*Ergod. Th. & Dynam. Sys.* (2006), **26**, 1931–1937
doi:10.1017/S0143385706000368


# Expanding maps of the circle rerevisited: positive Lyapunov exponents in a rich family

ENRIQUE R. PUJALS†, LEONEL ROBERT‡ and MICHAEL SHUB‡

† *IMPA Estrada Dona Castorina 110, Rio de Janeiro, Brazil 22460-320*
*(e-mail: enrique@impa.br)*
‡ *Math Dept, University of Toronto, 40 St. George Street, Toronto, ON M5S 2E4, Canada*
*(e-mail: lrobert@math.toronto.edu, michael.shub@utoronto.ca)*

(*Received* 5 *July* 2005 *and accepted in revised form* 2 *May* 2006)

*Abstract*. In this paper we revisit once again, see Shub and Sullivan (*Ergod. Th. & Dynam. Sys.* **5** (1985), 285–289), a family of expanding circle endomorphisms. We consider a family $\{B_\theta\}$ of Blaschke products acting on the unit circle, $\mathbb{T}$, in the complex plane obtained by composing a given Blaschke product $B$ with the rotations about zero given by multiplication by $\theta \in \mathbb{T}$. While the initial map $B$ may have a fixed sink on $\mathbb{T}$, there is always an open set of $\theta$ for which $B_\theta$ is an expanding map. We prove a lower bound for the average measure theoretic entropy of this family of maps in terms of $\int \ln |B'(z)|\, dz$.

## 1. *Introduction*

Several papers have suggested the possibility of giving lower bounds for the average entropy or Lyapunov exponents in a rich enough family of dynamical systems [**BPSW, LSSW**]. A particular consequence would establish the existence of positive entropy for a positive measure set of parameters in terms of comparatively easily computable quantities. A linear algebra analogue is proven in [**DS**]. In this paper we accomplish the task for families of (finite) Blaschke products. In these families it is fairly easy to establish the existence of positive measure sets of parameters which define expanding maps of the circle. Here we give a lower bound for the average entropy of these expanding maps with respect to the natural invariant measures which are absolutely continuous with respect to Lebesgue measure.

A (finite) Blaschke product is a map of the form

$$B(z) = \theta_0 \prod_{i=1}^{n} \frac{z - a_i}{1 - z\overline{a_i}}$$

where $n \geq 2$, $a_i \in \mathbb{C}$, $|a_i| < 1$ , $i = 1, \ldots, n$, and $\theta_0 \in \mathbb{C}$ with $|\theta_0| = 1$. $B$ is a rational mapping of $\mathbb{C}$, it is an analytic function in a neighborhood of the unit disc $\mathbb{D}$, and $B$ maps the unit circle $\mathbb{T}$ to itself. In this paper we consider the family of Blaschke products,

$$\{B_\theta\}_{\{\theta\in\mathbb{T}\}} = \{\theta B\}_{\{\theta\in\mathbb{T}\}}.$$

THEOREM 1.1. *Given a family of Blaschke products* $\{B_\theta\}_{\{\theta\in\mathbb{T}\}}$, *one of the next two options holds for any* $\theta \in \mathbb{T}$:

(1) $B_\theta$ *is an expanding map, i.e. there are* $n = n(\theta)$, *and* $\lambda = \lambda(\theta) > 1$ *such that*

$$|B_\theta^{n\prime}(x)| > \lambda;$$

(2) $B_\theta$ *has a unique attracting or indifferent fixed point in* $\mathbb{T}$.

*Moreover, the set of* $\theta \in \mathbb{T}$ *satisfying the first option is a non-empty open set.*

In the next theorem, we relate the previous option with the statistical behaviour of $B_\theta$. Let $\lambda$ be Lebesgue measure on $\mathbb{T}$ normalized to be a probability measure, $\lambda(\mathbb{T}) = 1$.

THEOREM 1.2. *Given a family of Blaschke products* $\{B_\theta\}_{\{\theta\in\mathbb{T}\}}$ *it follows that, for all* $\theta$, *the push forwards of Lebesgue measure* $B^n_{\theta\star}(\lambda)$ *converge to a measure* $\mu_\theta$ *which is:*

(1) *absolutely continuous with respect to Lebesgue if* $B_\theta$ *satisfies condition (1) of Theorem 1.1; or*

(2) *a Dirac delta measure supported on an attracting or indifferent fixed point of* $B_\theta$ *on* $\mathbb{T}$.

