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Implicit Gamma Theorems (I): Pseudoroots and Pseudospectra

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Abstract. Let $g\colon \mathbb{E} \to \mathbb{F}$ be an analytic function between two Hilbert spaces $\mathbb{E}$ and $\mathbb{F}$. We study the set $g(B(x, \varepsilon)) \subset \mathbb{F}$, the image under g of the closed ball about $x \in \mathbb{E}$ with radius ε. When $g(x)$ expresses the solution of an equation depending on x, then the elements of $g(B(x, \varepsilon))$ are ε-pseudosolutions. Our aim is to investigate the size of the set $g(B(x, \varepsilon))$. We derive upper and lower bounds of the following form:

$$g(x) + Dg(x)(B(0, c_1\varepsilon)) \subseteq g(B(x, \varepsilon)) \subseteq g(x) + Dg(x)(B(0, c_2\varepsilon)),$$

where $Dg(x)$ denotes the derivative of g at x. We consider both the case where g is given explicitly and the case where g is given implicitly. We apply our results to the

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implicit function associated with the evaluation map, namely the solution map, and to the polynomial eigenvalue problem. Our results are stated in terms of an invariant γ which has been extensively used by various authors in the study of Newton's method. The main tool used here is an implicit γ theorem, which estimates the γ of an implicit function in terms of the γ of the function defining it.

1. Introduction and Main Results

This work is motivated by the computation of lower and upper bounds for the pseudospectra of matrices or, more generally, of matrix polynomials

$$P(A, \lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0, \tag{1.1}$$

where $A = (A_0, A_1, \ldots, A_m)$ is an $(m+1)$-tuple of complex matrices. The ε-pseudospectrum of such a problem is the set of complex numbers λ that are eigenvalues of the perturbed matrix polynomial $P(A+E, \lambda)$ where the perturbation $E = (E_0, E_1, \ldots, E_m)$ is such that $\|E\| \leq \varepsilon$.

More generally, we consider an analytic function g between two Hilbert spaces $\mathbb{E}$ and $\mathbb{F}$,

$$g\colon \mathbb{E} \to \mathbb{F}.$$

$\mathbb{E}$ is viewed as a set of inputs and $\mathbb{F}$ as the set of outputs for a certain problem and $g(x)$ is the solution or output associated with the input $x \in \mathbb{E}$. The question of pseudospectrum may be generalized in this context as follows: we call an ε-pseudosolution any element of the set $g(B(x, \varepsilon))$, the image under g of the closed ball $B(x, \varepsilon)$ about x with radius ε. To estimate the size of this set a first approach is to consider its first-order approximation

$$g(B(x, \varepsilon)) \simeq g(x) + Dg(x)(B(0, \varepsilon)),$$

where Dg denotes the derivative of g. Our objective is to find two constants c_1 and c_2, which depend on g and x, such that the following inclusions hold:

$$g(x) + Dg(x)(B(0, c_1\varepsilon)) \subseteq g(B(x, \varepsilon)) \subseteq g(x) + Dg(x)(B(0, c_2\varepsilon)),$$

that is, to compute upper and lower estimates for the size of pseudosolutions in terms of the linear function $Dg(x)$ which, at least in principle, should be easier to compute than g.

We recall that for a surjective and bounded linear operator $L\colon \mathbb{E} \to \mathbb{F}$ between two Hilbert spaces, its pseudoinverse (or Moore–Penrose inverse) is defined by

$$L^\dagger = L^*(LL^*)^{-1}$$

with L^* the adjoint of L. We refer to Luenberger [**10, Section 6.11**] for properties of Moore–Penrose inverses. Let $x \in \mathbb{E}$ be such that $Dg(x)$ is onto. Shub and

Smale [13] defined

$$\gamma(g,x)=\sup_{k\geq 2}\left\|Dg(x)^{\dagger}\frac{D^k g(x)}{k!}\right\|^{1/(k-1)}, \tag{1.2}$$

where $\|\cdot\|$ is the norm defined by the standard Hermitian inner product.

By the root test we have that the radius of convergence r of the Taylor series of g at x satisfies $r \geq 1/\gamma(g,x)$. But $\gamma(g,x)$ is a finer invariant than r. It is, for example, used in measuring the domain of quadratic convergence of Newton's method, see [2].

Our first main result is the following:

Theorem 1.1. *Let $x \in \mathbb{E}$ be given such that $Dg(x)$ is onto. Let λ, c, and ε be three real numbers satisfying*

$$0 \leq \lambda < 1-\frac{\sqrt{2}}{2}, \qquad c=\frac{2-\lambda}{2(1-\lambda)^2}, \qquad 0 \leq \varepsilon\gamma(g,x) \leq \lambda.$$

Then

$$g(x)+Dg(x)(B(0,\varepsilon(1-c\varepsilon\gamma(g,x)))) \subseteq g(B(x,\varepsilon)),$$

and

$$g(B(x,\varepsilon)) \subseteq g(x)+Dg(x)\left(B\left(0,\frac{\varepsilon}{1-\varepsilon\gamma(g,x)}\right)\right).$$

Thus we have accomplished our goal with the constants $c_1 = 1 - c\varepsilon\gamma(g,x)$ and $c_2 = (1-\varepsilon\gamma(g,x))^{-1}$ expressed in terms of $\gamma(g,x)$. Note that εc_1 and εc_2 are asymptotically equal to ε when ε tends to 0.

Theorem 1.1 is not completely satisfactory for our purposes. In general, the solution map g is not given explicitly but via the implicit function theorem. To be more specific,

$$f\colon \mathbb{E}\times\mathbb{F}\to\mathbb{G}$$

is an analytic function between Hilbert spaces, $f(x,z)=0$, and $D_2 f(x,z)$ is an isomorphism where D_2 denotes the partial derivative with respect to the second variable. In such a case there is a unique analytic map g defined in a neighborhood V of x in $\mathbb{E}$, taking its values in a neighborhood of z in $\mathbb{F}$ and such that

$$g(x)=z \quad \text{and} \quad f(y,g(y))=0 \quad \text{for each} \quad y\in V.$$

Important examples of maps g defined via the implicit function theorem are given by:

(i) The evaluation map

$$\operatorname{eval}(u,x)=u(x)$$

which is defined for any u in a space of functions defined on a domain Ω contained in a Banach space $\mathbb{H}$ with range in $\mathbb{C}^n$, and any $x \in \Omega$. The implicit function g associated with the evaluation map is called the *solution map* and denoted by sol.

(ii) The polynomial eigenvalue problem which consists of finding scalars λ and nonzero vectors $x \in \mathbb{C}^n$ satisfying

$$f(A, (x, \lambda)) = P(A, \lambda)x = 0,$$

where $P(A, \lambda)$ is a matrix polynomial defined as in (1.1).

When $D_2 f(x, z)$ is an isomorphism we define

$$\gamma_2(f, x, z) = \sup_{k \geq 2} \left\| D_2 f(x, z)^{-1} \frac{D^k f(x, z)}{k!} \right\|^{1/(k-1)},$$

where the kth derivative is taken with respect to (x, z) and

$$\mu(f, x, z) = (1 + \|D_2 f(x, z)^{-1} D_1 f(x, z)\|^2)^{1/2}.$$

In addition, when $D_1 f(x, z)$ is onto, we let

$$\mu^\dagger(f, x, z) = \max(1, \|(D_2 f(x, z)^{-1} D_1 f(x, z))^\dagger\|).$$

We use these three invariants to compute a domain of definition for the implicit function g. The result is stated in the following proposition:

Proposition 1.2. *When $f(x, z) = 0$ and $D_2 f(x, z)$ is an isomorphism, the implicit function g is defined and analytic on the ball about x with radius*

$$\frac{3 - 2\sqrt{2}}{\mu(f, x, z)\gamma_2(f, x, z)}.$$

A serious difficulty is to compute $\gamma(g, x)$ as defined in (1.2) when g is given implicitly. In the next theorem we estimate this quantity in terms of $\gamma_2(f, x, z)$, $\mu(f, x, z)$, and $\mu^\dagger(f, x, z)$.

Theorem 1.3. *When $f(x, z) = 0$, $D_2 f(x, z)$ is an isomorphism, and $D_1 f(x, z)$ is onto, we have*

$$\gamma(g, x) \leq (3 + 2\sqrt{2})\mu^\dagger(f, x, z)\mu(f, x, z)^2\gamma_2(f, x, z),$$

and

$$\gamma_2(f, x, z) \leq \mu(f, x, z)\gamma(f, x, z).$$

We now rewrite Theorem 1.1 for the Implicit Function Theorem context.

Theorem 1.4. *Let $x \in \mathbb{E}$ and $z \in \mathbb{F}$ be such that $f(x, z) = 0$, $D_2 f(x, z)$ is an isomorphism, and $D_1 f(x, z)$ is onto. Let λ, c, r, and ε be such that*

$$0 \le \lambda < 1 - \frac{\sqrt{2}}{2}, \qquad c = \frac{2-\lambda}{2(1-\lambda)^2},$$

$$r = (3 + 2\sqrt{2})\mu^\dagger(f, x, z)\mu(f, x, z)^2\gamma_2(f, x, z),$$

and

$$0 \le \varepsilon r \le \lambda.$$

Then

$$\begin{aligned} g(x) + Dg(x)(B(0, \varepsilon(1 - c\varepsilon r))) &\subseteq g(B(x, \varepsilon)) \\ &\subseteq g(x) + Dg(x)\left(B\left(0, \frac{\varepsilon}{1-\varepsilon r}\right)\right). \end{aligned}$$

Note that since $\mu(f, x, z) \ge 1$ and $\mu^\dagger(f, x, z) \ge 1$ the ε in Theorem 1.4 satisfies

$$\varepsilon \le \frac{1 - \sqrt{2}/2}{(3 + 2\sqrt{2})\mu^\dagger(f, x, z)\mu(f, x, z)^2\gamma_2(f, x, z)} \le \frac{3 - 2\sqrt{2}}{\mu(f, x, z)\gamma_2(f, x, z)}.$$

Thus, thanks to Proposition 1.2, $g(B(x, \varepsilon))$ is well-defined and Theorem 1.4 makes sense.

