Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org
pp. **X–XX**

# ENTROPY ESTIMATES FOR A FAMILY OF EXPANDING MAPS OF THE CIRCLE

Rafael de la Llave

Mathematics Department
University of Texas at Austin, U.S.A.
1 University Station C1200
Austin TX 78712-0257

Michael Shub

Mathematics Department
University of Toronto
40 St. George Street, Toronto
ON M5S 2E4, Canada

Carles Simó

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585, 08007 Barcelona, Spain

(Communicated by )

Abstract. In this paper we consider the family of circle maps $f_{k,\alpha,\varepsilon} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ which when written mod 1 are of the form

$$f_{k,\alpha,\varepsilon} : x \mapsto kx + \alpha + \varepsilon \sin(2\pi x),$$

where the parameter $\alpha$ ranges in $\mathbb{S}^1$ and $k \geq 2$. We prove that for small $\varepsilon$ the average over $\alpha$ of the entropy of $f_{k,\alpha,\varepsilon}$ with respect to the natural absolutely continuous measure is smaller than $\int_0^1 \log |Df_{k,0,\varepsilon}(x)|dx$, while the maximum with respect to $\alpha$ is larger. In the case of the average the difference is of order of $\varepsilon^{2k+2}$. This result is in contrast to families of expanding Blaschke products depending on rotations where the averages are equal and for which the inequality for averages goes in the other direction when the expanding property does not hold, see [4]. A striking fact for both results is that the maximum of the entropies is greater than or equal to $\int_0^1 \log |Df_{k,0,\varepsilon}(x)|dx$. These results should also be compared with [3], where similar questions are considered for a family of diffeomorphisms of the two sphere.

2000 *Mathematics Subject Classification.* Primary: 37C40, 37D20, 37E10 ; Secondary: 37M25
.

*Key words and phrases.* entropy estimates, expanding circle maps.

We thank Carlangelo Liverani for conversations helping us to see that these experimentally small quantities are not equal to zero. Research of R. L has been partially supported by NSF and by MEC FEDER Grants MTM2006-00478. Research of M. S. has been partially supported by an NSERC Discovery Grant. Research of C. S. has been partially supported by grants BFM2003-09504-C02-01, MTM2006-05849/Consolider (Spain) and CIRIT 2005 SGR-1028 (Catalonia).

1. **Introduction.** Several papers have now studied lower bounds for the average (metric) entropy or Lyapunov exponents of families of dynamical systems [2],[3], [4]. The main point is that if the average of the entropy is bounded from below by a real number $r > 0$ then some elements of the family have entropy greater than $r$. So far family $\mathbb{F}$ of dynamical systems that have been considered are of the form of a composition of a fixed endomorphism $f : M \to M$ and the elements of $G$ a group of isometries of $M$ whose induced action on the projective or flag bundle of the tangent bundle of $M$ is transitive. Here the projective bundle is the bundle whose fiber at $m \in M$ is the projective space of $T_m M$ and the flag bundle the bundle whose fiber is the flag manifold of $T_m M$.

That is, $\mathbb{F} = \{g \circ f | g \in G\}$. The average is taken with respect to the measure on $\mathbb{F}$ which is the push forward of Haar measure on $G$. This average entropy is compared to the entropy of the random products $\ldots g_i f g_{i-1} f \ldots g_2 f g_1 f$ where the $g_i$ are chosen iid with respect to the Haar measure on $G$. This last quantity is usually "easily" computable from the derivative $Tf$ and usually easily seen to be positive, see in particular [2],[3] and the references therein.

In this paper we consider the family of circle maps $f_{k,\alpha,\varepsilon} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of the form

$$f_{k,\alpha,\varepsilon} : x \mapsto kx + \alpha + \varepsilon \sin(2\pi x), \tag{1}$$

for $k \in \mathbb{N}$, $k \geq 2$ and $\varepsilon$ small, where the parameter $\alpha$ ranges in $\mathbb{S}^1$. So for each $\varepsilon$ we are in the context above with $G$ equal to $\mathbb{S}^1$ acting on itself.

Let $\rho_{k,\alpha,\varepsilon}$ be the density of the invariant measure on $\mathbb{S}^1$ which is absolutely continuous with respect to Lebesgue measure. The existence follows immediately from uniform expansiveness. It can be averaged with respect to $\alpha$ to obtain $\hat{\rho}_{k,\varepsilon}$. Then the following has been observed from numerical computations:

**Claim 1.** $\hat{\rho}_{k,\varepsilon}(x) = 1 + \sum_{j\geq 1} C_{k,j} \cos(j2\pi x)$, *where* $C_{k,j} = \varepsilon^{2+j(2k-1)}(\hat{C}_{k,j} + \mathcal{O}(\varepsilon))$ *and* $\hat{C}_{k,1} < 0$.

We can also compute the average entropy of $f_{k,\alpha,\varepsilon}$ with respect to $\alpha$. It is given by

$$I(k,\varepsilon) = \int_0^1 \hat{\rho}_{k,\varepsilon}(x) \log |Df_{k,\alpha,\varepsilon}(x)| dx, \tag{2}$$

which is obviously independent of $\alpha$ for maps of the form (1).

On the other hand we can compute the integral

$$J(k,\varepsilon) = \int_0^1 \log |Df_{k,\alpha,\varepsilon}(x)| dx, \tag{3}$$

which is also obviously independent of $\alpha$. Let

$$\Delta(k,\varepsilon) = J(k,\varepsilon) - I(k,\varepsilon).$$

$J(k,\varepsilon)$ is the Lyapunov exponent of the random product as above and $\Delta(k,\varepsilon)$ the deviation of the average of the deterministic entropies from the random one. Ideally we would like to have $\Delta \leq 0$, so that the behaviour of present family is similar to Blaschke's one. Indeed if $f$ is an immediately expanding Blaschke product and $G = \mathbb{S}^1$, $\Delta = 0$ and for general Blaschke products $\Delta \leq 0$, [4]. Numerically (see Section 3 it has been observed that for our families as soon as $\varepsilon$ is small and positive, one has $\Delta$ non-negative. See also [3] for a similar result in two dimensions.