As a consequence of Theorem 1.2, it follows that, for any $\theta$ for which $B_\theta$ has an absolute continuous invariant measure, we can define the metric entropy, $h_\theta$, of $B_\theta$ with respect to $\mu_\theta$ and it satisfies

$$h_\theta = \int_{\mathbb{T}} \ln |B'(z)|\, d\mu_\theta.$$

In the next theorem we give a lower bound for the average measure theoretic entropy of this family of maps in terms of $\int \ln |B'(z)|\, dz$.

THEOREM 1.3. *Given a family of Blaschke products* $\{B_\theta\}_{\{\theta\in\mathbb{T}\}}$ *it follows that:*

(a)

$$\int h_\theta\, d\theta \geq \int_{\mathbb{T}} \ln |B'(z)|\, dz$$

*with equality if and only if* $|B'(z)| \geq 1$ *for all* $z \in \mathbb{T}$;

(b) *more precisely,*

$$\int h_\theta\, d\theta = \int_{\mathbb{T}} \ln^+ |B'(z)|\, dz + \int_{\mathbb{T}} |B'(z)| \ln^- |B'(z)|\, dz.$$

Here $\ln^+$ equals ln when it is positive and zero otherwise, while $\ln^-$ equals ln when it is negative and zero otherwise. When $h_\theta$ is positive it equals the Lyapunov exponent of $B_\theta$ with respect to $\mu_\theta$, i.e. for almost every point with respect to Lebesgue measure,

$$h_\theta = \lim_{n\to+\infty} \frac{1}{n} \ln |B^{n\prime}(z)|.$$

So we could equally well state our results with respect to Lyapunov exponents. The quantity $\int_{\mathbb{T}} \ln |B'(z)|\, dz$ is easily seen to be the Lyapunov exponent of the random product of elements of the family $\{B_\theta\}_{\{\theta\in\mathbb{T}\}}$. The inequality in part (a) of Theorem 1.3 proves that the mean of the deterministic exponents is greater than or equal to the random exponent. So we have achieved here in dimension one the unachieved goal of [**LSSW**] in dimension two.

Most of the proofs of the previous theorems could be assembled from results already in the literature. We give alternate largely self-contained proofs in the following sections. The proofs consist of three parts:

(1) for all $\theta$, Theorem 1.1 or 1.2 holds;

(2) $\int B^n_{\theta\star}(\lambda)\, d\theta = \lambda$ for all $n$;

(3) let $\phi : \mathbb{T} \to \mathbb{R}$ be continuous, then $\int_{\mathbb{T}} \phi\, d\lambda = \int \big( \int_{\mathbb{T}} \phi\, d\mu_\theta(z)\big)\, d\theta$.

The proofs are completed by applying (3) to $\ln |B'(z)|$, applying (1) and (2), and changing variables for those $\theta$ for which $\mu_\theta$ is supported on a contracting or indifferent fixed point. This proves (b) and (a) follows. The proofs are carried out in detail in the following sections.

## 2. *The fixed points of B*

For any Blaschke product $B$ as above, the equation $z = B(z)$ has at most $n+1$ zeros in the complex plane, $\mathbb{C}$. So $B : \mathbb{C} \to \mathbb{C}$ has at most $n+1$ fixed points in $\mathbb{C}$. The map $B : \mathbb{T} \to \mathbb{T}$ has degree $n$. By the Lefschetz formula $B$ has $-(n-1)$ fixed points counted with index on $\mathbb{T}$. Thus $B$ has at least $(n-1)$ expanding fixed points on $\mathbb{T}$ and at most $(n+1)$ fixed points in all.