We prove Theorem 1.1 in Section 2 and Theorem 1.3 and Proposition 1.2 in Section 3. Pseudoroots of the evaluation map are considered in Section 4. We give a version of Theorem 1.4 adapted to the polynomial eigenvalue problem in Section 5 and relate this result to the ε-pseudospectrum of the corresponding matrix polynomial.

2. Proof of Theorem 1.1

We first prove the inclusion

$$g(B(x, \varepsilon)) \subseteq g(x) + Dg(x)\left(B\left(0, \frac{\varepsilon}{1 - \varepsilon\gamma(g, x)}\right)\right).$$

Let $v \in \mathbb{E}$ be such that $\|v\| \le \varepsilon$ and let $u = Dg(x)^\dagger(g(x + v) - g(x))$. Since $Dg(x)Dg(x)^\dagger = \mathrm{id}_{\mathbb{F}}$, where $\mathrm{id}_{\mathbb{F}}$ denotes the identity operator on $\mathbb{F}$, we have

$$g(x + v) = g(x) + Dg(x)u.$$

From the Taylor expansion for g at x we get $u = \sum_{k\geq 1} Dg(x)^{\dagger}(D^k g(x)/k!)v^k$ so that

$$\|u\| \leq \|v\| \left(1 + \sum_{k\geq 2} \gamma(g,x)^{k-1}\|v\|^{k-1}\right) = \frac{\|v\|}{1-\gamma(g,x)\|v\|} \leq \frac{\varepsilon}{1-\gamma(g,x)\varepsilon}$$

which proves the first inclusion.

For the second inclusion we aim to compute a number $\kappa > 0$ such that

$$g(x) + Dg(x)(B(0,\varepsilon\kappa)) \subseteq g(B(x,\varepsilon)).$$

Suppose that $g(x) = 0$. Given u such that $\|u\| \leq \varepsilon\kappa$ we want to find y satisfying $\|x - y\| \leq \varepsilon$ and $Dg(x)u = g(y)$. Clearly we can suppose that $u \in \ker Dg(x)^{\perp}$ and we look for y in the same space. We define

$$h\colon \ker Dg(x)^{\perp} \to \ker Dg(x)^{\perp}, \qquad h(y) = u - Dg(x)^{\dagger} g(y) + y.$$

Suppose that h has a fixed point $y \in B(x,\varepsilon)$, that is, $h(y) = y$. This gives $u - Dg(x)^{\dagger}g(y) + y = y$ so that $u = Dg(x)^{\dagger}g(y)$. Since $Dg(x)Dg(x)^{\dagger} = \mathrm{id}_{\mathbb{F}}$ we have $Dg(x)u = g(y)$ which completes the proof. □

To prove the existence of such a fixed point we use the contraction principle: h has a fixed point as soon as this map is a contraction from $B(x,\varepsilon)$ into itself. A bound for the contraction constant is obtained from a bound on the norm of the derivative

$$\begin{aligned} Dh(y) &= \mathrm{id}_{\mathbb{F}} - Dg(x)^{\dagger}Dg(y) \\ &= \mathrm{id}_{\mathbb{F}} - \textstyle\sum_{k\geq 0} Dg(x)^{\dagger} \frac{D^{k+1}g(x)}{k!}(y-x)^k \\ &= -\sum_{k\geq 1} Dg(x)^{\dagger} \frac{D^{k+1}g(x)}{k!}(y-x)^k \end{aligned}$$

because $Dg(x)^{\dagger}Dg(x)$ is the identity on $\ker Dg(x)^{\perp}$. Thus

$$\begin{aligned} \|Dh(y)\| &\leq \sum_{k\geq 1}(k+1)\gamma(g,x)^k\|y-x\|^k \\ &= \frac{1}{(1-\gamma(g,x)\|y-x\|)^2} - 1 = \frac{\nu(2-\nu)}{(1-\nu)^2}, \end{aligned}$$

with $\nu = \gamma(g,x)\|y-x\|$. Since $\|y-x\| \leq \varepsilon$, $\nu \leq \gamma(g,x)\varepsilon \leq \lambda$ so that, for $\lambda < 1 - \sqrt{2}/2$,

$$\|Dh(y)\| \leq \frac{\nu(2-\nu)}{(1-\nu)^2} \leq \frac{\lambda(2-\lambda)}{(1-\lambda)^2} < 1.$$

Thus h is a contraction. From the same inequality we have

$$\|Dh(y)\| \leq \frac{(2-\lambda)}{(1-\lambda)^2}\gamma(g,x)\|y-x\|$$

and, by integrating,

$$\begin{aligned}\|h(y)-h(x)\| &= \left\|\int_0^1 Dh(x+t(y-x))(y-x)\,dt\right\| \\ &\le \int_0^1 \frac{(2-\lambda)}{(1-\lambda)^2}\gamma(g,x)\|y-x\|^2 t\,dt = c\gamma(g,x)\|y-x\|^2.\end{aligned}$$

This yields

$$\|h(y)-x\| \le \|h(y)-h(x)\| + \|h(x)-x\| \le c\gamma(g,x)\|y-x\|^2 + \|u\|.$$

If $\|u\| \le \varepsilon(1-c\gamma(g,x)\varepsilon)$, then $c\gamma(g,x)\|y-x\|^2+\|u\| \le \varepsilon$ and h is a contraction from $B(x,\varepsilon)$ into itself. This proves the existence of a fixed point for h and achieves the proof. □

3. Proof of Theorem 1.3 and Proposition 1.2

To prove Theorem 1.3, we need to estimate the γ of an implicit function. We first consider the case of an inverse function. Similar questions have already been investigated by Kim [**8**], [**9**] for complex functions of a single variable. The following lemma is a classical result [**1, Section 2.4.A**]. It gives the kth derivative of the composition of two maps.

Lemma 3.1. *Let f and h be smooth maps between Banach spaces. For any x, $x_1, \ldots, x_k$ in the source space for h we have*

$$\begin{aligned}&D^k(f\circ h)(x)(x_1,\ldots,x_k) \\ &\qquad = \sum_{i=1}^{k}\sum_{N} D^i f(h(x))(D^{\nu_1}h(x)(x_j,\, j\in N_1),\ldots,D^{\nu_i}h(x)(x_j,\, j\in N_i)),\end{aligned}$$

where the second sum is taken for all the partitions $N=\{N_1,\ldots,N_i\}$ of the set $\{1,\ldots,k\}$ and $\nu_j=|N_j|$ (thus $\nu_j\ge 1$ and $\nu_1+\cdots+\nu_i=k$).

The next two theorems are proved using standard majorant series techniques from complex analysis (see Cartan [**3**]) and Lemma 3.1.

Theorem 3.2. *Let $p(z)=z-\sum_{k=2}^{\infty}(\nu_k/k!)z^k$ be an analytic function of one variable and let $P\colon \mathbb{E}\to\mathbb{E}$ be analytic and such that*

$$P(0)=0,\qquad DP(0)=\mathrm{id}_{\mathbb{E}},\qquad \|D^kP(0)\|\le\nu_k.$$

Let q and Q be the compositional inverses of p and P, respectively, which are defined and analytic in a neighborhood of $0\in\mathbb{C}$ and $0\in\mathbb{E}$, respectively. Then

$$\|D^kQ(0)\|\le D^kq(0).$$

Proof. The proof is by induction. For $k = 1$,

$$\|DQ(0)\| = \|\mathrm{id}_{\mathbb{E}}\| = 1 = Dq(0).$$

For $k \geq 2$, since $0 = \|D^k \,\mathrm{id}_{\mathbb{E}}(0)\| = \|D^k(P \circ Q)(0)\|$, by Lemma 3.1 we have, with similar notations,

$$\begin{aligned}
\|D^k Q(0)\| &= \left\| -\sum_{i=2}^{k} \sum_{N} D^i P(0)(D^{\nu_1} Q(0), \ldots, D^{\nu_i} Q(0)) \right\| \\
&\leq \sum_{i=2}^{k} \sum_{N} \|D^i P(0)\| \prod_{j=1}^{i} \|D^{\nu_j} Q(0)\| \\
&\leq \left| -\sum_{i=2}^{k} \sum_{N} \nu_i \prod_{j=1}^{i} D^{\nu_j} q(0) \right| = D^k q(0)
\end{aligned}$$

also since $0 = D^k(p \circ q)$. □

Now we can estimate $\gamma(Q, 0)$ in terms of $\gamma(P, 0)$.

Theorem 3.3. *Let $P\colon \mathbb{E} \to \mathbb{E}$ be analytic, $P(0) = 0$, $DP(0) = \mathrm{id}_{\mathbb{E}}$, and let Q be the inverse of P. Then*

$$\gamma(Q, 0) \leq (3 + 2\sqrt{2})\gamma(P, 0).$$

Proof. This theorem is a consequence of Theorem 3.2. Since $\|D^k P(0)\| \leq \gamma(P, 0)^{k-1} k!$ we take here

$$p(z) = z - \sum_{k=2}^{\infty} \gamma(P, 0)^{k-1} z^k = z - z\frac{\gamma z}{1 - \gamma z} \equiv w,$$

where we write γ for $\gamma(P, 0)$. The corresponding inverse function $z = q(w)$ is given by the equation

$$2\gamma z^2 - (1 + \gamma w)z + w = 0$$

so that

$$q(w) = \frac{1 + \gamma w - \sqrt{1 - 6\gamma w + (\gamma w)^2}}{4\gamma}. \tag{3.1}$$

According to Theorem 3.2 we have $\|D^k Q(0)\| \leq D^k q(0)$, so that

$$\gamma(Q, 0) = \sup_{k \geq 2} \left(\frac{\|D^k Q(0)\|}{k!} \right)^{1/(k-1)} \leq \sup_{k \geq 2} \left(\frac{D^k q(0)}{k!} \right)^{1/(k-1)}.$$

The conclusion is obtained from the inequality

$$\sup_{k\geq 2}\left(\frac{D^k q(0)}{k!}\right)^{1/(k-1)} \leq (3+2\sqrt{2})\gamma(P,0). \tag{3.2}$$

