frac{D^{3/2}}{2}d_R(x,\zeta)\mu(f,x) \leqslant u_T = \frac{D^{3/2}}{2}d_T(x,\zeta)\mu(f,x) \leqslant \alpha\sigma(\alpha) \leqslant 0.049\sigma(0.049) = 0.0518\cdots < 1/19$. According to Lemma 3.10,

$$\mu(f,\zeta) \leqslant 1.2\mu(f,x).$$

Hence, we infer (4.1) from:

$$d_T(\zeta,x)\frac{D^{3/2}}{2}\mu(f,\zeta) \leqslant 1.2u_T < 0.1771\cdots = \frac{3-\sqrt{7}}{2}.$$

A similar argument holds to prove that y is an approximate root:

$$d_T(\zeta,y)\frac{D^{3/2}}{2}\mu(f,\zeta) \leqslant 1.2d_T(\zeta,y)\frac{D^{3/2}}{2}\mu(f,x) \leqslant 1.2\alpha(\sigma(\alpha)-\alpha) \leqslant \frac{3-\sqrt{7}}{2}.$$

□

In the following proposition we relate the invariant $\beta_0(f,x)$ for an approximate zero x to the distance from its associated zero ζ.

PROPOSITION 4.2 Let $f \in \mathscr{H}_{(d)}$ be fixed and $x \in \mathbb{C}^{n+1}$ be given. If

$$\frac{D^{3/2}}{2}\beta_0(f,x)\mu(f,x) \leqslant \alpha \leqslant 0.049,$$

then, $\beta_0(f,x) \leqslant 1.128d_T(x,\zeta)$, where ζ is the zero of f associated to x, given by Theorem 4.1.

Proof. We suppose that both f, x, and ζ are normalized. From Theorem 4.1, $d_T(x,\zeta) \leqslant \beta_0(f,x)\sigma(\alpha)$. Hence,

$$u \leqslant u_T = \frac{D^{3/2}}{2}\mu(f,x)d_T(x,\zeta) \leqslant \alpha\sigma(\alpha) \leqslant 0.0518.$$

From Theorem 3.1, we conclude for later use that

$$u_{T,\zeta} = \frac{D^{3/2}}{2}\mu(f,\zeta)d_T(x,\zeta) \leqslant u(1+3.805u)$$

Now we can bound:

$$\begin{aligned}
\beta_0(f,x) &= \|Df(x)|_{x^\perp}^{-1}f(x)\| \\
&\leqslant \|Df(x)|_{x^\perp}^{-1}Df(\zeta)_{\zeta^\perp}\|\|Df(\zeta)|_{\zeta^\perp}^{-1}f(x)\| \\
&\leqslant \left(\frac{1}{(1-u)^2}+u\right)\|Df(\zeta)|_{\zeta^\perp}^{-1}f(x)\|
\end{aligned}$$

using Lemmas 3.5 and 3.6. We further bound, as usual,

$$\|Df(\zeta)|_{\zeta^\perp}^{-1}f(x)\| \leqslant \|x-\zeta\| + \sum_{k\geqslant 2}\frac{1}{k!}\left\|Df(\zeta)|_{\zeta^\perp}^{-1}D^kf(\zeta)(x-\zeta)^k\right\| \leqslant d_T(x,\zeta)\frac{1}{1-u_{T\zeta}}.$$

Putting all together,

$$\beta_0(f,x) \leqslant \frac{1-u+u^2}{(1-u)^2}\,\frac{1}{1-u(1+3.805u)}d_T(x,\zeta) \leqslant 1.128d_T(x,\zeta);$$

the last step is obtained numerically, using $u \leqslant \alpha\sigma(\alpha) \leqslant 0.0518$. □

PROPOSITION 4.3 Assume that

$$\frac{D^{3/2}}{2}\beta_0(f,x)\mu(f,x) \leqslant a \leqslant 1/20$$

Let $y = N_f(x)$. Then,

$$\beta_0(f,y) \leqslant \frac{a(1-a)}{\psi(a)}\left(1+\frac{a}{1-3.805a}\right)\beta_0(f,x) < 1.23a\beta_0(f,x).$$