PROPOSITION 2.1. *One of the following three mutually exclusive cases holds.*

(a) *$B$ has all its fixed points on $\mathbb{T}$. There is exactly one of them, $z_0$, that is a sink and the other $n$ are expanding.*

(b) *$B$ has $n-1$ fixed points on $\mathbb{T}$, all expanding. It has one fixed point inside the disc which is a sink, and one outside. These two fixed points are related by the formula $z_0 \to 1/\overline{z_0}$ (hence, they lie on the same ray passing through the origin).*

(c) *$B$ has all its fixed points on $\mathbb{T}$. There is one that is an indifferent saddle node fixed point, $B(z_0) = z_0$ and $B'(z_0) = 1$.*

*In all three cases there is an open set of points in the disc which tend to $z_0$ under iteration of $B$.*

*Proof.* If $B'(z) \neq 1$ for all fixed points of $B$ on the circle, by the Lefschetz formula we can have $n$ expanding fixed points and one sink on the circle, or $n-1$ expanding points on the circle. In the first case we are in situation (a). In the second case, since $B(z)\overline{B(\overline{z}^{-1})} = 1$ for all $z \in \mathbb{C}$, we must be in case (b). The fact that the fixed point in the interior of the disc is attracting follows from direct calculation or the Schwarz lemma. Case (c) represents the remaining cases. □

### 2.1. *Iterates of B.*

The sequence $B^n(z)$ is uniformly bounded in the unit disc (i.e. it is a normal family). Let $z_0$ be the attracting or indifferent fixed point in Proposition 2.1.

Observe that a sink or an indifferent fixed point of a rational mapping of $\mathbb{C}$ always attracts an open set of points. Therefore, there is an open set of points in which $\{B^n(z)\}_n$ converges uniformly to $z_0$. Thus, by Vitali's convergence theorem the sequence $\{B^n(z)\}_n$ converges uniformly on compact sets of the open unit disc to $z_0$. Thus, $B^n(z) \to z_0$ for any $z$ in the open unit disc. Incidentally, this proves that the fixed point $z_0$ described in Proposition 2.1 is unique in the closed unit disc as an attracting or indifferent fixed point.

2.2. *B composed with rotations.* We now consider the one-parameter family of functions $B_\theta = \theta B$. Our main interest will be when $\theta$ goes around the circle, but we will also consider $c$ taking values in the disc, $\mathbb{D}$.

For every $\theta$ consider the set of fixed points of $B_\theta$. As $\theta$ goes around the circle the fixed points of $B_\theta$ will be in situations (a), (b) or (c) described before. Case (c) will happen at most a finite number of times. For every $\theta \in \mathbb{T}$ we define $\alpha(\theta)$ as the unique sink of $B$ if we are in situations (a) or (b). In case (c), $\alpha(\theta)$ is the unique indifferent fixed point of $B_\theta$ (but in fact this case is irrelevant for our ultimate discussion because it is measure zero in the parameter). For all $z_0 \in \mathbb{T}(\mathbb{C})$ such that $|B'(z_0)| \leq 1$, there is one value of $\theta$ (namely $\theta = z_0/B(z_0)$) such that $z_0$ is a fixed sink or indifferent point of $B_\theta$. Thus, all these values belong to the range of $\alpha$. Finally, if $|c| < 1$ we define $\alpha(c)$ as the unique fixed point of $B_c$ inside the unit disc.

PROPOSITION 2.2. *The function $\alpha$ is analytic in the open unit disc and continuous in the closed unit disc.*

*Proof.* By the implicit function theorem the attracting fixed points of $B_\theta$ vary analytically with $\theta$ in the closed disc minus the finite set of $\theta$ for which $B_\theta$ has an indifferent fixed point in $\mathbb{T}$, the values of which provide a continuous extension of the function. □

The next corollary is an obvious extension of our discussion of iterates to $B_c$ for $c \in \mathbb{D}$.

COROLLARY 2.3. *Let $z_0$ be inside the open unit disc and $c$ in the closed disc. Then $B_c^n(z_0)$ converges to $\alpha(c)$.*

3. *Expanding maps and proof of Theorem 1.1*

PROPOSITION 3.1. *If $\theta_0 \in \mathbb{T}$ and $\alpha(\theta_0)$ is in the open unit disc, then there is an $n > 0$ such that $|B_{\theta_0}^{n\prime}(z)| > 1$ for all $z \in \mathbb{T}$. That is, $B_{\theta_0}$ is expanding.*