To prove (3.2), we consider the function $V(x)=\sqrt{1-6x+x^2}$. This function has a Taylor series expansion at the origin with radius of convergence $3-2\sqrt{2}$. Let

$$V(x)=\sum_0^\infty b_n x^n = 1-xU(x) \tag{3.3}$$

with

$$U(x)=\sum_0^\infty a_n x^n = 3+4x+12x^2+\cdots. \tag{3.4}$$

Note that $b_0=1$, $-b_{n+1}=a_n$ for $n\geq 1$, and, along the real line, $V(x)$ is decreasing from 1 to 0 and $U(x)$ is increasing from 0 to $3+2\sqrt{2}$ as x increases from 0 to $3-2\sqrt{2}$. Since

$$V(x)^2=\left(\sum_0^\infty b_n x^n\right)^2=\sum_{n=0}^\infty\sum_{k=0}^n b_k b_{n-k}x^n$$

we get

$$-2b_{n+1}=\sum_{k=1}^n b_k b_{n+1-k}$$

for all $n\geq 2$ and, consequently,

$$2a_{n+1}=\sum_{k=0}^n a_k a_{n-k}$$

for all $n\geq 1$. From this identity we deduce $a_n>0$ for all $n\geq 0$ and also $a_{n+1}>3a_n$ for all $n\geq 1$. Suppose that $a_n\geq B^n$ for some $B>3+2\sqrt{2}$ and some n necessarily ≥ 3. Let x_0 be such that

$$\frac{1}{B}<x_0<\frac{1}{3+2\sqrt{2}} \qquad \text{and} \qquad 0.17<x_0.$$

This is possible since $0.17<1/(3+2\sqrt{2})$. Since $0.51<3x_0<1$ we have

$$\sum_{k=n}^\infty a_k x_0^k > \frac{B^n x_0^n}{1-3x_0} > 2B^n x_0^n > 2.$$

Thus

$$U(x_0)>3+4\times 0.17+12\times 0.17^2+2>3+2\sqrt{2}$$

which is an invalid value for U in the interval $[0, 3-2\sqrt{2})$. For this reason

$$0 < a_n \le (3+2\sqrt{2})^n \qquad \text{for each} \quad n > 0.$$

We now come back to the function $q(w)$ in (3.1) which we rewrite as

$$q(w) = \frac{w + w\sum_0^\infty a_n(\gamma w)^n}{4}$$

using (3.3) and (3.4). Hence, for all $k \ge 2$,

$$\frac{D^k q(0)}{k!} = \tfrac{1}{4} a_{k-1}\gamma^{k-1} \le \tfrac{1}{4}(3+2\sqrt{2})^{k-1}\gamma^{k-1}$$

so that

$$\left(\frac{D^k q(0)}{k!}\right)^{1/k-1} \le (\tfrac{1}{4})^{1/k-1}(3+2\sqrt{2})\gamma \le (3+2\sqrt{2})\gamma$$

which proves the inequality (3.2). □

Corollary 3.4. *Let $H\colon \mathbb{E} \to \mathbb{E}$ be analytic, $H(x) = y$, and let $DH(x)$ be an isomorphism. Then the inverse H^{-1} of H is defined and analytic in the ball about y with radius*

$$\frac{1}{(3+2\sqrt{2})\gamma(H,x)\|DH(x)^{-1}\|}.$$

Proof. Let $P = DH(x)^{-1}H$ so that $DP(x) = \mathrm{id}_{\mathbb{E}}$ and define $Q = P^{-1} = H^{-1} \circ DH(x)$. Then $H^{-1} = QDH(x)^{-1}$ so that

$$D^k H^{-1} = D^k Q(DH(x)^{-1}, \dots, DH(x)^{-1})$$

and

$$\|D^k H^{-1}\| \le \|D^k Q\| \|DH(x)^{-1}\|^k \le k!\,\gamma(Q)^{k-1}\|DH(x)^{-1}\|^k.$$

Then using Theorem 3.3 we obtain

$$\begin{aligned} \|D^k H^{-1}\| &\le k!((3+2\sqrt{2})\gamma(P))^{k-1}\|DH(x)^{-1}\|^k \\ &= k!((3+2\sqrt{2})\gamma(H))^{k-1}\|DH(x)^{-1}\|^k, \end{aligned}$$

so that

$$\limsup_{k\to\infty}\left(\frac{\|DH(x)^{-1}\|}{k!}\right)^{1/k} \le (3+2\sqrt{2})\gamma(H)\|DH(x)^{-1}\|.$$ □

3.1. *Proof of Theorem* 1.3

Let $H\colon \mathbb{E}\times\mathbb{F}\to\mathbb{E}\times\mathbb{G}$ be defined by

$$H(u,v)=\begin{pmatrix} u \\ f(u,v)\end{pmatrix}, \tag{3.5}$$

and let $\Pi_2\colon \mathbb{E}\times\mathbb{F}\to\mathbb{F}$, $\Pi_2(u,v)=v$ and $I\colon \mathbb{E}\to\mathbb{E}\times\mathbb{G}$, $I(u)=(u,0)$. According to these new notations, the implicit function g is equal to

$$g=\Pi_2\circ H^{-1}\circ I.$$

The derivative of H and its inverse are given by

$$\begin{aligned} DH(x,z) &= \begin{pmatrix} \mathrm{id}_{\mathbb{E}} & 0 \\ D_1 f(x,z) & D_2 f(x,z)\end{pmatrix}, \\ DH(x,z)^{-1} &= \begin{pmatrix} \mathrm{id}_{\mathbb{E}} & 0 \\ -D_2^{-1} f(x,z) D_1 f(x,z) & D_2^{-1} f(x,z)\end{pmatrix}. \end{aligned}$$

The kth derivative of H is equal to

$$D^k H(x,z)=\begin{pmatrix} 0 \\ D^k f(x,z)\end{pmatrix},$$

so that $\gamma(H,(x,z))=\gamma_2(f,(x,z))$. We have

$$\begin{aligned} \gamma(g,x) &= \gamma(\Pi_2 H^{-1} I, x) \\ &= \sup_{k\ge 2}\left\| (\Pi_2 DH^{-1}(x,0)I)^{\dagger}\Pi_2 \frac{D^k H^{-1}(x,0)}{k!}(I,\ldots,I)\right\|^{1/(k-1)} \\ &\le \sup_{k\ge 2} \|(\Pi_2 DH^{-1} I)^{\dagger}\|^{1/(k-1)}\|\Pi_2\|^{1/(k-1)} \\ &\qquad \times \left\| \frac{D^k(H^{-1}DH(x,z))(x,0)}{k!}\right. \\ &\qquad\qquad \left. \times\ (DH(x,z)^{-1}I,\ldots,DH(x,z)^{-1}I)\right\|^{1/(k-1)} \\ &\le \sup_{k\ge 2} \|(\Pi_2 DH^{-1} I)^{\dagger}\|^{1/(k-1)} \\ &\qquad \times \|\Pi_2\|^{1/(k-1)} \left\| \frac{D^k(H^{-1}DH(x,z))(x,0)}{k!}\right\|^{1/(k-1)} \\ &\qquad \times \|DH(x,z)^{-1}I\|^{k/(k-1)}. \end{aligned}$$

Since $\|\Pi_2\| \leq 1$ we get

$$\gamma(g, x) \;\leq\; \max(1, \|(\Pi_2 DH^{-1} I)^\dagger\|) \times \max(1, \|DH(x, z)^{-1} I\|^2) \sup_{k\geq 2} \left\| \frac{D^k(H^{-1}DH(x, z))(x, 0)}{k!} \right\|^{1/(k-1)},$$

and as

$$\|(\Pi_2 DH^{-1} I)^\dagger\| = \|(-D_2 f(x, z)^{-1} D_1 f(x, z))^\dagger\|,$$

we have

$$\max(1, \|(\Pi_2 DH^{-1} I)^\dagger\|) = \mu^\dagger(f, x, z).$$

From

$$DH(x, z)^{-1} I = \begin{pmatrix} \mathrm{id}_{\mathbb{E}} \\ -D_2^{-1} f(x, z) D_1 f(x, z) \end{pmatrix},$$

we obtain

$$\max(1, \|DH(x, z)^{-1} I\|^2) = (1 + \|D_2^{-1} f(x, z) D_1 f(x, z)\|^2) = \mu(f, x, z)^2.$$

For these reasons, and since $D(H^{-1}DH(x, z))(x, 0) = \mathrm{id}$, we have

$$\gamma(g, x) \leq \mu^\dagger(f, x, z)\mu(f, x, z)^2 \gamma(H^{-1}DH(x, z), (x, 0)).$$

From Theorem 3.3, $\gamma(H^{-1}DH(x, z), (x, 0)) \;\leq\; (3 + 2\sqrt{2})\gamma(DH(x, z)^{-1}H, (x, z))$ and since

$$\gamma(DH(x, z)^{-1}H, (x, z)) = \gamma(H, (x, z)) = \gamma_2(f, (x, z)),$$

we conclude that

$$\gamma(g, x) \leq (3 + 2\sqrt{2})\mu^\dagger(f, x, z)\mu(f, x, z)^2 \gamma_2(f, (x, z)).$$

We now estimate $\gamma_2(f, (x, z))$ in terms of $\gamma(f, x, z)$. Since $Df(x, z)$ is onto we have $Df(x, z)Df(x, z)^\dagger = \mathrm{id}$ so that

$$\begin{aligned} \gamma_2(f, (x, z)) \;&=\; \sup_{k\geq 2} \left\| D_2 f(x, z)^{-1} \frac{D^k f(x, z)}{k!} \right\|^{1/(k-1)} \\ &=\; \sup_{k\geq 2} \left\| D_2 f(x, z)^{-1} Df(x, z) Df(x, z)^\dagger \frac{D^k f(x, z)}{k!} \right\|^{1/(k-1)} \\ &\leq\; \max(1, \|D_2 f(x, z)^{-1} Df(x, z)\|) \\ &\qquad \times \sup_{k\geq 2} \left\| Df(x, z)^\dagger \frac{D^k f(x, z)}{k!} \right\|^{1/(k-1)}. \end{aligned}$$

□

3.2. *Proof of Proposition* 1.2

This proof uses Corollary 3.4 applied to the function H defined in (3.5). We use the same notations as in the proof of Theorem 1.3. We have $\|D^k g(x)\| \le \|D^k H^{-1}(x, 0)\|$. Thus g is defined and analytic in the ball about x with radius

$$1/((3+2\sqrt{2})\gamma(H, x, z)\|DH(x, z)^{-1}\|) \le (3-2\sqrt{2})/(\gamma_2(f, x, z)\mu(f, x, z)). \ \square$$

Remark 3.5. Proposition 1.2 is also valid in the context of Banach spaces.