Proof. We assume $\|x\| = 1$. Let $F : X \in x^\perp \mapsto f(x+X)$ be the affine polynomial system associated with f. Then, $\beta(F,0) = \beta_0(f,x)$. Moreover, we can scale $y = x+Y$, for $Y = N_F(0)$. By Proposition 3, p.478 in Shub & Smale (1993),

$$\beta(F,Y) \leqslant \frac{a(1-a)}{\psi(a)}\beta_0(f,x). \tag{4.2}$$

Moreover,

$$\beta_0(f,y) \leqslant \|Df(y)|_{y^\perp}^{-1} Df(y)|_{x^\perp}\| \|y\|^{-1} \beta(F,Y). \tag{4.3}$$

Clearly, $\|y\| \geqslant 1$. It remains to bound the norm of the first term in the rhs of (4.3). By hypothesis,

$$u = \frac{D^{3/2}}{2}\mu(f,x)d_R(f,x) \leqslant a \leqslant 1/20$$

so Theorem 3.1 implies that

$$\mu(f,y) \leqslant \frac{1}{1-3.805a}\mu(f,x).$$

In particular, $Df(y)|_{y\perp}$ has full rank, and we can apply Lemma 3.7 and then Lemma 3.6 to bound

$$\|Df(y)|_{y^\perp}^{-1} Df(y)|_{x^\perp}\| \leqslant 1 + \delta(f,y)d_P(x,y) \leqslant 1 + D^{1/2}\mu(f,y)\beta_0(f,x) \leqslant 1 + \frac{a}{1-3.805a}$$

Combining with (4.2) and (4.3), we obtain:

$$\beta_0(f,y) \leqslant \frac{a(1-a)}{\psi(a)}\left(1+\frac{a}{1-3.805a}\right)\beta_0(f,x). \leqslant 1.23a\beta_0(f,x)$$

□

5. The homotopy

The objective of this section is to prove Theorem 1.1. Through this section the considered systems and zeros are normalized: $f(\zeta) = 0$ with $\|f\| = \|\zeta\| = 1$.

Through this section, we assume without loss of generality that the homotopy path is scaled such that $\langle f_t, \dot{f}_t \rangle = 0$. This is justified as follows.

Assume that a $\mathscr{C}^1$ path f_t is given Let $t(\tau)$ be a $\mathscr{C}^1$ increasing function. Then define a new path $g_\tau = f_{t(\tau)}$. Also, if z_t was such that $f_t(z_t) \equiv 0$, set $w_\tau = z_{t(\tau)}$. The quantities β_0, μ, ϕ, l and L are invariant by parameter change. For instance,

$$L(f_t, z_t; t(a), t(b)) = L(g_\tau, w_\tau; a, b)$$

and if $t = t(\tau)$, $t' = t(\tau')$,

$$\begin{aligned}\phi_{t,t'}(x) &= \min_{|\lambda|=1} \|x\|^{-1} \|Df_t(x)|_{x^\perp}^{-1}(f_t(x) - \lambda f_{t'}(x))\| = \\ &= \min_{|\lambda|=1} \|x\|^{-1} \|Dg_\tau(x)|_{x^\perp}^{-1}(g_\tau(x) - \lambda g_{\tau'}(x))\| = \phi_{\tau,\tau'}(x).\end{aligned}$$

Any statement depending on those quantities can be proved without loss of generality by assuming a parametrization f_t by constant arc length with $\|f_t\| = \|z_t\| = 1$. This will be the case Lemma 5.1 below. However, no change of parameter is actually necessary in the algorithm.