*Proof.* Suppose $z_0$ is a fixed point of $B_{\theta_0}$ inside the disc. Let $C_r$ be a disk of radius $r$, $r < 1$, and center 0 that contains $z_0$. Since $B_{\theta_0}^n$ converges uniformly to $z_0$, there is some $n$ such that $B_{\theta_0}^n(C_r) \subset C_r$. This implies that $\theta B_{\theta_0}^n$ has a fixed point in $C_r$ for all $\theta \in \mathbb{T}$. This means that $B_{\theta_0}^n$ never has an attracting or indifferent fixed point on the unit circle; hence, the set $\{z \in \mathbb{T} : |B_{\theta_0}^{n\prime}(z)| \leq 1\}$ is empty. □

Observe that this finishes the first part of Theorem 1.1. In fact, if the attracting fixed point of $B_\theta$ is in the open unit disc then the map is expanding; if not, it has a unique fixed point in the circle which is either attracting or an indifferent saddle-node point. Now we proceed to finish the proof of Theorem 1.1.

*Proof of Theorem 1.1.* By Proposition 3.1 it is enough to show that there exists $\theta_0$ such that $\alpha(\theta_0)$ is in the open unit disc. Let us assume that there is an $x_0$ such that $|B'(x_0)| = 1$ (otherwise, the thesis of the theorem holds for every $\theta \in \mathbb{T}$). Therefore, there exists $\theta_0$ such that $B_{\theta_0}(x_0) = x_0$ and so $x_0$ is an indifferent saddle node. This implies that there is an $\epsilon_0 > 0$ and an open interval $J_0$ in $\mathbb{T}$ containing $x_0$ such that either, for every $\theta \in (\theta_0, \theta_0 + \epsilon_0)$, $B_\theta$ does not have a fixed point in $J_0$ and, for every $\theta \in (\theta_0 - \epsilon_0, \theta_0)$, $B_\theta$ has a sink in $J_0$, or, for every $\theta \in (\theta_0 - \epsilon_0, \theta_0)$, $B_\theta$ does not have a fixed point in $J_0$ and, for every $\theta \in (\theta_0, \theta_0 + \epsilon_0)$, $B_\theta$ has a sink in $J_0$. Let us assume that the first option holds. To conclude the theorem, it is enough to show that there exists $\epsilon_1$ such that for, every $\theta \in (\theta_0, \theta_0 + \epsilon_1)$, $B_\theta$ does not have a sink or an indifferent fixed point in the complement of $J_0$. If not, there is a sequence $\theta_n \to \theta_0$ such that $B_{\theta_n}$ has a sink or indifferent fixed point contained in $J_0^c$. But then so does $B_{\theta_0}$, which contradicts the uniqueness of the fixed point $x_0$ among indifferent or attracting fixed points of $B_{\theta_0}$. □

## 4. *Push forwards of Lebesgue measure. Proof of Theorems 1.2 and 1.3*

If $B$ has a fixed point $z_0$ on the circle then the Dirac measure, $\mu_{z_0}$, corresponding to that point is left invariant by $B$. Given a point $z_0$ in the interior of the unit disc, we let $\mu_{z_0}$ denote the absolutely continuous measure on the circle $\mathbb{T}$ defined in any of three equivalent ways.

- Let $h : \mathbb{T} \to \mathbb{C}$ be continuous and let $\tilde{h}$ be its harmonic extension to the disc. Then $\int_{\mathbb{T}} h\, d\mu_{z_0} = \tilde{h}(z_0)$.
- Let $\int_{\mathbb{T}} h\, d\mu_{z_0} = \int_{\mathbb{T}} h P_{z_0}\, d\lambda$, where $P_{z_0}$ is the Poisson kernel and $\lambda$ is the Lebesgue measure.
- Let $A_{z_0}$ be a fractional linear transformation mapping 0 to $z_0$ that preserves the unit disc. Then $\mu_{z_0} = A_{z_0\star}(\lambda)$.