**Claim 2.** *For* $\varepsilon$ *small* $\Delta(k,\varepsilon) > 0$ *and it behaves like* $\varepsilon^{2k+2}$.

Yet it is striking that, even if the inequality for the average and the random goes in the opposite direction to the heuristics, there is an inequality for the maximum, which goes in the direction of the heuristics.

**Claim 3.** *For $\varepsilon$ small and positive, $max_{\alpha\in\mathbb{S}^1} h(f_{k,\alpha,\varepsilon}) > J(k,\varepsilon)$, where $h(f_{k,\alpha,\varepsilon})$ is the measure theoretic entropy of $f_{k,\alpha,\varepsilon}$ with respect its absolutely continuous invariant measure.*

The main goal of this note is to give analytic proofs of all three Claims 1, 2 and 3. In fact Claim 2 follows immediately from Claim 1. It turns out that a main ingredient has essentially been available for a couple of centuries, that is, the classical formulae for the expansions of elliptic motion in terms of the mean anomaly. In particular a key role is played by Bessel functions. Of course, this relies strongly on the particular format of the family (1). In Section 2 we give the proofs. In Section 3 we give the numerics which are somewhat subtle because of the high powers of $\varepsilon$ involved.

2. **Proofs.** The main goal of this note is

**Theorem 1.** *The density of the invariant measure of (1), averaged with respect to $\alpha$, is of the form given by Claim 1. Furthermore the coefficients $C_{k,j}$ can be expressed in terms of Bessel functions.*

Before giving the proof, let us introduce the parameter $e = -2\pi\varepsilon/k$, to be used in what follows. The reason to name it $e$ will be clear later. We can also state the following

**Corollary 1.** *Claim 2 is true.*

**Proof**. As

$$\log|Df_{j,\alpha,\varepsilon}(x)| = \log k + \log(1 - e\cos(2\pi x)) = \log k - e\cos(2\pi x) + \mathcal{O}(e^2),$$

by using Theorem 1 and noting that $\log k$ multiplies periodic terms with zero average, from (2) and (3) we have

$$\begin{aligned}\Delta(k,\varepsilon) &= \int_0^1 (1-\hat\rho_{k,\varepsilon}(x))\log|Df_{j,\alpha,\varepsilon}(x)|dx \\ &= \int_0^1 \varepsilon^{2k+1}(-\hat C_{k,1}\cos(2\pi x) + \mathcal{O}(\varepsilon))(-e\cos(2\pi x)+\mathcal{O}(e^2))dx,\end{aligned}$$

which is obviously $\mathcal{O}(\varepsilon^{2k+2})$ with coefficient $-\pi\hat C_{k,1}/k > 0$. □

To prove Theorem 1 we look directly for the invariant measure $\rho_{k,\alpha,\varepsilon}$. That is, for $y\in\mathbb{S}^1$, we ask for invariance

$$\rho_{k,\alpha,\varepsilon}(y+\alpha) = \frac{1}{k}\sum_{j=1}^{k}\frac{\rho_{k,\alpha,\varepsilon}(x_j)}{1-e\cos(2\pi x_j)}, \tag{4}$$

where $x_j$, $j = 1,\ldots,k$ are the preimages of $y+\alpha$ under the map. More concretely, $x_j$ is the unique solution (in $\mathbb{R}^1$) of

$$kx + \alpha + \varepsilon\sin(2\pi x) = y + j + \alpha, \tag{5}$$

where uniqueness follows from the smallness of $\varepsilon$. Dividing by $k$, multiplying by $2\pi$ and introducing $M_j = 2\pi\dfrac{y+j}{k}$, $E_j = 2\pi x_j$ and $e$, as defined above, equation (5) reads as

$$E_j - e\sin E_j = M_j, \tag{6}$$

that is, the classical *Kepler's equation* of the elliptic two-body problem. The variables $M, E, e$ are the well-known mean anomaly, eccentric anomaly and eccentricity, respectively. It is possible to express in closed form the solution of (6) as a function of $e, M_j$ and also $(1-e\cos E_j)^{-1}$, to be used in (4). A convenient reference for our purposes is Chapter II in [1].

The solution of (6) is given by

$$E = M + 2\sum_{s\geq 1}\frac{1}{s}J_s(se)\sin(sM),$$

where $J_s$ denotes the Bessel function of the first kind and order $s$. We recall that these are entire functions with Taylor series, for $s \geq 0$,

$$J_s(u) = \sum_{r\geq 0}\frac{(-1)^r(u/2)^{s+2r}}{r!(s+r)!}$$

and $J_{-s}(-u) = J_s(u)$. In particular $J_s(u) = \dfrac{(u/2)^s}{s!}(1+\mathcal{O}(u^2))$ around $u = 0$, for $s \geq 0$.

At this point it is better to pass to exponential form. Let $v = \exp(\mathrm{i}\,M)$, $w = \exp(\mathrm{i}\,E)$. Derivation of Kepler's equation gives $(1-e\cos E)\dfrac{dE}{dM} = 1$ and then $(1-e\cos E)^{-1} = \sum_{s\in\mathbb{Z}} c_s v^s$, where $c_s = J_s(se)$ for $s\in\mathbb{Z}$. In particular $c_0 = 1$.