Let $t \in [a,b]$, and $x_t \in \mathbb{C}^{n+1}$ be given with $\|x_t\| = 1$. We suppose that

$$\operatorname{rank} Df_t(x_t)|_{x_t^\perp} = n, \tag{5.1}$$

and that

$$\frac{D^{3/2}}{2}\beta_0(f_t,x_t)\mu(f_t,x_t) \leqslant \alpha. \tag{5.2}$$

According to Theorem 4.1, for α small enough, x_t is an approximate zero of f_t. We call ζ_t the associated zero and extend it continuously for $s \in [t,t']$ so that $f_s(\zeta_s) = 0$.

The main difficulty to prove Theorem 1.1 is to transfer the properties (5.1) and (5.2) supposed to be true at $t = t_i$ onto a similar property at $t' = t_{i+1}$. Moreover, we must show that if x_t is an approximate zero associated to ζ_t, then the same is true for t', for a continuous path ζ_s. For this purpose we study this transfer in a general context.

Through this section, $\varepsilon \leqslant 1/6$. Let $t' > t$ be given and assume that

$$l(t') - l(t) \leqslant \frac{\varepsilon}{5D^{1/2}\mu(f_t,x_t)} \tag{5.3}$$

and

$$\max_{s\in[t,t']} \phi_{t,s}(x) \leqslant \frac{\varepsilon}{5D^{3/2}\mu(f_t,x_t)}. \tag{5.4}$$

For any $s \in]t,t']$ let us define $x_s = N_{f_s}(x_t)$. Notice that x_s is not necessarily normalized.

LEMMA 5.1 Let $\varepsilon \leqslant 1/6$ and set $\alpha = \varepsilon^2/2$. Under the hypotheses above, for any $s \in]t,t']$, one has

1. $\mu(f_s,x_t) \leqslant \frac{1}{1-\varepsilon}\mu(f_t,x_t)$.
2. $\frac{1}{1+\varepsilon/5}\left(\phi_{t,s}(x_t) - \frac{2\alpha}{D^{3/2}\mu(f_t,x_t)}\right) \leqslant \beta_0(f_s,x_t) \leqslant \frac{\varepsilon/5+2\alpha}{(1-\varepsilon/5)D^{3/2}\mu(f_t,x_t)}$,
3. $\frac{D^{3/2}}{2}\beta_0(f_s,x_t)\mu(f_s,x_t) \leqslant 0.049$. In particular, x_t and x_s are approximate zeros of f_s associated with ζ_s,
4. $\mu(f_s,x_s) \leqslant \frac{1}{1-\varepsilon}\mu(f_t,x_t)$.
5. $(1-\varepsilon)\mu(f_s,\zeta_s) \leqslant \mu(f_t,x_t) \leqslant (1+\varepsilon)\mu(f_s,\zeta_s)$. In particular, ζ_s is non-degenerate zero of f_s, and hence $s \mapsto \zeta_s$ in continuous for $s \in [t,t']$.
6. $\beta_0(f_s,x_s) \leqslant 1.23\alpha(f_s,x_t)\beta_0(f_s,x_t)$.
7. Hypothesis (5.1) and a strong version of (5.2) hold at s: rank $Df_s(x_s)|_{x_s^\perp} = n$, and

$$\frac{D^{3/2}}{2}\beta_0(f_s,x_s)\mu(f_s,x_s) \leqslant 0.128\alpha$$

Proof. We assumed without loss of generality that the homotopy path is scaled such that $\langle f_t, \dot{f}_t\rangle = 0$.

1. From equation (5.3),

$$\|f_t - f_s\| \leqslant \int_t^s \|\dot{f}_\sigma\| \,\mathrm{d}\sigma = l(s) - l(t) \leqslant l(t') - l(t) \leqslant \frac{\varepsilon}{5D^{1/2}\mu(f_t,x_t)},$$

so that $v \leqslant \sqrt{D}\mu(f_t,x_t)\|f_t - f_s\| \leqslant \varepsilon/5 \leqslant 1/20$, and Corollary 3.1 gives $(1-\varepsilon)\,\mu(f_s,x_t) \leqslant \mu(f_t,x_t)$.