4. The Evaluation Map

4.1. *General Results*

In this section we apply our theorems to the function eval. For any i, $1 \le i \le n$, let $\mathbb{H}_i$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_i$ contained in $\mathbb{C}^\Omega$, the space of functions from Ω to $\mathbb{C}$ with Ω a given open subset in $\mathbb{C}^n$, and such that, for any $x \in \Omega$, there exists a positive constant $C_{i,x}$ with

$$|u_i(x)| \le C_{i,x}\|u_i\|_i$$

for any $u_i \in \mathbb{H}_i$. In other words, Dirac's map

$$\delta_x\colon\ \mathbb{H}_i \to \mathbb{C}, \qquad \delta_x(u_i) = u_i(x),$$

is a linear and bounded operator on $\mathbb{H}_i$. According to Riesz's Representation Theorem, there exists a two variable function

$$h_i\colon\ \Omega \times \Omega \to \mathbb{C}$$

such that:

(1) $h_i(\cdot, x) \in \mathbb{H}_i$ for any $x \in \Omega$; and
(2) $u_i(x) = \langle u_i, h_i(\cdot, x)\rangle_i$ for any $u_i \in \mathbb{H}_i$ and for any $x \in \Omega$.

We call h_i the reproducing kernel (or Aronszajn–Bergman reproducing kernel) of $\mathbb{H}_i$, see K. Yosida [**19**]. Note the following elementary facts:

(1) $h_i(x, y) = \overline{h_i(y, x)} = \langle h_i(\cdot, y), h_i(\cdot, x)\rangle_i$.
(2) $\sum_{k,l=1}^n \bar{\lambda}_k \lambda_l h_i(x_k, x_l) \ge 0$, for any positive integer n, and for any $\lambda_k \in \mathbb{C}$ and $x_k \in \Omega$, $1 \le k \le n$.
(3) The subspace in $\mathbb{H}_i$, generated by $h_i(\cdot, x)$, $x \in \Omega$, is dense in $\mathbb{H}_i$.

We now consider $\mathbb{H} = \mathbb{H}_1 \times \mathbb{H}_2 \times \cdots \times \mathbb{H}_n$ endowed with the product structure and define eval: $\mathbb{H} \times \Omega \to \mathbb{C}^n$ by

$$\mathrm{eval}(u, x) = u(x) = [u_1(x), u_2(x), \ldots, u_n(x)]^T.$$

We have the following lemma:

Lemma 4.1. *Let $u \in \mathbb{H}$ and $x \in \Omega$ be given. Suppose that $u(x) = 0$, $Du(x)$ is an isomorphism, and $v \in \mathbb{H} \to v(x) \in \mathbb{C}^n$ is onto. Then*

$$D_1 \operatorname{eval}(u, x)v = v(x), \tag{4.1}$$

$$D_2 \operatorname{eval}(u, x) = Du(x), \tag{4.2}$$

$$\gamma_2(\operatorname{eval}, u, x) = \gamma(u, x), \tag{4.3}$$

$$\begin{aligned} \mu(\operatorname{eval}, u, x) &= (1 + \|Du(x)^{-1} \operatorname{diag}(\langle \cdot, h_i(\cdot, x)\rangle_i)\|^2)^{1/2} \\ &\le \left(1 + \|Du(x)^{-1}\|^2 \max_{1 \le i \le n} h_i(x, x)\right)^{1/2}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \mu^\dagger(\operatorname{eval}, u, x) &= \max(1, \|\operatorname{diag}(h_i(x, x)^{-1/2}) Du(x)\|) \\ &\le \max\left(1, \|Du(x)\| \left(\min_{1 \le i \le n} h_i(x, x)\right)^{-1/2}\right). \end{aligned} \tag{4.5}$$

Here, $\operatorname{diag}(\langle \cdot, h_i(\cdot, x)\rangle_i)$ *is the linear operator defined by*

$$v \in \mathbb{H} \to (\langle v_i, h_i(\cdot, x)\rangle_i) \in \mathbb{C}^n$$

and $\operatorname{diag}(h_i(x, x)^{-1/2})$ *is the diagonal matrix with entries equal to* $h_i(x, x)^{-1/2}$, $1 \le i \le n$.

Note that $h_i(x, x) \neq 0$ otherwise $v \to v(x)$ cannot be onto. Thus the diagonal matrix $\operatorname{diag}(h_i(x, x)^{1/2})$ is nonsingular and its inverse is well defined.

Proof. Equations (4.1), (4.2), (4.3), and (4.4) are obvious. To prove (4.5) we first compute the pseudoinverse of

$$D_2 \operatorname{eval}(u, x)^{-1} D_1 \operatorname{eval}(u, x) = Du(x)^{-1} \operatorname{diag}(\langle \cdot, h_i(\cdot, x)\rangle_i).$$

The kernel of this operator is the set of $v \in \mathbb{H}$ with $v(x) = 0$. This last equation may be interpreted as orthogonality relations between v and

$$(0, \ldots, h_i(\cdot, x), \ldots, 0)^T, \qquad 1 \le i \le n.$$

For this reason the orthogonal complement to this kernel is the set of

$$(\lambda_1 h_1(\cdot, x), \ldots, \lambda_i h_i(\cdot, x), \ldots, \lambda_n h_n(\cdot, x))^T$$

with $\lambda \in \mathbb{C}^n$. The pseudoinverse of $D_2^{-1} D_1$ satisfies

$$(D_2^{-1} D_1)^\dagger \mu = (\lambda_1 h_1(\cdot, x), \ldots, \lambda_i h_i(\cdot, x), \ldots, \lambda_n h_n(\cdot, x))^T$$

and

$$D_2^{-1} D_1 (\lambda_1 h_1(\cdot, x), \ldots, \lambda_i h_i(\cdot, x), \ldots, \lambda_n h_n(\cdot, x))^T = \mu.$$

This gives

$$Du(x)^{-1}\,\mathrm{diag}(\langle\cdot, h_i(\cdot,x)\rangle_i)(\lambda_1 h_1(\cdot,x),\ldots,\lambda_i h_i(\cdot,x),\ldots,\lambda_n h_n(\cdot,x))^T$$
$$= Du(x)^{-1}\,\mathrm{diag}(h_i(x,x))\lambda = \mu,$$

so that

$$\lambda = \mathrm{diag}(h_i(x,x)^{-1})Du(x)\mu.$$

The norm of this pseudoinverse is given by

$$\begin{aligned}\|(D_2^{-1}D_1)^\dagger\mu\| &= \left(\sum |\lambda_i|^2 h_i(x,x)\right)^{1/2}\\ &= \|\,\mathrm{diag}(h_i(x,x)^{1/2})\lambda\|\\ &= \|\mathrm{diag}(h_i(x,x)^{-1/2})Du(x)\mu\|\\ &\le \|Du(x)\|\left(\min_{1\le i\le n} h_i(x,x)^{-1/2}\right).\end{aligned}$$ □

Let $u \in \mathbb{H}$ and $x \in \Omega$ such that $u(x) = 0$, $Du(x)$ is an isomorphism, and $v \in \mathbb{H} \to v(x) \in \mathbb{C}^n$ is onto. The implicit function associated with the evaluation map is called the *solution map* and denoted by sol. Thus,

$$\mathrm{sol}(u) = x \qquad \text{and} \qquad \mathrm{eval}(v, \mathrm{sol}(v)) = 0$$

for any v in a neighborhood of u.

In what follows, we consider two important examples of Hilbert spaces and associated reproducing kernels.

4.2. *Analytic Systems in the Unit Ball*

Let $(L^2 \cap \mathcal{H})(B)$ be the space of square-integrable holomorphic maps defined on the unit ball B in $\mathbb{C}^n$. It is a Hilbert space when endowed with the product $\langle f, g\rangle = \int_B f(z)\overline{g(z)}\,d\nu(z)$. Here $d\nu$ is the normalized Lebesgue measure on B so that $d\nu(B) = 1$. We also have $d\nu = (n!/\pi^n)d\mu$ with $d\mu$ the Lebesgue measure normalized so that the unit cube has measure 1. The reproducing kernel is the Bergman kernel $h(\cdot, z) = (1 - \langle\cdot, z\rangle)^{-n-1}$ and

$$f(z) = \int_B \frac{f(w)\,d\nu(w)}{(1-\langle z, w\rangle)^{n+1}}.$$

See Rudin [**12**]. When we take for $\mathbb{H}$ the product of n copies of $(L^2 \cap \mathcal{H})(B)$ we have

$$\begin{aligned}\|\mathrm{diag}(\langle\cdot, h_i(\cdot,z)\rangle)\| &= (1-\|z\|^2)^{-(n+1)/2},\\ \|\mathrm{diag}(h_i(z,z)^{-1/2})\| &= (1-\|z\|^2)^{(n+1)/2}.\end{aligned}$$

According to Lemma 4.1 we obtain the following estimates:

$$\begin{aligned}\gamma_2(\text{eval}, u, z) &= \gamma(u, z),\\ \mu(\text{eval}, u, z) &\le (1 + \|Du(z)^{-1}\|^2(1 - \|z\|^2)^{-n-1})^{1/2},\\ \mu^\dagger(\text{eval}, u, z) &\le \max(1, \|Du(z)\|(1 - \|z\|^2)^{(n+1)/2}).\end{aligned}$$

In the context of analytic systems of equations defined on the unit ball, Theorem 1.4 becomes

Theorem 4.2. *Let $u \in \mathbb{H}$ and $z \in B$ be such that $u(z) = 0$, $Du(z)$ is an isomorphism. Let λ, c, r, and ε be such that*

$$\begin{aligned}0 &\le \lambda < 1 - \frac{\sqrt{2}}{2}, \qquad c = \frac{2-\lambda}{2(1-\lambda)^2},\\ r &= (3 + 2\sqrt{2}) \max(1, \|Du(z)\|(1 - \|z\|^2)^{(n+1)/2})\\ &\quad \times (1 + \|Du(z)^{-1}\|^2(1 - \|z\|^2)^{-n-1})\gamma(u, z),\end{aligned}$$

and

$$0 \le \varepsilon r \le \lambda.$$

Then, $\gamma(\text{sol}, u) \le r$ and, for any $v \in \mathbb{H}$ with $\|v\| \le \varepsilon(1 - c\varepsilon r)$, there exists $w \in \mathbb{H}$, $\|u - w\| \le \varepsilon$ such that

$$z - Du(z)^{-1}v(z) = \text{sol}(w).$$

Moreover, for any $w \in \mathbb{H}$ with $\|u - w\| \le \varepsilon$, the following inequality holds:

$$\|\text{sol}(u) - \text{sol}(w)\| \le \|Du(z)^{-1}\|(1 - \|z\|^2)^{-(n+1)/2}\frac{\varepsilon}{1 - r\varepsilon}.$$