In a similar way it is possible to express $\cos(pE), \sin(pE)$ as functions of $e, M$, or, in exponential form

$$w^n = \sum_{r\in\mathbb{Z}} d_{n,r}v^r, \quad d_{n,r} = \frac{n}{r}J_{r-n}(re) \ \text{ for } \ r\neq 0,$$

$$d_{0,0} = 1,\ d_{\pm 1,0} = -\frac{e}{2},\ d_{n,0} = 0 \ \text{ otherwise}\ .$$

We substitute these representations of the different functions in (4). Let $z = \exp(2\pi\,\mathrm{i}\,y)$, $\theta = \exp(2\pi\,\mathrm{i}\,\alpha)$, and let us represent the density as

$$\rho_{k,\alpha,\varepsilon}(y) = \sum_{m\in\mathbb{Z}} a_m z^m. \tag{7}$$

Then (4) is written as

$$\sum_{m\in\mathbb{Z}} a_m z^m\theta^m = \frac{1}{k}\sum_{j=1}^{k}\left(\sum_{s\in\mathbb{Z}} c_s z^{s/k}\mathbf{1}^{js/k}\right)\left(\sum_{n\in\mathbb{Z}} a_n \sum_{r\in\mathbb{Z}} d_{n,r}z^{r/k}\mathbf{1}^{jr/k}\right), \tag{8}$$

where $\mathbf{1}^{p/q}$ denotes the complex number of modulus 1 and argument $\frac{2\pi p}{q}$. As $\frac{1}{k}\sum_{j=1}^{k}\mathbf{1}^{j(r+s)/k}$ equals zero unless $r+s = pk$, $p\in\mathbb{Z}$, in which case it equals 1, we can collect terms in $z^p$ in (8) to obtain

$$a_p\theta^p = \sum_{n\in\mathbb{Z}} a_n g_{p,n}, \quad \text{where} \quad g_{p,n} = \sum_{r\in\mathbb{Z}} c_{pk-r}d_{n,r}. \tag{9}$$

Using the properties of $J_s$ it is clear that $g_{p,n}$ is exactly of order $e^{|pk-n|}$. Furthermore $g_{0,0} = 1$, $g_{p,0} = c_{pk}$.

Now we want to solve (9) for the $a_p$. It is clear from (7) that $a_0 = 1$ (normalisation of the density). To solve (9) we shall use an iterative procedure. Let $a_p^{(0)} = 0$ if $p \neq 0$. Then, for $j \geq 0$, we compute

$$a_p^{(j+1)} = \theta^{-p} \sum_{n \in \mathbb{Z}} g_{p,n} a_n^{(j)}. \tag{10}$$

At the first step we obtain $a_p^{(1)} = \theta^{-p} g_{p,0} = \mathcal{O}(e^{|pk|})$. Every new iteration of (10) improves the correct terms at least by an additional factor $e$.

But we are interested in $\hat{\rho}$. Hence we need the terms independent of $\theta$ in $a_p$ and, for concreteness, we assume $p > 0$. These are terms of the form

$$g_{p,p_1}\, g_{p_1,p_2} \,\cdots\, g_{p_{q-1},p_q}\, g_{p_q,0} \tag{11}$$

with $p+p_1+p_2+\ldots+p_q = 0$. Using the order in $e$ of the $g$ coefficients, as defined in (9), we have that the minimal order is obtained using the choice $q = p$, $p_1 = \ldots = p_q = -1$. The corresponding order is $pk + 1 + (p-1)(k-1) + k = 2 + p(2k-1)$, as stated in Claim 1. A symmetric choice must be used for $p < 0$, giving the same coefficient.

If $p = 1$ the coefficient in (11) is $g_{1,-1}g_{-1,0}$. As $g_{-1,0} = c_k = J_{-k}(-ke) = J_k(-2\pi\varepsilon)$ it has sign $(-1)^k$. Concerning $g_{1,-1}$ all the terms in (9) with $r$ between $-1$ and $k$ contribute with the common factor $(e/2)^{k+1}$. Hence, it is enough to prove that, for $k \geq 2$, the coefficient

$$A := \frac{(k+1)^{k+1}}{(k+1)!} - \frac{k^k}{k!} - \frac{(k-1)^{k-1}}{(k-1)!}\frac{1}{1\cdot 1!} - \frac{(k-2)^{k-2}}{(k-2)!}\frac{1}{2\cdot 2!} - \ldots - \frac{1^1}{1!}\frac{1}{(k-1)\cdot(k-1)!} - \frac{1}{k\cdot k!} \tag{12}$$

is positive to show that $\hat{C}_{k,1}$ has negative sign. Skipping last term in (12) $A$ is majorated by $\dfrac{(k+1)^{k+1}}{(k+1)!} - \displaystyle\sum_{j=1}^{k} \frac{j^j}{j!}$. The quotient of the terms $j+1$ and $j$ in the sum is $\left(1 + \frac{1}{j}\right)^j \geq 2$ for all $j \geq 1$. Hence, $\sum_{j=1}^k \leq 2\dfrac{k^k}{k!} - 1$ and, therefore, the sum of all the negative terms in $A$ has absolute value $\leq 2\dfrac{k^k}{k!}$. Then $A\dfrac{k!}{k^k} \geq \left(1 + \dfrac{1}{k}\right)^k - 2$. Taking logarithms $k \log\left(1 + \frac{1}{k}\right) > 1 - \dfrac{1}{2k} > \log 2$ for $k \geq 2$, as desired. This ends the proof of Theorem 1. □

Up to now we have proved that the entropy of the family (1), averaged with respect to $\alpha$ and given by $I(k,\varepsilon)$ in (2) is less than the entropy of the randomized map, given by $J(k,\varepsilon)$, when for each new iteration the value of $\alpha$ is taken at random with uniform probability in $\mathbb{S}^1$. Now we want to prove Claim 3 that the maximum of the entropies, with respect to $\alpha$, exceeds $J(k,\varepsilon)$. More concretely