PROPOSITION 4.1. *Let $B$ be a Blaschke product. Then $B_\star(\mu_{z_0}) = \mu_{B(z_0)}$. Thus, if $B$ has a fixed point $z_0$ inside the disc then the absolutely continuous measure given by $\mu_{z_0}$ is left invariant by $B$.*

*Proof.* Let $h : \mathbb{T} \to \mathbb{C}$ be continuous and let $\tilde{h}$ be its harmonic extension to the disc. Then $\int_{\mathbb{T}} h\, dB_\star(\mu_{z_0}) = \int_{\mathbb{T}} h \circ B\, d(\mu_{z_0}) = \widetilde{h \circ B}(z_0)$. Since $B$ is analytic, $\tilde{h} \circ B$ is harmonic, and thus $\widetilde{h \circ B}(z_0) = \tilde{h} \circ B(z_0) = \int_{\mathbb{T}(\mathbb{C})} h\, d\mu_{B(z_0)}$. □

For every $c \in \mathbb{D}$ we write $\nu_c = \mu_{\alpha(c)}$. Then $|\alpha(c)| < 1$ if and only if $\nu_c$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ and for $\theta \in \mathbb{T}$ it follows that $|\alpha(\theta)| < 1$ if and only if $B_\theta$ is expanding. If $|\alpha(\theta)| = 1$ then $\nu_\theta$ is the Dirac measure supported on $\alpha(\theta)$.

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* By Corollary 2.3, $B_\theta^n(0)$ converges to $\alpha(\theta)$. It follows that $B^n_{\theta\star}(\lambda)$ converges to the measure $\nu_\theta$ defined above. When $B_\theta$ is expanding, then $\nu_\theta$ is absolutely continuous with respect to Lebesgue, and $h_\theta = \int_{\mathbb{T}} \ln |B_\theta'(z)|\, d\nu_\theta = \int_{\mathbb{T}} \ln |B'(z)|\, d\nu_\theta$ (see [**L**]). In the case that $B_\theta$ has an attracting or indifferent fixed point, it follows that the push forward converges to a Dirac measure supported on this point. □

*Remark 4.2.* Theorem 1.2 has a version for $C^2$ dynamical systems which we could have used here with a little work (see [**M**]).

Now we prove item (2) of the proof of the theorems as mentioned in the introduction.

PROPOSITION 4.3. $\int_{\mathbb{T}} B^n_{\theta\star}(\lambda)\, d\theta = \lambda$ *for all* $n$.

*Proof.* For any continuous function $h : \mathbb{T} \to \mathbb{R}$, $\int\int_{\mathbb{T}} h\, dB^n_{\theta\star}(\lambda)\, d\theta = \int \tilde{h}(B^n_\theta(0))\, d\theta$ and, since the map $c \to B^n_c(0)$ is an analytic function of $c$ in the unit disc and at $c = 0$, $B^n_c(0) = 0$, it follows that $\int \tilde{h}(B^n_\theta(0))\, d\theta = \tilde{h} \circ B^n_0(0) = \tilde{h}(0)$. □

*Remark 4.4.* Proposition 4.3 can also be proved by Fourier series as was done in [**LSSW**, §§4.11 and 4.12].

PROPOSITION 4.5. *Let* $\phi : \mathbb{T} \to \mathbb{R}$ *be continuous. Then* $\int_{\mathbb{T}} \phi\, d\lambda = \int \big(\int_{\mathbb{T}} \phi\, d\nu_\theta(z)\big)\, d\theta$.

*Proof.* By the Lebesgue dominated convergence theorem,

$$\int \bigg(\int_{\mathbb{T}} \phi\, d\nu_\theta(z)\bigg)\, d\theta = \lim \int \bigg(\int_{\mathbb{T}} \phi\, dB^n_{\theta\star}(\lambda)\bigg)\, d\theta = \int_{\mathbb{T}} \phi\, d\lambda. \qquad \square$$

Now we proceed to give the proof of Theorem 1.3.

*Proof of Theorem 1.3.* We consider the sets $\mathbb{T}_l = \{\theta \in \mathbb{T} \mid \nu_\theta \text{ is absolutely continuous}\}$, $\mathbb{T}_d = \{\theta \in \mathbb{T} \mid \nu_\theta \text{ is Dirac}\}$ and $\mathbb{T}_a = \{z \in \mathbb{T} \mid |B'(z)| \leq 1\}$:

$$\begin{aligned}\int_{\mathbb{T}} \ln|B'(z)|\, d\lambda &= \int_{\mathbb{T}_l} \ln|B'(z)|\, d\nu_\theta\, d\theta + \int_{\mathbb{T}_d} \ln|B'(\alpha(\theta))|\, d\theta \\ &= \int_{\mathbb{T}_l} h_\theta\, d\theta + \int_{\mathbb{T}_a} (1 - |B'(z)|) \ln|B'(z)|\, d\lambda\end{aligned}$$

where this last equality follows from the fact that $d\theta = (1 - |B'(z)|)\, d\lambda$. Finally, subtract the last term on the right from the term on the left to prove the theorem. □