Proof. We use Lemma 4.1 and the estimates for $\gamma_2(\text{eval}, u, z)$, $\mu(\text{eval}, u, z)$, $\mu^\dagger(\text{eval}, u, z)$. We also notice that

$$\|D\,\text{sol}(u)\| \le \|Du(z)^{-1}\|(1 - \|z\|^2)^{-(n+1)/2}.$$

The inequality for γ comes from Theorem 1.3. $\square$

4.3. *Polynomial Systems*

Let $\mathcal{P}_d$ be the space of polynomials $f\colon \mathbb{C}^n \to \mathbb{C}$ of degree less than or equal to d. We consider the Hermitian structure given by

$$\langle f, g\rangle = \sum_{|\alpha|\le d} \binom{d}{\alpha}^{-1} f_\alpha \bar{g}_\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, is an n-tuple of nonnegative integers,

$$\binom{d}{\alpha} = \frac{d!}{\alpha_1! \cdots \alpha_n!\,(d - |\alpha|)!}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

and $f = \sum_{|\alpha| \le d} f_\alpha z^\alpha$ with $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Its reproducing kernel is given by $h(x, z) = (1 + \langle x, z\rangle)^d$ so that

$$f(z) = \langle f, (1 + \langle \cdot, z\rangle)^d \rangle.$$

In this context, Theorem 1.4 becomes

Theorem 4.3. *Let $u \in \mathcal{P}_{(d)} = \mathcal{P}_{d_1} \times \cdots \times \mathcal{P}_{d_n}$. Let us denote $D = \max_i d_i$ and $\delta = \min_i d_i$. Let $z \in \mathbb{C}^n$ be such that $u(z) = 0$, $Du(z)$ is an isomorphism. Let λ, c, r, and ε be such that*

$$0 \le \lambda < 1 - \frac{\sqrt{2}}{2}, \qquad c = \frac{2 - \lambda}{2(1 - \lambda)^2},$$

$$r = (3 + 2\sqrt{2}) \max(1, \|Du(z)\|(1 + \|z\|^2)^{-\delta/2}) \times (1 + \|Du(z)^{-1}\|^2(1 + \|z\|^2)^D)\gamma(u, z),$$

and

$$0 \le \varepsilon r \le \lambda.$$

Then, $\gamma(\mathrm{sol}, u) \le r$ and, for any $v \in \mathcal{P}_{(d)}$ with $\|v\| \le \varepsilon(1 - c\varepsilon r)$, there exists $w \in \mathcal{P}_{(d)}$, $\|u - w\| \le \varepsilon$ such that

$$z - Du(z)^{-1} v(z) = \mathrm{sol}(w).$$

Moreover, for any $w \in \mathcal{P}_{(d)}$ with $\|u - w\| \le \varepsilon$, the following inequality holds:

$$\|\mathrm{sol}(u) - \mathrm{sol}(w)\| \le \|Du(z)^{-1}\|(1 + \|z\|^2)^{D/2} \frac{\varepsilon}{1 - r\varepsilon}.$$

The proof of Theorem 4.3 is similar to Theorem 4.2. $\square$

In what follows we illustrate graphically the upper and lower estimates in Theorem 1.4 in the simple case of complex polynomials of one variable ($n = 1$). We denote by $Z(u) = \{z_1, \ldots, z_d\}$ the set of zeros of $u(z) = \sum_{k=0}^d a_k z^k$. The distance over $\mathcal{P}_d$ is given by

$$\|u\| = \left(\sum_{k=0}^{d} \binom{d}{k}^{-1} |a_k|^2 \right)^{1/2}.$$

We define the ε-pseudozero set of u by

$$Z_\varepsilon(u) = \{z \in \mathbb{C}, z \in Z(v) \text{ for some } v \in \mathcal{P}_d \text{ with } \|u - v\| \le \varepsilon\}.$$

This set can be numerically computed using Mosier [**11**] or Toh and Trefethen [**15**] algebraic characterization,

$$Z_\varepsilon(u) = \{z \in \mathbb{C}, |u(z)| \le \varepsilon \|\Delta \tilde{z}\|_2\},$$

where $\tilde{z} = (1, z, \ldots, z^n)^T$ and $\Delta = \operatorname{diag}(1, \binom{d}{1}, \ldots, \binom{d}{d-1}, 1)$. A plot of the boundaries of $Z_\varepsilon(u)$ for

$$u(z) = z(z-1)(z-2) \tag{4.6}$$

is given in Figure 4.1 for several values of ε. The three roots are marked with points.

Let $z_k \in Z(u)$. In the context of Theorem 4.3 we have

$$\begin{aligned}\gamma(u, z_k) &= \max_{2\le\ell\le d} \left| \frac{u^{(\ell)}(z_k)}{\ell!\, u'(z_k)} \right|^{1/(\ell-1)},\\ r_k &= (3+2\sqrt{2}) \max\left(1, \frac{|u^{(k)}(z_k)|}{(1+|z_k|^2)^{d/2}}\right)\left(1 + \frac{(1+|z_k|^2)^d}{|u'(z_k)|^2}\right)\gamma(u, z_k).\end{aligned}$$

For $\varepsilon \le \lambda/r_k$, Theorem 1.4 gives for the lower estimate

$$\bigcup_{k=1}^{d} \left\{ z_k - \frac{v(z_k)}{u'(z_k)} : \text{for some } v \in \mathcal{P}_d \text{ with } \|v\| \le \varepsilon(1 - c\varepsilon r_k) \right\} \subset Z_\varepsilon, \tag{4.7}$$

and for the upper estimate

$$Z_\varepsilon \subset \bigcup_{k=1}^{d} \left\{ z_k - \frac{v(z_k)}{u'(z_k)} : \text{for some } v \in \mathcal{P}_d \text{ with } \|v\| \le \frac{\varepsilon}{1 - r_k\varepsilon} \right\}. \tag{4.8}$$

For the particular polynomial in (4.6) and the particular root $z_k = 1$ we have $r_k = 9(3 + 2\sqrt{2})$. If we take $\lambda = \frac{1}{4}$ and $c = \frac{14}{9}$ then since $u'(1) = 1$ the lower

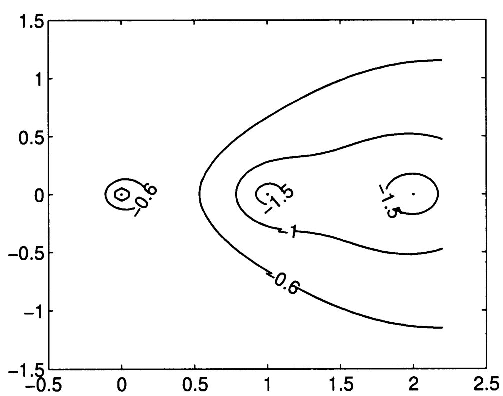

Fig. 4.1. ε-Pseudozero sets ($\varepsilon = 10^{-0.6}, 10^{-1}$, and $10^{-1.5}$) for the polynomial $u(z) = z(z-1)(z-2)$.

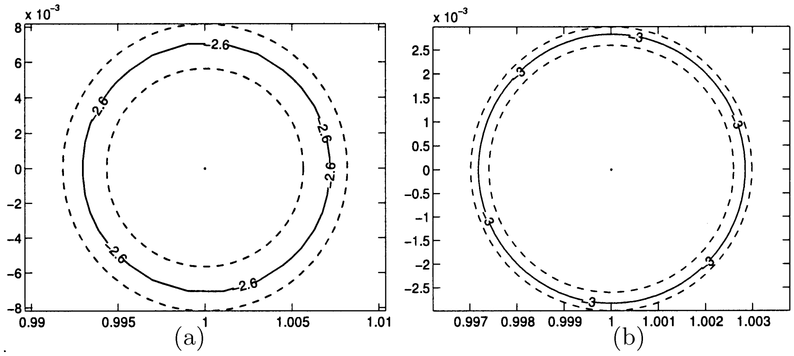

Fig. 4.2. Upper and lower bounds (dashed line) for $Z_\varepsilon(u)$ (solid line) with $u = z(z-1)(z-2)$. (a) $\varepsilon = 10^{-2.6}$; (b) $\varepsilon = 10^{-3}$.

and upper estimates for Z_ε in (4.7) and (4.8) are valid for $\varepsilon \le 10^{-2.6}$. Note that $|v(1)| \le \|v\|\kappa$ with $\kappa = \|(1, \sqrt{3}, \sqrt{3}, 1)\|_2$ and that the bound can be attained. Hence, the sets inside the curly brackets in (4.7) and (4.8) are disks of center 1 and radius $\kappa\varepsilon(1 - c\varepsilon r_k)$ and $\kappa\varepsilon/(1 - r_k\varepsilon)$, respectively. Figure 4.2 shows the lower and upper estimates of Z_ε for two different values of ε.

5. The Homogeneous Polynomial Eigenvalue Problem

We consider the map defined by

$$F(A, x, \alpha, \beta) = P(A, \alpha, \beta)x = \left(\sum_{k=0}^{m} \alpha^k \beta^{m-k} A_k\right) x.$$

Here $A = (A_0, A_1, \ldots, A_m) \in \mathcal{M}_n(\mathbb{C})^{m+1}$, where $\mathcal{M}_n(\mathbb{C})^{m+1}$ denotes the set of $(m+1)$-tuples of $n \times n$ complex matrices, $x \in \mathbb{C}^n$ and $(\alpha, \beta) \in \mathbb{C}^2$. $F(A, x, \alpha, \beta)$ is linear in A, linear in x, and homogeneous of degree m in (α, β). Such a map is called multihomogeneous with degrees $(1, 1, m)$.