2. For all λ with $|\lambda| = 1$,

$$\beta_0(f_s,x_t) = \|Df_s(x_t)|_{x_t^\perp}^{-1}\lambda f_s(x_t)\| \leqslant$$

$$\|Df_s(x_t)|_{x_t^\perp}^{-1}Df_t(x_t)|_{x_t^\perp}\|\left(\|Df_t(x_t)|_{x_t^\perp}^{-1}\left(\lambda f_s(x_t) - f_t(x_t)\right)\| + \|Df_t(x_t)|_{x_t^\perp}^{-1}f_t(x_t)\|\right).$$

By Lemma 3.11, (5.2) and (5.4) we obtain:

$$\beta_0(f_s,x_t) \leqslant \frac{1}{1-v}\left(\phi_{t,s}(x_t)+\beta_0(f_t,x_t)\right) \leqslant$$

$$\frac{1}{1-\varepsilon/5}\left(\frac{\varepsilon}{5D^{3/2}\mu(f_t,x_t)}+\frac{2\alpha}{D^{3/2}\mu(f_t,x_t)}\right)=\frac{\varepsilon/5+2\alpha}{(1-\varepsilon/5)D^{3/2}\mu(f_t,x_t)}.$$

For the lower bound,

$$\beta_0(f_s,x_t)=\|Df_s(x_t)|_{x_t^\perp}^{-1}f_s(x_t)\| \geqslant$$

$$\left\|\left(Df_s(x_t)|_{x_t^\perp}^{-1}Df_t(x_t)|_{x_t^\perp}\right)^{-1}\right\|^{-1}\left(\|Df_t(x_t)|_{x_t^\perp}^{-1}(f_s(x_t)-f_t(x_t))\|-\|Df_t(x_t)|_{x_t^\perp}^{-1}f_t(x_t)\|\right).$$

By Lemma 3.11, (5.2) and (5.4) we obtain:

$$\beta_0(f_s,x_t) \geqslant \frac{1}{1+v}\left(\phi_{t,s}(x_t)-\beta_0(f_t,x_t)\right) \geqslant$$

$$\geqslant \frac{1}{1+\varepsilon/5}\left(\phi_{t,s}(x_t)-\frac{2\alpha}{D^{3/2}\mu(f_t,x_t)}\right).$$

3. Combining the the two preceding items,

$$\frac{D^{3/2}}{2}\beta_0(f_s,x_t)\mu(f_s,x_t) \leqslant \frac{\varepsilon/10+\alpha}{(1-\varepsilon)(1-\varepsilon/5)} \leqslant 0.037931\cdots < 0.049. \tag{5.5}$$

Thus, by Theorem 4.1, x_t and x_s are approximate zeros of f_s associated with ζ_s.

4. Using item 2,

$$d_R(x_t,x_s) \leqslant \beta_0(f_s,x_t) \leqslant \frac{\varepsilon/5+2\alpha}{(1-\varepsilon/5)D^{3/2}\mu(f_t,x_t)}..$$

Thus,

$$u = D^{3/2}d_R(x_t,x_s)\mu(f_t,x_t)/2 \leqslant \frac{\varepsilon/10+\alpha}{(1-\varepsilon/5)} < \varepsilon/5.$$

Then, we can use Corollary 3.1 again to bound

$$\mu(f_s,x_s) \leqslant \frac{1}{1-\varepsilon}\mu(f_t,x_t).$$

5. From Theorem 4.1,

$$d_R(x_t,\zeta_s) \leqslant d_T(x_t,\zeta_s) \leqslant \sigma(\alpha(f_s,x_t))\beta_0(f_s,x_t) \leqslant 1.0429\cdots\beta_0(f_s,x_t).$$

Thus,

$$u = \frac{D^{3/2}}{2}\mu(f_t,x_t)d_R(x_t,\zeta_s) \leqslant 0.1978026\cdots\varepsilon < \varepsilon/5.$$