## 5. *Remarks, questions and conclusions*

We have given a lower bound and exact integral estimates for the average entropy of a family of Blaschke products with respect to the SRB measures determined by iterates of members of the family. In [**LSSW**], similar estimates for a family of diffeomorphisms of the sphere were discussed, but nothing positive was proven for deterministic products as were considered here. The success with Blaschke products suggests other families of examples.

(1) What about the family $\theta f$ where $f$ is an immersion of the circle of finite smoothness, or even a $C^r$ topological covering with a cubic singularity?

(2) What about similar estimates for the quadratic family of maps of the unit interval, normalized to have the unit interval as the image? Is there a meaningful measure on the space of parameters which is absolutely continuous with respect to Lebesgue and for which positive estimates of the mean entropy can be relatively easily proven?

(3) Let $A_1$, $A_2$, and $A_3$ be fractional linear transformations of the unit disc. Let $(\theta, \psi) \in \mathbb{T} \times \mathbb{T}$ and consider the family of diffeomorphisms of the two torus $\mathbb{T} \times \mathbb{T}$ defined by

$$B_{\theta,\psi}(w, z) = (\theta A_1(w) A_2(w) \psi A_3(z), \theta A_2(w) \psi A_3(z)).$$

Then these diffeomorphisms are all isotopic to the usual linear Anosov diffeomorphism of the two torus, which is usually written additively (in the Anosov case, $A_1(w) = A_2(w) = w$ and $A_3(z) = z$). Can one estimate the average entropy of SRB measures associated to this family of diffeomorphisms? Is the set of $(\theta, \psi)$ for which $B_{\theta,\psi}$ is Anosov non-empty? Is the set of $(\theta, \psi)$ for which $B_{\theta,\psi}$ has an SRB measure of positive entropy of positive measure?

(4) Our theorem involves a probability measure $\mu$ on a space of parameters $P$ of dynamical systems of a manifold M with a probability measure $\nu$. What can be said about the existence of measures satisfying: for almost all $p \in P$, $\lim[(1/n) \sum f^j_{p\star}(\nu)]$ converges to a measure $\nu_p$ and $\lim[(1/n) \sum \int f^j_{p\star}(\nu)\, d\mu] = \nu$ or, even as in item (2) of the introduction, that $\int f^n_{p\star}(\nu)\, d\mu = \nu$ for all $n$?

*Acknowledgement.* Research partially supported by an NSERC Discovery Grant.

## REFERENCES

**[BPSW]** K. Burns, C. Pugh, M. Shub and A. Wilkinson. Recent results about stable ergodicity. *Smooth Ergodic Theory and Its Applications (Proceedings of Symposia in Pure Mathematics, 69).* Eds. A. Katok, R de la Llave, Y. Pesin and H. Weiss. American Mathematical Society, Providence, RI, 2001, pp. 327–366.

**[DS]** J. P. Dedieu and M. Shub. On random and mean exponents for unitarily invariant probability measures on GL($n$, **C**). *Geometric Methods in Dynamical Systems (II)—Volume in Honor of Jacob Palis, Asterisque* **287** (2003), 1–18.

**[L]** F. Ledrappier. Some properties of absolutely continuous invariant measures on an interval. *Ergod. Th. & Dynam. Sys.* **1** (1981), 77–93.

**[LSSW]** F. Ledrappier, M. Shub, C. Simó and A. Wilkinson. Random versus deterministic exponents in a rich family of diffeomorphisms. *J. Statist. Phys.* **113** (2003), 85–149.

**[M]** R. Mañé. Hyperbolicity, sinks and measure in one-dimensional dynamics. *Comm. Math. Phys.* **100**(4) (1985), 495–524.

**[ShSu]** M. Shub and D. Sullivan. Expanding endomorphisms of the circle revisited. *Ergod. Th. & Dynam. Sys.* **5** (1985), 285–289.