**Theorem 2.** *For $\varepsilon$ small the entropy $h_{k,\alpha,\varepsilon}$ of $f_{k,\alpha,\varepsilon}$ is greater than $J(k,\varepsilon)$ if $\alpha = 0$ for $k$ even or if $\alpha = 1/2$ for $k$ odd.*

*Furthermore the difference $h_{k,\alpha,\varepsilon} - J(k,\varepsilon)$, for these choices of $\alpha$ depending on $k$, is of the form $B_k\varepsilon^{k+1}(1 + \mathcal{O}(\varepsilon))$ with $B_k > 0$.*

**Proof**. First let us compute $J(k,\varepsilon)$. Expanding as in the proof of Corollary 1 but up to order 3 in $\varepsilon$, one has

$$\log |Df_{k,\alpha,\varepsilon}(x)| = \log k - e\cos(2\pi x) - \frac{e^2}{2}\cos^2(2\pi x) - \frac{e^3}{3}\cos^3(2\pi x) + \mathcal{O}(e^4),$$

and it is clear that $J(k,\varepsilon) = \log k - \dfrac{(\pi\varepsilon)^2}{k^2} + \mathcal{O}(\varepsilon^4)$. It turns out that the dominant terms of the entropy $h_{k,\alpha,\varepsilon}$ coincide with the corresponding terms in $J(k,\varepsilon)$ to order $\varepsilon^0$ and $\varepsilon^2$ (even more terms coincide if $k > 2$). Hence, we proceed by looking directly for the difference. It is given by

$$\int_0^1 \log |Df_{k,\alpha,\varepsilon}(x)|(\rho_{k,\alpha,\varepsilon}(x) - 1)dx. \tag{13}$$

If $k$ is even and $\alpha = 0$ (i.e., $\theta = 1$), it is enough to use the first approximation for the coefficient of the first harmonic (which is the dominant one in $(\rho_{k,\alpha,\varepsilon}(x) - 1)$): $a_1^{(1)} = c_k = J_k(ke)$. Hence, the dominant contribution to (13) is

$$\int_0^1 (-e\cos(2\pi x))J_k(ke)2\cos(2\pi x)dx.$$

As $e = -2\pi\varepsilon/k < 0$ and $J_k(ke) = (ke/2)^k(1 + \mathcal{O}(e^2)) > 0$, the result follows for $k$ even.

For $k$ odd it is enough to use $\alpha = 1/2$ (i.e., $\theta = -1$) and again $-J_k(ke) > 0$.

Finally, we have that the difference $h_{k,\alpha,\varepsilon} - J(k,\varepsilon)$, if $\alpha = 0$ for $k$ even or $\alpha = 1/2$ for $k$ odd, has a dominant term of the form $B_k\varepsilon^{k+1}$ with $B_k = \dfrac{1}{2}\dfrac{2\pi}{k}(-1)^k 2\dfrac{(-\pi)^k}{k!} = 2\dfrac{\pi^{k+1}}{k\cdot k!} > 0$, as we wanted to prove. □

3. **Numerics.** As mentioned at the Introduction the numerical computations of entropy must be done with enough accuracy due to the very small values of $\Delta(k,\varepsilon)$ for small $\varepsilon$.

First we have considered a subfamily of the expanding Blaschke products as presented in [4]. As the results are known analytically, this is used as a check of the numerical methods.

We consider the family of maps $T_{a_1,a_2,\theta} : \mathbb{S}^1 \to \mathbb{S}^1$ given by

$$z \mapsto \theta \frac{z - a_1}{1 - a_1 z}\frac{z - a_2}{1 - a_2 z}, \tag{14}$$

where $\theta \in \mathbb{S}$. The parameters $a_1, a_2$ are taken real and in $[0,1)$. Due to the symmetry it is enough to compute for $a_1 \le a_2$. It is also clear that if $\theta = \exp(2\pi\,\mathrm{i}\,\alpha)$ then it is sufficient to study the dynamics for $\alpha \in [0, 1/2]$.

Let $h_{a_1,a_2,\theta}$ be the entropy of the map $T_{a_1,a_2,\theta}$. It coincides with the Lyapunov exponent if this one is positive. Otherwise it is zero. We want to check the formula

$$\int_0^1 h_{a_1,a_2,\theta}d\alpha = \int_0^1 g_{a_1,a_2}(z)ds, \tag{15}$$

where $z = \exp(2\pi\,\mathrm{i}\,s)$ and the function $g_{a_1,a_2}$ is defined as

$$g_{a_1,a_2}(z) = \log^+(|T'_{a_1,a_2,\theta}(z)|) + |T'_{a_1,a_2,\theta}(z)|\log^-(|T'_{a_1,a_2,\theta}(z)|).$$

As usual, $\log^+$ is log if it is positive and zero otherwise, and $\log^-$ is log if it is negative and zero otherwise. It is clear that the value of $\theta \in \mathbb{S}^1$ in the previous formula plays no role.