The polynomial eigenvalue problem is to find pairs of scalars $(\alpha, \beta) \neq (0, 0)$ and nonzero vectors $x, y \in \mathbb{C}^n$ satisfying

$$P(A, \alpha, \beta)x = 0, \qquad y^* P(A, \alpha, \beta) = 0.$$

The vectors x, y are called the right and left eigenvectors corresponding to the eigenvalue (α, β). Special instances are the generalized eigenvalue problem ($m = 1$) and the quadratic eigenvalue problem ($m = 2$).

When $(\alpha, \beta) \in \mathbb{C}^2$ and $x \in \mathbb{C}^n$ is an eigenpair, since F is multihomogeneous in A, x and α, β, for any scalars ρ and τ, $\rho(\alpha, \beta)$ and τx is also an eigenpair. For this reason, it is natural to consider the eigenvalues and eigenvectors in the projective spaces $\mathbb{P}_1(\mathbb{C})$ and $\mathbb{P}_{n-1}(\mathbb{C})$, respectively. Here we denote by $\mathbb{P}_{n-1}(\mathbb{C}) = \mathbb{P}(\mathbb{C}^n)$ the set of vector lines in $\mathbb{C}^n$. Let us define

$$\mathcal{V}_P = \{(A, x, \alpha, \beta) \in \mathcal{M}_n(\mathbb{C})^{m+1} \times \mathbb{P}_{n-1} \times \mathbb{P}_1 : P(A, \alpha, \beta)x = 0\}.$$

$\mathcal{V}_P$ is an algebraic variety and a smooth manifold.

5.1. *General Results*

Our aim in this section is to give a version of Theorem 1.4 adapted to the context of the homogeneous polynomial eigenvalue problem. First we define the implicit function g.

Definition 5.1. Let Π_1 be the restriction to $\mathcal{V}_P$ of the projection $(A, x, \alpha, \beta) \to A$. We say that (A, x, α, β) is *well-posed* when the derivative of the first projection Π_1 at (A, x, α, β) is an isomorphism.

Note that this derivative is itself a projection, $D\Pi_1(A, x, \alpha, \beta)(E, \dot{x}, \dot{\alpha}, \dot{\beta}) = E$. In the following we use intensively the matrix

$$\mathcal{J}_P = \begin{pmatrix} P(A, \alpha, \beta) & D_\alpha P(A, \alpha, \beta)x & D_\beta P(A, \alpha, \beta)x \\ x^* & 0 & 0 \\ 0 & \bar{\alpha} & \bar{\beta} \end{pmatrix}.$$

The next lemma is from Dedieu and Tisseur [**4**] and characterizes well-posed problems.

Lemma 5.2 [4, Theorem 3.3]. *Let (α, β) be an eigenvalue of $P(A, \alpha, \beta)$ with corresponding left and right eigenvectors y and x, respectively, and let $v = (\bar{\beta} D_\alpha P - \bar{\alpha} D_\beta P)x$. The following conditions are equivalent*:

- (i) $D\Pi_1(A)$: $T_{(A,x,\alpha,\beta)}\mathcal{V}_P \to \mathcal{M}_n(\mathbb{C})^{m+1}$ *is an isomorphism, that is*, (A, x, α, β) *is well-posed*.
- (ii) *The matrix $\mathcal{J}_P$ is nonsingular.*
- (iii) $\operatorname{rank} P(A, \alpha, \beta) = n - 1$ *and* $y^*v \neq 0$.
- (iv) (α, β) *is a simple eigenvalue.*

When (A, x, α, β) is well-posed, by the Inverse Function Theorem there exist a neighborhood $U(x, \alpha, \beta) \subset \mathbb{P}_{n-1} \times \mathbb{P}_1$ of (x, α, β) and a neighborhood $U(A) \subset \mathcal{M}_n(\mathbb{C})^{m+1}$ of A such that

$$\Pi_1\colon\ \mathcal{V}_P \cap (U(A) \times U(x, \alpha, \beta)) \to U(A)$$

is invertible. Its inverse gives rise to a smooth map

$$G = (G_1, G_2) = \Pi_2 \circ \Pi_1^{-1}\colon\ U(A) \to U(x, \alpha, \beta), \tag{5.1}$$

such that

$$\operatorname{Graph}(G) = \mathcal{V}_P \cap (U(A) \times U(x, \alpha, \beta)).$$

We now take two charts in $\mathbb{P}_{n-1}$ at x and in $\mathbb{P}_1$ at (α, β). See Hirsch [**7**] for a description of these manifolds and their charts. We suppose that $\|x\| = 1$ and $\left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} = 1$. We denote by $x^{\perp}$ the set of vectors in $\mathbb{C}^n$ perpendicular to x. We consider the chart ψ defined on

$$\{y : \langle y, x\rangle \neq 0\} \times \{(\lambda, \mu) : \langle(\lambda, \mu), (\alpha, \beta)\rangle \neq 0\} \subset \mathbb{P}_{n-1} \times \mathbb{P}_1$$

taking its values in

$$(x + x^{\perp}) \times ((\alpha, \beta) + (\alpha, \beta)^{\perp}) \subset \mathbb{C}^n \times \mathbb{C}^2$$

defined by

$$\psi(y, (\lambda, \mu)) = \left(\frac{y}{\langle y, x\rangle}, \frac{(\lambda, \mu)}{\langle z(\lambda, \mu), z(\alpha, \beta)\rangle}\right),$$

where $z(\alpha, \beta) = (\alpha^m, \alpha^{m-1}\beta, \ldots, \alpha\beta^{m-1}, \beta^m)$. The set $x^{\perp}$ is the usual model for the tangent space at x to $\mathbb{P}_{n-1}$.

Let us denote by f the restriction of F to $\mathcal{M}_n(\mathbb{C})^{m+1} \times (x + x^{\perp}) \times ((\alpha, \beta) + (\alpha, \beta)^{\perp})$ so that $f(B, y, (\lambda, \mu)) = \sum \lambda^k \mu^{m-k} B_k y$. When $(x, (\alpha, \beta))$ is an eigenpair associated with A and when the problem $(A, x, (\alpha, \beta))$ is well-posed then, $f(A, x, (\alpha, \beta)) = 0$ and the derivative

$$D_2 f(A, x, (\alpha, \beta)) : x^{\perp} \times (\alpha, \beta)^{\perp} \to \mathbb{C}^n$$

is invertible. The corresponding implicit function is the map $g = \psi \circ G$. We refer to Figure 5.1 for a sketch of the solution maps G and g.

In the following lemmas we compute the invariants μ, $\mu^{\dagger}$, and γ_2. Notice that, for any $u \in \mathbb{C}^n$, the partial derivative with respect to (x, α, β) satisfies

$$D_2 f(A, x, (\alpha, \beta))^{-1} u = \mathcal{J}_P^{-1} \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}.$$

Lemma 5.3. *We have*

$$\|F(A, x, \alpha, \beta)\| \leq \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} \|A\|_F \|x\|,$$

with

$$\|A\|_F^2 = \sum_{k=0}^{m} \|A_k\|_F^2 = \sum_{k=0}^{m} \sum_{i,j=1}^{n} |A_{k,ij}|^2$$

the Frobenius norm of the matrix tuple A.

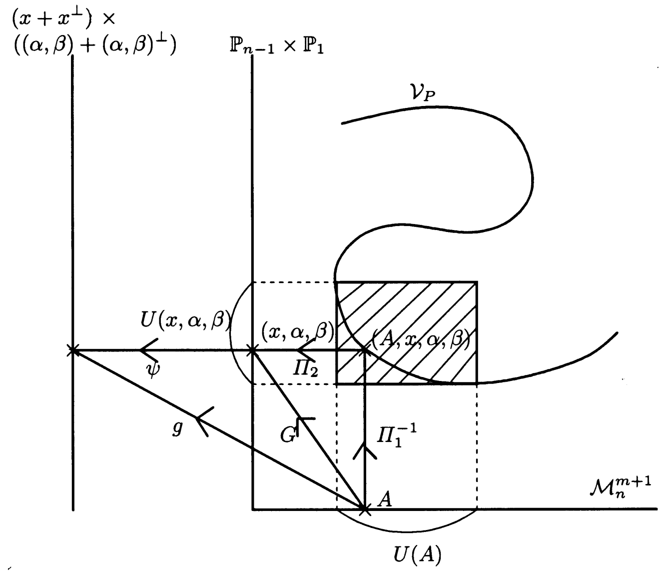

Fig. 5.1. The solution maps G and g.

Proof. First we have $\|F(A, x, \alpha, \beta)\| \leq \|P(A, \alpha, \beta)\|_F \|x\|$. Writing $P(A, \alpha, \beta)$ as

$$P(A, \alpha, \beta) = [A_m, A_{m-1} \cdots A_0][\alpha^m I, \alpha^{m-1}\beta I, \ldots, \beta^m I]^T,$$

and using $\|EF\|_F \leq \|E\|_F \|F\|_2$ we obtain

$$\|P(A, \alpha, \beta)\|_F \leq \|A\|_F \|[\alpha^m I, \alpha^{m-1}\beta I, \ldots, \beta^m I]\|_2.$$

The result follows from $\|[\alpha^m I, \alpha^{m-1}\beta I, \ldots, \beta^m I]\|_2 = (\sum_{k=0}^m |\alpha|^{2k}|\beta|^{2(m-k)})^{1/2}$. □

The following lemma is taken from Blum, Cucker, Shub, and Smale [**2, Section 14.2, Proposition 1**].

Lemma 5.4. *Let h: $\mathbb{C}^n \to \mathbb{C}$ be a degree d homogeneous polynomial. Let us define its norm as*

$$\|h\|^2 = \sum_{\alpha} \binom{d}{\alpha_1, \ldots, \alpha_n}^{-1} |h_\alpha|^2,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers satisfying

$$\alpha_1 + \cdots + \alpha_n = d, \quad \binom{d}{\alpha_1, \ldots, \alpha_n} = \frac{d!}{\alpha_1! \cdots \alpha_n!}, \quad \textit{and} \quad h = \sum_{\alpha} h_\alpha z^\alpha.$$

Then, for any $x, w_1, \dots, w_k \in \mathbb{C}^n$ one has

$$|D^k h(x)(w_1, \dots, w_k)| \le d(d-1)\cdots(d-k+1)\|h\|\|x\|^{d-k}\|w_1\|\cdots\|w_k\|.$$

Remark 5.5. A similar result holds for homogeneous polynomial systems $H\colon \mathbb{C}^n \to \mathbb{C}^m$, $H = (H_1, \dots, H_m)$. In that case

$$\|D^k H(x)(w_1, \dots, w_k)\| \le d(d-1)\cdots(d-k+1)\|H\|\|x\|^{d-k}\|w_1\|\cdots\|w_k\|$$

with $\|H\|^2 = \sum_i \|H_i\|^2$ and $d = \max_i \deg H_i$. For this reason

$$\begin{aligned}\|D^k H(x)\| &= \max_{\|w_i\|\le 1} \|D^k H(x)(w_1, \dots, w_k)\| \\ &\le d(d-1)\cdots(d-k+1)\|H\|\|x\|^{d-k}.\end{aligned}$$

Note that since $F(A, x, (\alpha, \beta))$ is linear in A there is no loss of generality in assuming that $\|A\|_F = 1$.