We bounded in item 1 the quantity $v \leqslant D^{1/2}\mu(f_t,x_t)\|f_t-f_s\| < \varepsilon/5$. Hence, by Corollary 3.1 again:

$$(1-\varepsilon)\mu(f_s,\zeta_s) \leqslant \mu(f_t,\zeta_t) \leqslant (1+\varepsilon)\mu(f_s,\zeta_s).$$

6. From item 3 and Proposition 4.3,

$$\beta_0(f_s,x_s) \leqslant 1.23\alpha_0(f_s,x_t)\beta_0(f_s,x_t).$$

7. Because $\mu(f_s,x_s)$ is finite, $Df_s(x_s)$ has full rank. From items 1, 6 and (5.5),

$$\frac{D^{3/2}}{2}\beta_0(f_s,x_s)\mu(f_s,x_s) \leqslant 1.23\left(\frac{\varepsilon/10+\alpha}{(1-\varepsilon)(1-\varepsilon/5)}\right)^2 \leqslant 0.128\alpha.$$

□

Recall that our algorithm allows for an approximate computation of the Newton iteration. The robustness Lemma below shows that if a point x satisfies (5.2) and conclusion 7 of Lemma 5.1, then an approximation y of x satisfies (5.2) and (5.3).

LEMMA 5.2 Assume that $\|f\| = 1$ and $\|x\| = \|y\| = 1$. Let $\alpha \leqslant 1/72$ and $c \leqslant 0.8$. Suppose that $Df(x)_{|x^\perp}$ has rank n, and

$$\frac{D^{3/2}}{2}\beta_0(f,x)\mu(f,x) \leqslant 0.128\alpha \tag{5.6}$$

$$u = \frac{D^{3/2}}{2}d_R(x,y)\mu(f,x) \leqslant \frac{c\alpha}{\sqrt{D}\mu(f,x)}. \tag{5.7}$$

Then, $Df(y)_{|y^\perp}$ has rank n, and

$$\frac{D^{3/2}}{2}\beta_0(f,y)\mu(f,y) \leqslant \alpha \tag{5.8}$$

and furthermore, x and y are approximate zeros associated to the same exact zero ζ.

Proof. By using $D^{3/2}\mu(f,x) \geqslant 4$ (see Lemma 3.2 and the hypothesis $D \geqslant 2$) we obtain that $u \leqslant 0.0055\cdots < 1/19$. Therefore, Lemma 3.10 implies that

$$\mu(f,y) \leqslant (1+3.805)\mu(f,x)$$

and in particular, $Df(y)_{|y^\perp}$ has rank n.

To estimate $\beta_0(f,y)$, we decompose

$$\beta_0(f,y) = \left\|Df(y)^{-1}_{|y^\perp}f(y)\right\| \leqslant \left\|Df(y)^{-1}_{|y^\perp}Df(x)_{|x^\perp}\right\| \left\|Df(x)^{-1}_{|x^\perp}f(y)\right\|.$$

The first term is bounded by Lemma 3.8,

$$\left\|Df(y)^{-1}_{|y^\perp}Df(x)_{|x^\perp}\right\| \leqslant 1+3.805u.$$

Taylor's exansion gives $Df(x)^{-1}_{|x^\perp}f(y) =$

$$Df(x)^{-1}_{|x^\perp}f(x) + Df(x)^{-1}_{|x^\perp}Df(x)(y-x) + \sum_{k\geqslant 2}\frac{1}{k!}Df(x)^{-1}_{|x^\perp}D^kf(x)(y-x)^k.$$

Taking norms,

$$\left\|Df(x)^{-1}_{|x^\perp}f(y)\right\| \leqslant \beta_0(f,x) + \delta(f,x)\|y-x\| + \frac{\|y-x\|^2\gamma_0(f,x)}{1-\|y-x\|\gamma_0(f,x)}.$$