To compute the Lyapunov exponent $\Lambda$ we take a random initial value of $s \in [0,1)$, compute $z$ and $T$ transient iterates. Then we compute up to a maximum of $N$ iterates as follows: Every $L$ iterates (after the transient) we estimate the

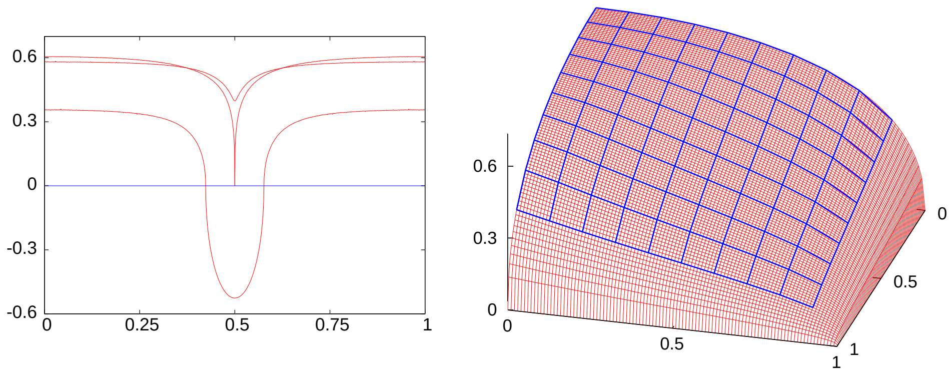


FIGURE 1. Left: Plots of the Lyapunov exponents for the maps (14) and values of $(a_1, a_2)$ equal to $(0.1, 0.6)$, $(0.2, 0.5)$ and $(0.3, 0.9)$ as a function of $\alpha \in [0, 1]$. The values at $\alpha = 0.5$ decrease from the first couple of parameters to the third one. Right: Values of the integrals appearing in (15) as a function of $(a_1, a_2)$. The integral in the right (resp. left) hand side is shown in a fine (resp. rough) grid. The agreement is clearly seen.

Lyapunov exponent. Let $\{\Lambda_j\}$ denote these estimates. When $j$ is a multiple of 4 the maximum of $|\Lambda_j - \Lambda_{j/2}|$ and $|\Lambda_j - \Lambda_{3j/4}|$ is computed. The value $\Lambda_j$ is accepted as Lyapunov exponent if this maximum is less than some tolerance $\eta$. If $N$ iterates are carried out without stopping the process, then the last estimate of $\Lambda_j$ is taken as Lyapunov exponent. Typical values for $T, N, L, \eta$ are in the ranges $10^5 - 10^6$, $10^7 - 10^8$, $10^4 - 10^5$, $10^{-5} - 10^{-4}$, respectively. Experimentally one needs to reach the maximal value of $N$ in around 1% of the cases.

Other approaches to estimate the Lyapunov exponents (see, e.g., [3] and references therein) have also been tested with similar results.

This has been used for values of $a_1, a_2$ in $[0, 0.9]$ with stepsize 0.1. It is clear that for $a_1 = a_2 = 0$ the value is $\log(2)$ and if $a_1 = 0$ the value is independent of $\alpha$. Figure 1 left shows the value of the Lyapunov exponent as a function of $\alpha$ for values of $(a_1, a_2)$ equal to $(0.1, 0.6)$, $(0.2, 0.5)$ and $(0.3, 0.9)$. The three curves can be easily identified because the values at $\alpha = 0.5$ decrease from the first couple of values to the third one.

The point $z = -1$ for $\theta = -1$ (i.e., $\alpha = 0.5$) is a fixed point. It becomes attracting when crossing the curve $3a_1a_2 + a_1 + a_2 = 1$ from left to right. That is, when $a_2 > a_2^*(a_1) = (1 - a_1)/(1 + 3a_1)$. The other fixed points do not play any role, because they are non attracting. Hence, from that value of $a_2 = a_2^*(a_1)$ on, the Lyapunov exponent is negative at $\alpha = 0.5$ (compare with the third curve in the figure).

Numerically one observes several phenomena. For $a_2 < a_2^*(a_1)$ the Lyapunov exponent is positive for all $\alpha$ while for $a_2 > a_2^*(a_1)$ it changes sign. For $a_2 = a_2^*(a_1)$ it is positive everywhere except at $\alpha = 0.5$, where it becomes zero.

It is worth to remark the behaviour of the Lyapunov exponent around the value of $\theta$ (or $\alpha$) where it becomes zero. Let $\alpha(a_1, a_2)$ be such that $\Lambda(a_1, a_2, \alpha(a_1, a_2)) =$

0, assuming $a_2 > a_2^*(a_1)$ and $\alpha(a_1, a_2) < 1/2$ (there is a symmetric point with $\alpha(a_1, a_2) > 1/2$). It is easy to obtain an explicit (but cumbersome) formula for $\alpha(a_1, a_2)$. Then, for nearby $\alpha$ the value of $\Lambda$ behaves as $c|\alpha - \alpha(a_1, a_2)|^{1/2}$, with $c > 0$ to the left and $c < 0$ to the right. The absolute value of these constants $c$, to the left and to the right of $\alpha(a_1, a_2) < 1/2$, is different.

Furthermore, in the critical case $a_2 = a_2^*(a_1)$, the local behaviour around $\alpha = 1/2$ is of the form $\Lambda = c|\alpha - 1/2|^{1/4}$ for some $c > 0$. See middle curve in Figure 1.

The integrals of the entropy, i.e., in the left hand side of (15), have been computed by evaluation of the Lyapunov exponent on a grid of stepsize $10^{-3}$ and use of the trapezoidal rule. It is clear that, due to the low regularity near $\alpha(a_1, a_2)$, the integrals become affected by a relatively large error when $a_2 \geq a_2^*(a_1)$. This is easy to correct, by splitting the integral in pieces, estimating the values of the different $c$ and carrying out analytically the integrals in a vicinity of the critical values of $\alpha$. But even without these improvements the results are quite remarkable. Figure 1 right shows, in a grid of stepsize 0.01, the numerical value of the integral in the right hand side of (15) using Simpson rule and extrapolation. Superimposed one can see the values of the integrals of the left hand side of (15) computed as described above in a grid of stepsize 0.1. The maximal observed difference has absolute value below $10^{-4}$. As expected, the largest values of the differences occur for points near the line $a_2 = a_2^*(a_1)$.