Lemma 5.6. *When $\|x\| = 1$, $\|A\|_F = 1$, and $\sum_{k=0}^m |\alpha|^{2k}|\beta|^{2(m-k)} = 1$ we have the following estimate*:

$$\sup_{k\ge 2}\left\|\frac{D^k F(A, x, (\alpha, \beta))}{k!}\right\|^{1/k-1} \le \frac{(m+1)(m+2)}{2}.$$

Proof. When $k \ge 2$, the kth derivative of F is given by

$$\begin{aligned}
&D^k F(A, x, (\alpha, \beta))((E_1, \dot{x}_1, (\dot{\alpha}_1, \dot{\beta}_1)), \dots, (E_k, \dot{x}_k, (\dot{\alpha}_k, \dot{\beta}_k)))\\
&= D^k_{(\alpha,\beta)} F(A, x, (\alpha, \beta))((\dot{\alpha}_1, \dot{\beta}_1), \dots, (\dot{\alpha}_k, \dot{\beta}_k))\\
&+ \sum_{i=1}^k D^{k-1}_{(\alpha,\beta)} F(A, \dot{x}_i, (\alpha, \beta))((\dot{\alpha}_1, \dot{\beta}_1), \dots, \widehat{(\dot{\alpha}_i, \dot{\beta}_i)}, \dots, (\dot{\alpha}_k, \dot{\beta}_k))\\
&+ \sum_{i=1}^k D^{k-1}_{(\alpha,\beta)} F(E_i, x, (\alpha, \beta))((\dot{\alpha}_1, \dot{\beta}_1), \dots, \widehat{(\dot{\alpha}_i, \dot{\beta}_i)}, \dots, (\dot{\alpha}_k, \dot{\beta}_k))\\
&+ \sum_{\substack{i,j=1\\ i\ne j}}^k D^{k-2}_{(\alpha,\beta)} F(E_i, \dot{x}_j, (\alpha, \beta))((\dot{\alpha}_1, \dot{\beta}_1), \dots, \widehat{(\dot{\alpha}_i, \dot{\beta}_i)}, \dots, \widehat{(\dot{\alpha}_j, \dot{\beta}_j)}, \dots, (\dot{\alpha}_k, \dot{\beta}_k)).
\end{aligned}$$

In these formulas we overlined the missing terms and the derivatives are taken with respect to (α, β). According to Remark 5.5 and Lemma 5.3 we obtain the following bound:

$$\begin{aligned}\|D^k F(A, x, (\alpha, \beta))\| &\le m(m-1)\cdots(m-k+1)\\ &\quad + km(m-1)\cdots(m-(k-1)+1)\\ &\quad + km(m-1)\cdots(m-(k-1)+1)\\ &\quad + (k^2-k)m(m-1)\cdots(m-(k-2)+1)\\ &= (m+2)(m+1)\cdots(m-k+3).\end{aligned}$$

To conclude, we notice, as in Blum, Cucker, Shub, and Smale [**2, Section 14.2, Lemma 10**], that $\sup_{k\ge 2}((m+2)(m+1)\cdots(m-k+3)/k!)^{1/k-1}$ is achieved for $k=2$. $\square$

Lemma 5.7. *When $\|x\| = 1$ and $\|A\|_F = 1$ and $\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)} = 1$, we have*

$$\gamma_2(f, (A, x, (\alpha, \beta)) \le \|\mathcal{J}_P^{-1}\| \frac{(m+1)(m+2)}{2}.$$

Proof. This is a consequence of Lemma 5.6 and the definition of γ_2. $\square$

Lemma 5.8. *When $\|x\| = 1$ and $\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)} = 1$ we have*

$$\mu(f, (A, x, (\alpha, \beta)) \le \sqrt{1 + \|\mathcal{J}_P^{-1}\|^2}.$$

Proof. This is a consequence of Lemma 5.3 and the equality

$$D_2 f(A, x, (\alpha, \beta))^{-1} D_1 f(A, x, (\alpha, \beta))E = \mathcal{J}_P^{-1}\begin{pmatrix} P(E, \alpha, \beta)x \\ 0 \\ 0 \end{pmatrix}. \qquad \square$$

Lemma 5.9. *When $\|x\| = 1$ and $\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)} = 1$ we have*

$$\mu^{\dagger}(f, (A, x, (\alpha, \beta)) = \|\mathcal{J}_P\|.$$

Proof. Our aim is to estimate the norm of the operator

$$(D_2 f(A, x, (\alpha, \beta))^{-1} D_1 f(A, x, (\alpha, \beta)))^{\dagger}.$$

The generalized inverse of a linear surjective operator is its right inverse of minimum norm, that is, the inverse taking its values in the orthogonal to the kernel. Using the unitary invariance of the homogeneous polynomial eigenvalue problem we may suppose $x = e_1$ the first vector of the canonical basis in $\mathbb{C}^n$. This kernel is the set of $(m+1)$-tuples of matrices $\dot{A}$ such that $\sum_k \alpha^k \beta^{m-k} E_{k,i1} = 0$ for each $i = 1, \ldots, n$. This is an orthogonality relation between E and the $(m+1)$-tuples

$$B^{(i)} = (B_0^{(i)}, \ldots, B_m^{(i)}), \qquad i = 1, \ldots, n,$$

defined by $B_{k,i1}^{(i)} = \bar{\alpha}^k \bar{\beta}^{m-k}$ and 0 for the other entries. Given $(\dot{x}, \dot{\alpha}, \dot{\beta}) \in x^{\perp} \times (\alpha, \beta)^{\perp}$, we want to find an $(m+1)$-tuple of matrices $E = \sum_i \nu_i B^{(i)}$ such that

$$D_2 f(A, x, (\alpha, \beta))^{-1} D_1 f(A, x, (\alpha, \beta)) E = \mathcal{J}_P^{-1} \begin{pmatrix} P(E, \alpha, \beta) e_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix}.$$

This equation is equivalent to

$$\begin{pmatrix} \sum_k \alpha^k \beta^{m-k} E_k e_1 \\ 0 \\ 0 \end{pmatrix} = \sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \\ 0 \\ 0 \end{pmatrix} = \mathcal{J}_P \begin{pmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix}.$$

Thus

$$\|E\|_F = \left\| \mathcal{J}_P \begin{pmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix} \right\|. \qquad \square$$

We now have all the ingredients necessary to translate Theorem 1.4 to this new context.

Theorem 5.10. *Let $(x, (\alpha, \beta))$ be an eigenpair associated with the homogeneous polynomial eigenvalue problem*

$$\left(\sum_{k=0}^{m} \alpha^k \beta^{m-k} A_k \right) x = 0,$$

where $\|A\|_F = 1$. We suppose this problem well-posed and also $\|x\| = 1$, $\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} = 1$. Let λ, c, r, and ε be such that

$$0 \le \lambda < 1 - \frac{\sqrt{2}}{2}, \qquad c = \frac{2 - \lambda}{2(1-\lambda)^2},$$

$$r = \frac{3 + 2\sqrt{2}}{2} (m+1)(m+2) \kappa(\mathcal{J}_P)(1 + \|\mathcal{J}_P^{-1}\|),$$

where $\kappa(\mathcal{J}_P) = \|\mathcal{J}_P\| \|\mathcal{J}_P^{-1}\|$ *is the condition number of* $\mathcal{J}_P$ *and*

$$0 \leq \varepsilon r \leq \lambda.$$

Then $\gamma(\psi \circ G, A) \leq r$,

$$\psi^{-1}\left(g(A) + Dg(A)(B_F(0, \varepsilon(1 - c\varepsilon r)))\right) \subseteq G(B_F(A, \varepsilon)),$$

and

$$G(B_F(A, \varepsilon)) \subseteq \psi^{-1}\left(g(A) + Dg(A)\left(B_F\left(0, \frac{\varepsilon}{1 - \varepsilon r}\right)\right)\right).$$

Proof. This theorem comes from Theorem 1.4 and Lemmas 5.7, 5.8, and 5.9. The inequality $\gamma(\psi \circ G, A) \leq r$ comes from Theorem 1.3. □

The upper and lower estimates for the set $G(B_F(A, \varepsilon))$ in Theorem 5.10 are given in terms of $Dg(A) : \mathcal{M}_n(\mathbb{C})^{m+1} \to x^{\perp} \times (\alpha, \beta)^{\perp}$. In the following lemma we compute this derivative.