By Lemma 3.1c,$\|y-x\|\gamma_0(f,x) \leqslant u$. Hence,

$$\beta_0(f,x) \leqslant (1+3.805u)\left(\beta_0(f,x) + \delta(f,x)\|y-x\| + \frac{\|y-x\|u}{1-u}\right).$$

Using Lemma 3.6, $\delta(f,x) \leqslant \sqrt{D}\mu(f,x)$. Thus,

$$\begin{aligned}
\frac{D^{3/2}}{2}\beta_0(f,y)\mu(f,y) &\leqslant (1+3.805u)^2\left(0.128\alpha+\sqrt{D}\mu(f,x)u+\frac{u^2}{1-u}\right)\\
&\leqslant (1+3.805u)^2\left(0.128+c+\frac{cu}{\sqrt{D}\mu(f,x)(1-u)}\right)\alpha\\
&\leqslant 0.97\alpha<\alpha
\end{aligned}$$

Since $\alpha \leqslant 1/72 \leqslant 0.049$, Theorem 4.1 implies that both x and y are approximate zeros of f. As this is also the case for all the points in the shortest arc of circle between x and y, the associated zero must be the same. □

LEMMA 5.3 Assume the Hypotheses of Lemma 5.1. Let $0<\xi\leqslant 1$. If furthermore

$$l(t')-l(t) \geqslant \frac{\xi\varepsilon}{5\sqrt{D}\mu(f_t,x_t)},$$

then

$$L(t')-L(t) \geqslant \frac{\xi\varepsilon}{5D^{3/2}}.$$

Proof.

$$\begin{aligned}
L(t')-L(t) &= \int_t^{t'}\left\|(\dot f_s,\dot\zeta_s)\right\|\mu(f_s,\zeta_s)\,ds\\
&\geqslant \int_t^{t'}\left\|\dot f_s\right\|\mu(f_s,\zeta_s)\,ds\\
&\geqslant \frac{\mu(f_t,x_t)}{1+\varepsilon}\int_t^{t'}\left\|\dot f_s\right\|\,ds \quad \text{using Lemma 5.1(5).}\\
&= \frac{\mu(f_t,x_t)}{1+\varepsilon}(l(t')-l(t))\\
&\geqslant \frac{\xi\varepsilon}{5(1+\varepsilon)\sqrt{D}}.\\
&\geqslant \frac{\xi\varepsilon}{5D^{3/2}}.
\end{aligned}$$

□

LEMMA 5.4 Assume the Hypotheses of Lemma 5.1 and choose ε and ξ so that $20\varepsilon\leqslant\xi\leqslant 1$. If furthermore

$$\phi_{t,t'}(x) \geqslant \frac{\xi\varepsilon}{5D^{3/2}\mu(f_t,x_t)},$$

then

$$L(t')-L(t) \geqslant \frac{\xi\varepsilon}{13D^{3/2}}.$$

Proof.

$$\begin{aligned}
L(t')-L(t) &= \int_t^{t'}\left\|(\dot f_s,\dot\zeta_s)\right\|\mu(f_s,\zeta_s)\,ds\\
&\geqslant \int_t^{t'}\left\|\dot\zeta_s\right\|\mu(f_s,\zeta_s)\,ds\\
&\geqslant \frac{\mu(f_t,x_t)}{1+\varepsilon}\int_t^{t'}\left\|\dot\zeta_s\right\|\,ds \quad \text{using Lemma 5.1(5).}\\
&\geqslant \frac{\mu(f_t,x_t)}{1+\varepsilon}d_R(\zeta_t,\zeta_{t'}).
\end{aligned}$$