Now we pass to the numerical study of (1). In fact this numerical study preceded the statement of results in Section 1 and the proofs in Section 2. The behaviour is different from the Blaschke case. The map (14) can be put in the form $s \mapsto \psi_{a_1,a_2}(s) + \alpha$. Figure 2 shows the graphs of $\psi_{a_1,a_2}$ for the three couples of parameters of Figure 1 and the ones of $x \mapsto 2x + \varepsilon \sin(2\pi x)$ for the values $\varepsilon$ such that the derivatives coincide at the central point. From the graphs it is unclear how to understand the different behaviour of both families of maps.

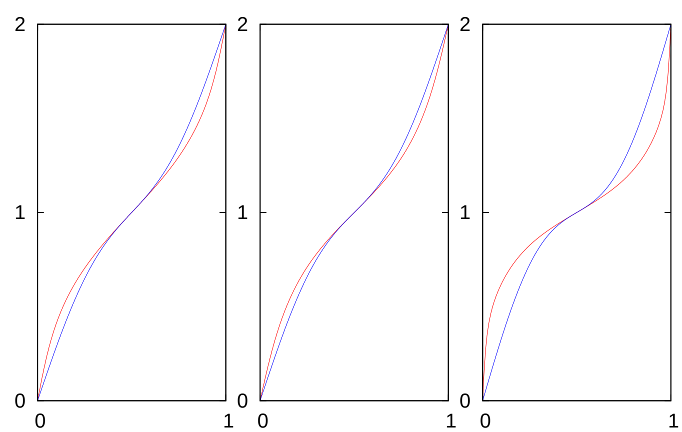


FIGURE 2. Graphs of the maps (14) corresponding to Figure 1 and $\alpha = 0$ (red) and of the maps (1) with the same derivative at the central point (blue).

As we shall be faced with very small values of $\Delta(k, \varepsilon)$ the proposed algorithm to compute $\Lambda$ is not accurate enough. For small $\varepsilon$ one can use expansions of the invariant measure as we did in Section 2. But we wanted to proceed in a purely numerical way to obtain independent checks.

To compute invariant measures Perron algorithm has been used for a lattice of values of $x$, computation of backwards iterates and transport of the measure. It has been required that the computed densities of the invariant measure at a given point, $\rho_{k,\alpha,\varepsilon}(x)$, in two successive iterates of the process (i.e., by going back $m$ times under $f$, with a total of $k^m$ preimages, and going back $m+1$ times under $f$, with a total of $k^{m+1}$ preimages) have differences which are less than some prescribed tolerance $\eta_m$. Then the computation of $h_{k,\alpha,\varepsilon}$ proceeds as in formula (2) by using $\rho_{k,\alpha,\varepsilon}(x)$ instead of $\hat{\rho}_{k,\varepsilon}(x)$. The integrations are done using Simpson rule and extrapolation until different estimates give differences bounded by $\eta_i$. The selection of the tolerances $\eta_m$ and $\eta_i$ depends on the values of $k$ and $\varepsilon$ and it is a delicate question if we want to obtain the desired accuracy in reasonable computing time.

For fixed $k$ and $\varepsilon$ we can look for the average measure $\hat{\rho}_{k,\varepsilon}(x)$. Figure 3 shows, on the left, the densities (in the vertical variable) for given values of $\alpha$ (tics spaced by 0.2) and as a function of $x$ (tics spaced 0.25). The plot corresponds to $k = 2, \varepsilon = 0.05$. On the right part we show the isodensity curves, for values of the density between 0.97 and 1.03 with step 0.005.

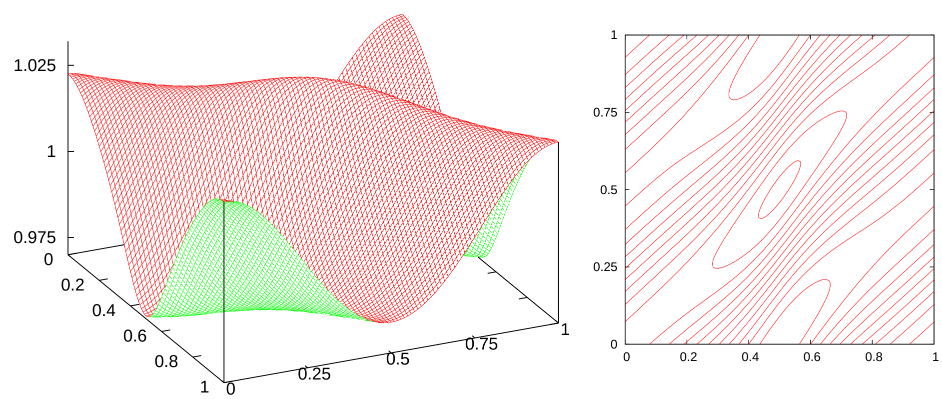


FIGURE 3. The density $\hat{\rho}_{k,\varepsilon}(x)$ for $k = 2, \varepsilon = 0.05$ and isodensity curves. See the text for details.

We can average now with respect to $\alpha$ for given $x$. Figure 4 shows the results for $k = 2$ and $\varepsilon = 0.05, 0.06$ and 0.07. A Fourier analysis of these average measures, $\hat{\rho}_{k,\varepsilon}(x)$, shows that they are even functions of $x$ and that the harmonic of order $j$ scales as $\varepsilon^{2+3j}$ in perfect agreement with Theorem 1.

As one can expect, if we compute $I(k, \varepsilon)$ but using this average measure, the result coincides with $J(k, \varepsilon)$ within the current tolerances. This is used as additional check.