Lemma 5.11. *Let* (x, α, β) *be a simple eigenpair of our polynomial eigenvalue problem. Assume that* $\|x\| = 1$ *and* $\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} = 1$. *We have*

$$Dg(A)(E) = \mathcal{J}_P^{-1} \begin{pmatrix} -P(E, \alpha, \beta)x \\ 0 \\ 0 \end{pmatrix}.$$

Proof. Recall that $G = \Pi_2 \circ \Pi_1^{-1}$ so that

$$DG(A) = D\Pi_2(x, \alpha, \beta) \circ (D\Pi_1(A, x, \alpha, \beta))^{-1}.$$

The last derivative is given by

$$D\Pi_1(A, x, \alpha, \beta) : T_{(A,x,\alpha,\beta)}\mathcal{V}_P \to \mathcal{M}_n(\mathbb{C})^{m+1},$$
$$D\Pi_1(A, x, \alpha, \beta)(E, \dot{x}, \dot{\alpha}, \dot{\beta}) = E.$$

Hence $D\Pi_1(A, x, \alpha, \beta)^{-1}(E) = (E, \dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(A,x,\alpha,\beta)}\mathcal{V}_P$ so that

$$DG(A)(E) = (\dot{x}, \dot{\alpha}, \dot{\beta}).$$

According to the description of this tangent space we have

$$\begin{pmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \mathcal{J}_P^{-1} \begin{pmatrix} -P(E, \alpha, \beta)x \\ 0 \\ 0 \end{pmatrix}.$$

Since $g = \psi \circ G$, $Dg(A) = D\psi(x, \alpha, \beta) \circ DG(A)$ and this proves our lemma because $D\psi(x, \alpha, \beta) = \mathrm{id}$ when x and (α, β) are normalized. □

5.2. *Bounding Regions of ε-Pseudospectrum*

In what follows we relate the result in Theorem 5.10 to the ε-pseudospectrum of $P(A, \alpha, \beta)$ defined by

$$\Lambda_\varepsilon(P) = \{(\alpha, \beta) \in \mathbb{C}^2 \backslash (0,0) : P(A+E, \alpha, \beta)x = 0 \\ \text{for some } x \neq 0 \text{ and } E \in \mathcal{M}_n(\mathbb{C})^{m+1} \text{ with } \|E\|_F \leq \varepsilon\}. \quad (5.2)$$

Pseudospectra are a valuable tool for assessing the global sensitivity of matrix eigenvalues to perturbations in the matrix. The literature on pseudospectra is large and growing and we refer to Trefethen [**16**], [**17**], [**18**] for thorough surveys of pseudospectra and their computation for a single matrix; see also the Web site [**5**]. We refer to Higham and Tisseur [**14**], [**6**] for definitions and characterizations of pseudospectra for square and rectangular matrix polynomials. Note that the definition of $\Lambda_\varepsilon(P)$ in (5.2) differs from the one given in [**6**] since it uses the Frobenius norm rather than a subordinate matrix norm $\|\cdot\|_s$ and the perturbations are measured together $\|[E_0, \ldots, E_m]\|_F \leq \varepsilon$ rather than individually $\|E_k\|_s \leq \varepsilon \nu_k$, $0 \leq k \leq m$, for some nonnegative parameter ν_k.

Let (α_i, β_i), $i = 1, 2, \ldots, mn$, be eigenvalues of $P(A, \alpha, \beta)$ and assume that all of them are simple. The map $g = (g_1, g_2)$ depends on the considered eigenvalue (α_i, β_i) and we denote this dependence by g^i. Then

$$\Lambda_\varepsilon(P) = \bigcup_{i=1}^{mn} g_2^i(B_F(A, \varepsilon)). \quad (5.3)$$

For $\varepsilon \leq \lambda/r$ with λ and r as defined in Theorem 5.10, we obtain lower and upper bounds for $\Lambda_\varepsilon(P)$:

$$\bigcup_{i=1}^{mn} \{(\alpha_i, \beta_i) + Dg_2^i(A)(B_F(0, \varepsilon(1 - c\varepsilon r)))\} \subseteq \Lambda_\varepsilon(P),$$

and

$$\Lambda_\varepsilon(P) \subseteq \bigcup_{i=1}^{mn} \{(\alpha_i, \beta_i) + Dg_2^i(A)(B_F(0, \varepsilon/(1 - \varepsilon r)))\}$$

in terms of $Dg_2^i(A) : \mathcal{M}_n(\mathbb{C})^{m+1} \to (\alpha_i, b_i)^\perp$.

Lemma 5.12. *Let (α, β) be a simple eigenvalue with corresponding left and right eigenvectors y and x, respectively. Assume that the eigenpair is normalized so that $\left(\sum_{k=0}^m |\alpha|^{2k} |\beta|^{2(m-k)}\right)^{1/2} = 1$ and $\|x\| = \|y\| = 1$. Then*

$$Dg_2(A)(E) = \frac{y^* P(E, \alpha, \beta)x}{y^* v}(-\bar{\beta}, \bar{\alpha}),$$

where $v = \bar{\beta} D_\alpha P(A, \alpha, \beta)x - \bar{\alpha} D_\beta P(A, \alpha, \beta)x$. Moreover,

$$\max_{\|E\|_F \leq \eta} \|Dg_2(A)(E)\| = \frac{\eta}{|y^* v|}$$

and the maximum is attained for $E = (E_0, \dots, E_m)$ *with*

$$E_k = \eta \bar{\alpha}^k \bar{\beta}^{m-k} y x^*, \qquad k = 1, 2, \dots, m.$$

Proof. See Dedieu and Tisseur [**4, Theorem 4.2**]. □

Lemma 5.12 provides a way of computing the boundary of $(\alpha, \beta) + Dg_2(A)(B_F(0, \eta))$. We form the matrices $E_k = \eta \bar{\alpha}^k \bar{\beta}^{m-k} y x^*$ and the set

$$\left\{ (\alpha, \beta) + \eta e^{i\theta} \frac{y^* P(E, \alpha, \beta) x}{y^* v} (-\bar{\beta}, \bar{\alpha}),\ \theta \in [0, 2\pi] \right\}.$$

The next lemma gives a characterization of $\Lambda_\varepsilon(P)$ in terms of a scaled resolvent norm and is the basis of our algorithm for computing $\Lambda_\varepsilon(P)$ or parts of it.

Lemma 5.13.

$$\Lambda_\varepsilon(P) = \left\{ (\alpha, \beta) \in \mathbb{C}^2 \backslash (0, 0) : \right.$$

$$\left. \| P(A, \alpha, \beta)^{-1} \|_F^{-1} \le \varepsilon \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right)^{1/2} \right\} \tag{5.4}$$

with $\| P(A, \alpha, \beta)^{-1} \|_F^{-1} = 0$ *when* (α, β) *is an eigenvalue.*

Proof. Let $\mathcal{S}$ denote the set on the right-hand side of (5.4). We first show that $(\alpha, \beta) \in \Lambda_\varepsilon(P)$ implies $(\alpha, \beta) \in \mathcal{S}$. If (α, β) is an eigenvalue of P this is immediate, so we can assume that (α, β) is not an eigenvalue of P and hence that $P(A, \alpha, \beta)$ is nonsingular. Let E and $y \neq 0$ be such that $\|E\|_F \le \varepsilon$ and $P(A + E, \alpha, \beta) y = 0$. Since

$$P(A + E, \alpha, \beta) = P(A, \alpha, \beta)(I + P(A, \alpha, \beta)^{-1} P(E, \alpha, \beta))$$

is singular, we have

$$1 \le \| P(A, \alpha, \beta)^{-1} P(E, \alpha, \beta) \|_F \le \varepsilon \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right)^{1/2} \| P(A, \alpha, \beta)^{-1} \|_F$$

using Lemma 5.3 so that $(\alpha, \beta) \in \mathcal{S}$.

Now let $(\alpha, \beta) \in \mathcal{S}$. Again we can assume that $P(A, \alpha, \beta)$ is nonsingular. Choose y with $\|y\| = 1$ so that $\| P(A, \alpha, \beta)^{-1} y \| = \| P(A, \alpha, \beta)^{-1} \|_F$ and let $x = P(A, \alpha, \beta)^{-1} y / \| P(A, \alpha, \beta)^{-1} \|_F$, so that $\|x\| = 1$. Let $E = -yx^* / \| P(A, \alpha, \beta)^{-1} \|_F$. Then

$$(P(A, \alpha, \beta) + E) x = \frac{y}{\| P(A, \alpha, \beta)^{-1} \|_F} - \frac{y}{\| P(A, \alpha, \beta)^{-1} \|_F} = 0,$$

and

$$\|E\|_F = 1/\|P(A,\alpha,\beta)^{-1}\|_F \leq \varepsilon \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2}.$$

We now apportion E between the E_k by defining

$$E_k = -\operatorname{sign}(\alpha^k \beta^{m-k}) \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{-1} E,$$

where, for complex z,

$$\operatorname{sign}(z) = \begin{cases} \bar{z}/|z|, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Then

$$P(E,\alpha,\beta) = \sum_{k=0}^{m} \alpha^k \beta^{m-k} E_k = E,$$

and $\|E\|_F \leq \varepsilon$. Hence $(\alpha,\beta) \in \Lambda_\varepsilon(P)$. $\square$

Note that the standard way to numerically compute and visualize pseudospectra is to evaluate $\|P(A,z,1)^{-1}\|_F^{-1}/(\sum_{k=0}^{m}|z|^{2k})^{1/2}$ over a finite region of the complex plane and plot level curves.

Let us illustrate the results in this section with the following example. We consider the quadratic eigenvalue problem

$$\alpha^2 A_2 + \alpha\beta A_1 + \beta_0 A_0 = 0$$

given by the matrices

$$A_0 = \begin{pmatrix} 2 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The six eigenvalues are (0, 1), (1, 1), (1, 1), (2, 1), (3, 1), and (1, 0). We consider the first one $(\alpha,\beta) = (0,1)$ whose right and left eigenvectors are given by $x = y = (0,1,0)^T$. We take in Theorem 5.10, $\lambda = 0.29289$. This gives $c = 1.7071$, $r = 2.86 \times 10^{-4}$, and $\lambda/r \approx 10^{-4}$. Pairs of complex numbers (α,β) are plotted as α/β. The straight line in Figure 5.2 represents the boundary of $\Lambda_\varepsilon(P)$ for $\varepsilon = 10^{-4}$ in a neighborhood of the zero eigenvalue. The dotted lines are boundaries of the upper and lower estimates of $\Lambda_\varepsilon(P)$ in this region.

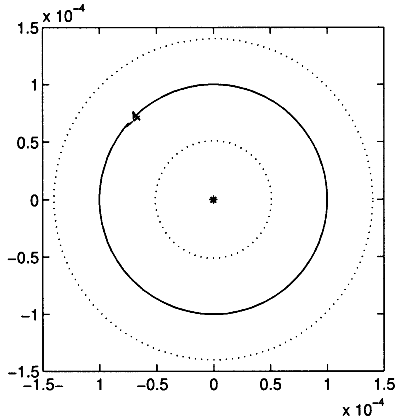

Fig. 5.2. Boundaries of $\Lambda_\varepsilon(P)$ (solid line) and boundaries of the upper and lower estimates of $\Lambda_\varepsilon(P)$ (dotted line) for $\varepsilon = 10^{-4}$ as seen in the $(z, 1)$-plane in $\mathbb{C}^2$.

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