By the triangle inequality,

$$d_R(\zeta_t, \zeta_{t'}) \geqslant d_R(\zeta_{t'}, x_t) - d_R(x_t, \zeta_t)$$

We know from Theorem 4.1 that $d_R(x_t, \zeta_t) \leqslant \sigma(\alpha)\beta_0(f_t, x_t)$. Lemma 5.1(3) says that $\alpha_0(f_{t'}, x_{t'}) \leqslant 0.049$ and hence, from Proposition 4.2, we obtain:

$$d_T(x_t, \zeta_{t'}) \geqslant \beta_0(f_{t'}, x_t)/1.128.$$

Thus, $d_T(x_t, \zeta_{t'}) \geqslant \frac{\xi\varepsilon/5-2\alpha}{(1+\varepsilon/5)D^{3/2}\mu(f_t,x_t)}\frac{1}{1.128}$ by Lemma 5.1(2). We use now the bound $20\varepsilon \leqslant \xi$ and the fact that $\alpha = \varepsilon^2/2$ to deduce that

$$d_T(x_t, \zeta_{t'}) \geqslant \omega = \frac{3\xi\varepsilon/20}{1.128(1+\varepsilon/5)D^{3/2}\mu(f_t, x_t)}$$

Since $\varepsilon \leqslant \xi/20 \leqslant 1/20$, we can bound $\omega \leqslant 0.001645\cdots$. Of course, $d_R(x_t, \zeta_{t'}) \geqslant \arctan\omega$. We may bound $\arctan(\omega) \geqslant \omega \arctan'(0.001645\cdots) = \frac{1}{1+0.001645\cdots^2}\omega > 0.999\omega$. Now,

$$\begin{aligned} L(t') - L(t) &\geqslant \frac{\mu(f_t, x_t)}{1+\varepsilon}\left(0.999\frac{3\xi\varepsilon/20}{1.128(1+\varepsilon/5)D^{3/2}\mu(f_t,x_t)} - \frac{2\sigma(\alpha)\alpha}{D^{3/2}\mu(f_t,x_t)}\right) \\ &\geqslant 0.0775\frac{\xi\varepsilon}{D^{3/2}} > \frac{\xi\varepsilon}{13D^{3/2}} \end{aligned}$$

□

Proof of Theorem 1.1

We take $\xi = 20\varepsilon$ and $\alpha = \varepsilon^2/2$. Assume that $L(b)$ is finite. By hypothesis and Theorem 4.1, x_0 is an approximate zero of f_a. $(f_t, \zeta_t)_{t\in[a,b]}$ denotes the unique lifting of the path f_t corresponding to ζ_0 zero of f_a associated to x_0.

Induction hypothesis:

$$\frac{D^{3/2}}{2}\mu(f_{t_i}, x_i)\beta_0(f_{t_i}, x_i) < \alpha$$

and furthermore, x_i is an approximate zero associated to ζ_{t_i}.

The induction hypothesis holds by hypothesis at $i = 0$, so we assume it is verified up to step i. We are in the hypotheses of Lemma 5.1 for $t = t_i$, $t' = t_{i+1} = \min(s, s', b)$ and $x = x_{t_i} = x_i$. Thus,

$$\frac{D^{3/2}}{2}\mu\left(f_{t_{i+1}}, N_{f_{t_i}}(x_i)\right)\beta_0\left(f_{t_{i+1}}, N_{f_{t_i}}(x_i)\right) < 0.128\alpha$$

and $N_{f_{t_i}}(x_i)$ is an approximate zero for $f_{t_{i+1}}$ associated to $\zeta_{t_{i+1}}$.

Now we are in the hypotheses of Lemma 5.2. Thus, $y = x_{i+1}$ picked at (1.4) satisfies the induction hypothesis.

In order to bound the number of iterations, we remark that at each step i, one of the following alternatives is true:

1. This is the last step: $t_{i+1} = b$.
2. Condition (1.1) is true. In that case, we are under the hypotheses of Lemma 5.3.
3. Condition (1.2) is true. Then we are under the hypotheses of Lemma 5.4.