Keeping the analysis restricted to the case $k = 2$ we see that the difference of densities between the Lebesgue measure and the average measure $\hat{\rho}_{k,\varepsilon}$ has, as dominant term, a harmonic of the form $c\varepsilon^5 \cos(2\pi x)$, where $c$ is a negative constant. If we multiply by $\log |Df_{k,\alpha,\varepsilon}(x)| = \log(2) + \pi\varepsilon \cos(2\pi x) + \mathcal{O}(\varepsilon^2)$, the terms which do not average to zero are $\mathcal{O}(\varepsilon^6)$, in agreement with the previous results. This has

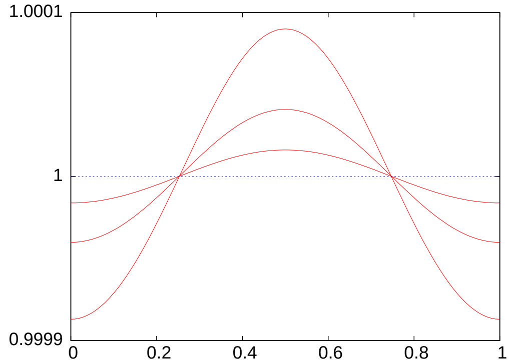


FIGURE 4. The average density $\hat{\rho}_{k,\varepsilon}(x)$ for $k = 2$ and $\varepsilon = 0.05, 0.06, 0.07$.

inspired the proof of Theorem 2. For arbitrary $k \geq 2$ when we take the average of $|Df_{k,\alpha,\varepsilon}(x)|$ over the corresponding preimages to obtain the density at $x$ and then average with respect to $\alpha$, the terms up to order $2k$ in $\varepsilon$ average to zero, but not next one.

The computed values of $h_{k,\alpha,\varepsilon}$ for $k = 2$ as a function of $(\alpha, \varepsilon)$ are displayed in Figure 5 which has been obtained from direct computation of the Lyapunov exponent. Values of $\varepsilon \in [0, 0.1]$ have been used. For $\varepsilon = 0.1$ the dependence with respect to $\alpha$ is clearly seen, while for small values of $\varepsilon$ it is hard to see any variation as a function of $\alpha$ on this scale.

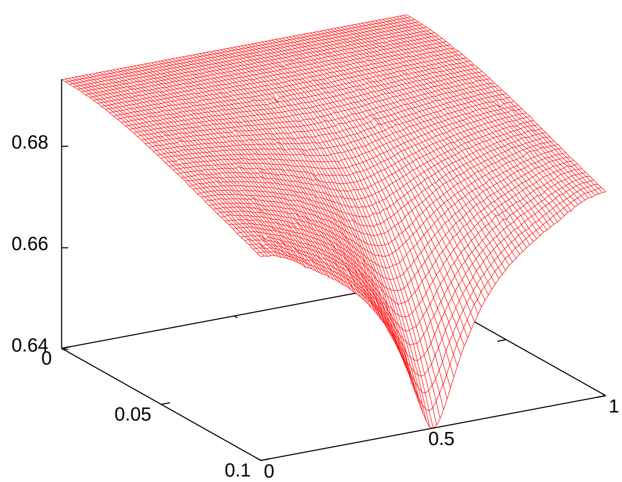


FIGURE 5. Values of the entropy $h_{2,\alpha,\varepsilon}$ as a function of $\alpha \in [0, 1], \varepsilon \in [0, 0.1]$.

From values of $\hat{\rho}_{k,\varepsilon}$, computed as described before, the values of $I(k, \varepsilon)$ and then $\Delta(k, \varepsilon)$ can be obtained. Figure 6 shows the behaviour of $\Delta(k, \varepsilon)$ as a function of $\varepsilon$ for different values of $k$. We use logarithmic scales in both axes. From left to right the plots correspond to $k = 2, \dots, 10$. Fitting by lines one finds that the slopes are of the form $2k+2$ as proved in corollary 1. Note that the range of values of $\Delta(k,\varepsilon)$ is quite wide (roughly from $10^{-14}$ to $10^{-4}$. We recall that map (1) is expanding if $|\varepsilon| < \varepsilon^*(k) = \dfrac{k-1}{2\pi}$. For every $k \in [2,10]$ the last value of $\varepsilon$ shown in Figure 6 largely exceeds $\frac{1}{2}\varepsilon^*(k)$. The numerical evidence is that $\Delta(k,\varepsilon)$ is always positive, and not only for small $\varepsilon$.

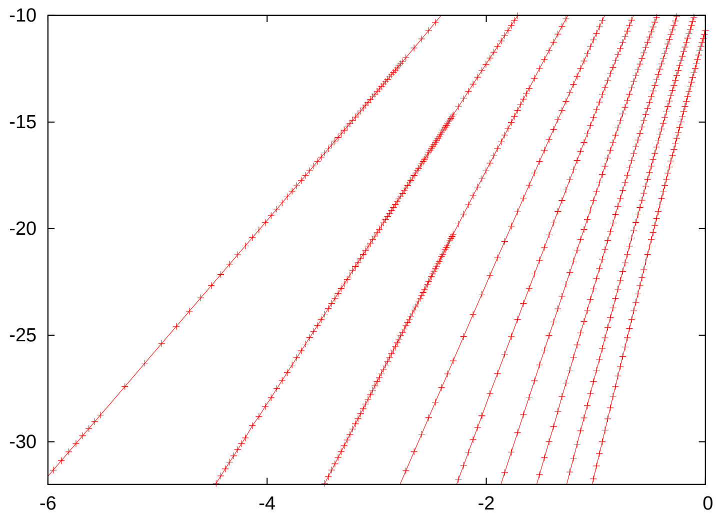


FIGURE 6. Values of $\hat{\rho}_{k,\varepsilon}$ for $k = 2,\ldots,10$, from left to right, as functions of $\varepsilon$. Logarithmic scales are used in both axes.

Finally, to give a numerical evidence that the results of Theorem 2 are also valid for large values of $\varepsilon$ we display, in Figure 7, values of $h_{k,\alpha,\varepsilon}$ as a function of $\alpha$. Maxima occur for $\alpha = 0$ (resp. $\alpha = 1/2$) if $k$ is even (resp. odd), in agreement with the case $\varepsilon$ small as given by the theorem. The figure also shows that the difference $\max_{\alpha\in\mathbb{S}^1}\{h_{k,\alpha,\varepsilon}\} - J(k,\varepsilon)$ is much larger than $\Delta(k,\varepsilon)$.

To have a more global view we have computed $J(k,\varepsilon)$ and $\max_{\alpha\in\mathbb{S}^1}\{h_{k,\alpha,\varepsilon}\}$ for $k = 2,\ldots,10$ and $\varepsilon$ between 0 and $\varepsilon^*(k)$. The values are shown in Figure 8. In all cases the maximum exceeds the value of $J$. Note that with the scale used in the plot the function $I(k,\varepsilon)$ can not be distinguished from $J(k,\varepsilon)$. We recall that beyond $\varepsilon^*(k)$ the maps are no longer everywhere expanding.

## REFERENCES


[1] Brouwer, D. and Clemence, G.M., *Methods of Celestial Mechanics*, Academic Press, 1961.

[2] Dedieu,J.P. and Shub,M., On random and mean exponents for unitarily invariant probability measures on GL(n, **C**), in "Geometric Methods in Dynamical Systems (II)-Volume in Honor of Jacob Palis", *Astérisque*, **287** (2003) 1–18, Soc. Math. de France.

[3] Ledrappier,F. Shub,M., Simó,C.,and Wilkinson,A., Random versus deterministic exponents in a rich family of diffeomorphisms, *Journal of Statistical Physics*, **113** (2003), 85–149.

[4] Robert,L., Pujals,E. and Shub,M., Expanding maps of the circle revisited: Positive Lyapunov exponents in a rich family, *Ergodic Theory & Dynamical Systems* **26** (2006), 1931–1937.


*E-mail address*: llave@math.utexas.edu; michael.shub@utoronto.ca; carles@maia.ub.es

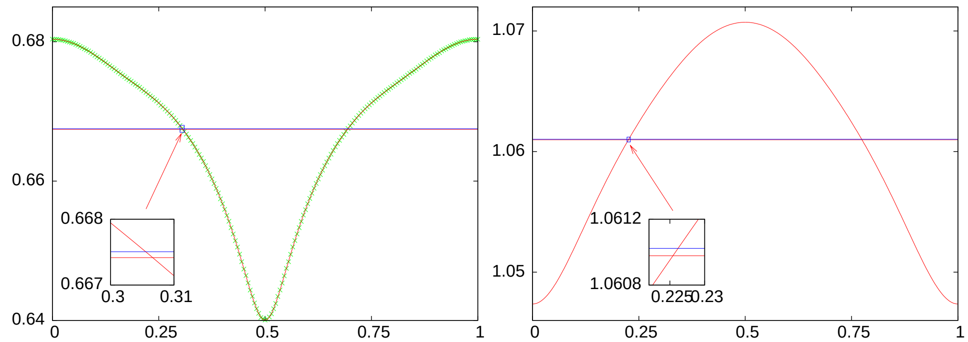


FIGURE 7. Plots of $h_{k,\alpha,\varepsilon}$ as a function of $\alpha$ showing also the values of $J(k,\varepsilon)$ (upper horizontal line) and $I(k,\varepsilon)$ (lower horizontal line). To see the differences $\Delta(k,\varepsilon)$ between both lines a magnification is displayed. Left plot corresponds to $k=2, \varepsilon=0.1$. Right one to $k=3, \varepsilon=0.18$. The values of $h_{k,\alpha,\varepsilon}$ shown as continuous lines have been computed using the invariant measure. In the left plot the values marked as points (in green) have been computed directly from estimates of $\Lambda$.

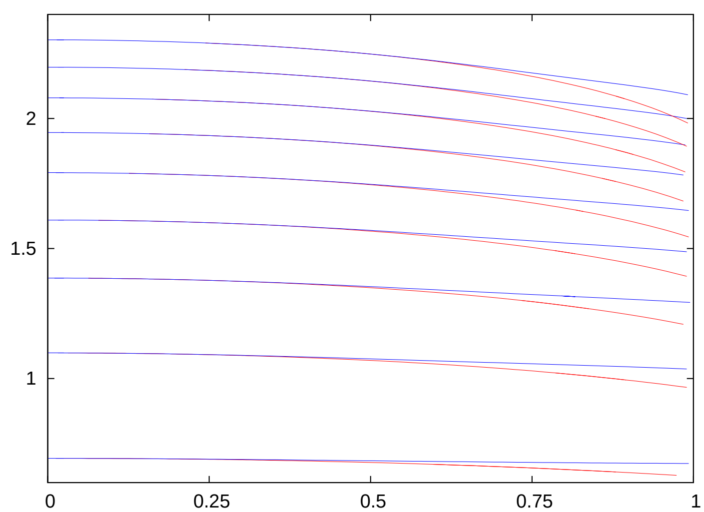


FIGURE 8. Plots of $J(k,\varepsilon)$ (in red) and $\max_{\alpha\in\mathbb{S}^1}\{h_{k,\alpha,\varepsilon}\}$ (in blue) for $k=2,\ldots,10$, from bottom to top. In the horizontal axis we use the variable $\varepsilon/\varepsilon^*(k)$.