Therefore, we may infer that at each non-terminal step,

$$L(t_{i+1}) - L(t_i) > \frac{\xi\varepsilon}{13D^{3/2}}.$$

Therefore, there can be no more than $260\xi^{-2}L(b)D^{3/2}$ non-terminal steps. There is only one terminal step, so the total number of steps is at most

$$1 + 260\xi^{-2}L(b)D^{3/2} = 1 + 0.65\varepsilon^{-2}L(b)D^{3/2}$$

6. Some remarks on how to implement the algorithm

In Theorem 1.1, we gave a complexity bound for approximating a lifting (f_t, ζ_t) for an arbitrary homotopy path f_t of class C^1, and for an initial solution ζ_0 for $f_a(\zeta_a) = 0$.

In that theorem, it is assumed that we know how to perform each step of the **Homotopy** algorithm. In particular, we must be able to solve the inequalities (1.1), (1.2) and (1.3).

In this section, we explain how to do this for the case of a linear homotopy. Then, a nonlinear homotopy may be approximated by a piecewise linear one.

The homotopy will be parametrized by arc length. Therefore, we assume first that $[a,b] = [0, \theta_{\text{Max}}]$ and $f_t = \cos(t) f_0 + \sin(t) g$, with of course $g \perp f_0$ and $\|f_0\| = \|g\| = 1$ and $\theta_{\text{Max}} \leqslant \pi/2$. With that convention,

$$l(t) = t.$$

Assuming $\varepsilon = 1/20$, we solve explicitly (1.1) by

$$s - t = \frac{\varepsilon}{5D^{1/2}\mu(f_{t_i}, x_i)}.$$

Solving (1.2) and (1.3) is more tricky. Define:

$$\begin{aligned} X &= Df_t(x_t)|_{x_t^\perp}^{-1} f_t(x) \\ Y &= Df_t(x_t)|_{x_t^\perp}^{-1} f_\sigma(x) \\ Z &= Df_t(x_t)|_{x_t^\perp}^{-1} \dot{f}_t(x) \end{aligned}$$

Also, let $c = \cos(\sigma - t)$ and $s = \sin(\sigma - t)$, so that

$$Y = cX + sZ.$$

We expand

$$\begin{aligned} \phi_{t,\sigma}(x)^2 &= \|x\|^{-2}\left(\|X\|^2 + \|Y\|^2 - 2|\langle X, Y\rangle|\right) \\ &= \|x\|^{-2}\left((1+c^2)\|X\|^2 + s^2\|Z\|^2 + 2cs\mathrm{Re}(\langle X, Z\rangle) - 2|c\|X\|^2 + s\langle X, Z\rangle|\right) \\ &= \|x\|^{-2}\Bigg((1+c^2)\|X\|^2 + s^2\|Z\|^2 + 2cs\mathrm{Re}(\langle X, Z\rangle) \\ &\qquad -2\sqrt{(c\|X\|^2 + s\mathrm{Re}(\langle X, Z\rangle))^2 + (s\mathrm{Im}(\langle X, Z\rangle))^2}\Bigg). \end{aligned}$$

The equation $\phi_{t,\sigma}(x)^2 = C$ becomes, after clearing the square root, a polynomial in c and s. Replacing c by $\sqrt{1-s^2}$ and clearing the square root again, we obtain a polynomial in s of which only the smallest positive root is relevant. This way, solving (1.2) and (1.3) can be reduced to univariate polynomial solving for a fixed degree.

The other steps of the **Homotopy** algorithm pose no difficulty for the implementation.

7. Conclusions

We gave a rigorous homotopy algorithm for polynomial systems. The correctness of the result is guaranteed, in the sense that there is no 'path jumping'.

Some algorithms such as the one by Li (2003) proceed by setting a step size in an heuristic way, and then checking the results. Our algorithm allows to eliminate the need for checking.

This way, it is possible to consider a massive parallel implementation of the algorithm, each processor tracing one or a few roots. The running time will be given by the maximum (over all paths) of the condition length.

It is also possible to use the algorithm for a rigorous computation of the root permutation associated to a closed path, and for investigating the corresponding Galois group as in Leykin & Sottile (2009).

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