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Stably ergodic approximation: two examples

MICHAEL SHUB†§ and AMIE WILKINSON‡

† *IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598, USA*
(e-mail: mshub@us.ibm.com)
‡ *Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA*
(e-mail: wilkinso@math.nwu.edu)

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Abstract. It has been conjectured that the stably ergodic diffeomorphisms are open and dense in the space of volume-preserving, partially hyperbolic diffeomorphisms of a compact manifold. In this paper we deal with two recalcitrant examples; the standard map cross Anosov and the ergodic automorphisms of the 4-torus. In both cases we show that they may be approximated by stably ergodic diffeomorphisms which have the stable accessibility property.

0. *Introduction*

It was conjectured in [**PS2**] that the stably ergodic diffeomorphisms are open and dense in the space of volume-preserving, partially hyperbolic diffeomorphisms of a compact manifold. Recall that a diffeomorphism $f : M \to M$ of a compact manifold M is *partially hyperbolic* if the tangent bundle TM splits as a Whitney sum of Tf-invariant subbundles:

$$TM = E^u \oplus E^c \oplus E^s,$$

and there exist a Riemannian (or Finsler) metric on M and constants $\lambda < 1$ and $\mu > 1$ such that for every $p \in M$,

$$m(T_p f|_{E^u}) > \mu > \|T_p f|_{E^c}\| \geq m(T_p f|_{E^c}) > \lambda > \|T_p f|_{E^s}\| > 0.$$

(The co-norm $m(A)$ of a linear operator A between Banach spaces is defined by $m(A) := \inf_{\|v\|=1} \|A(v)\|$.) The bundles E^u, E^c and E^s are referred to as the *unstable*, *center* and *stable* bundles of f, respectively. A degenerate example of a partially hyperbolic diffeomorphism is an Anosov diffeomorphism, for which $E^c = \{0\}$. We give more examples below.

If f is C^k and partially hyperbolic, then its stable and unstable bundles are uniquely integrable and are tangent to foliations $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$, whose leaves are C^k. A partially hyperbolic diffeomorphism is said to have the *accessibility property* if, for every pair of points $p, q \in M$, there is a continuous path $\gamma : [0, 1] \to M$ such that:

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- $\gamma(0) = p$,
- $\gamma(1) = q$,
- there exist $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_i, t_{i+1}]) \subseteq \mathcal{W}_f^{a_i}(\gamma(t_i))$, where $a_i = u$ or s, for $i = 0, \ldots, n-1$.

The path γ is called a (n-legged) $\mathcal{W}_f^u$, $\mathcal{W}_f^s$-path.

The unstable and stable foliations of a volume-preserving Anosov diffeomorphism have the accessibility property, since they are transverse. More generally, a pair of non-transverse foliations can have the accessibility property; in this case accessibility is a global version of the non-integrability of the pair of foliations.

Partial hyperbolicity is an open property in the C^1 topology on M, and so any diffeomorphism g of M that is sufficiently C^1-close to the partially hyperbolic diffeomorphism f has stable and unstable foliations $\mathcal{W}_g^u$ and $\mathcal{W}_g^s$. We say that f has the *stable accessibility property* if every g sufficiently C^1-close to f has the accessibility property. It is an open question whether accessibility implies stable accessibility.

A volume-preserving C^2 diffeomorphism is *stably ergodic* if every sufficiently C^1-small, volume-preserving perturbation of it is ergodic. In [**PS2**] it was shown that C^2, volume-preserving, partially hyperbolic diffeomorphisms with the stable accessibility property and which satisfy certain other technical hypotheses are stably ergodic. In the direction of proving the stable ergodicity conjecture, it is further conjectured in [**PS2**] that the stable accessibility property is open and dense among partially hyperbolic diffeomorphisms.

In this paper, we consider two examples of partially hyperbolic diffeomorphisms that do not have the accessibility property. In fact, in these examples, the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are non-transverse and jointly integrable. We prove that these examples can be arbitrarily closely approximated in the C^r topology $2 \le r \le \infty$ by diffeomorphisms that are stably ergodic and that have the stable accessibility property.

0.1. *Non-trivial center behavior.* Let $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ be the n-torus. We will write this group additively.

Let λ be a real parameter. The *standard map* g_λ of the 2-torus is defined by

$$g_\lambda(z, w) = (z + w, w + (\lambda \sin(2\pi(z + w)))).$$

It preserves the symplectic form $dz \wedge dw$. By KAM theory, for all values of λ near zero, g_λ has a positive-measure set of invariant circles. For such parameter values, this map is persistently not ergodic; any sufficiently nearby C^∞ symplectic map will fail to be ergodic.

If we add some transverse hyperbolicity to this example a very different phenomenon appears. If $f : \mathbf{T}^{2n} \to \mathbf{T}^{2n}$ is a C^r, symplectic Anosov diffeomorphism, then $f \times g_\lambda$ is not ergodic for small λ; it has a positive measure set of invariant, codimension-1 tori. However, now, $f \times g_\lambda$ may be approximated by a stably ergodic diffeomorphism, and all of these invariant tori disappear. This result should be contrasted with the work of Cheng and Sun [**CS**], Herman (summarized in [**Y**]), and Xia [**X**], showing the persistence of codimension-1 invariant tori in non-hyperbolic situations.

THEOREM A. *Let $f : \mathbf{T}^{2n} \to \mathbf{T}^{2n}$ be a C^r symplectic Anosov diffeomorphism, $r \geq 2$, and let $g_0 : \mathbf{T}^{2m} \to \mathbf{T}^{2m}$ be a symplectic linear map whose eigenvalues lie on the unit circle in $\mathbf{C}$.*

Then there is a neighborhood $\mathcal{U}$ of g_0 in the space of symplectic C^r diffeomorphisms $\mathrm{Diff}^r_\omega(\mathbf{T}^{2m})$ such that for every $g \in \mathcal{U}$, the diffeomorphism $f \times g : \mathbf{T}^{2(m+n)} \to \mathbf{T}^{2(m+n)}$ is partially hyperbolic. Furthermore, for every neighborhood $\mathcal{V}$ of $f \times g$ in $\mathrm{Diff}^r_\omega(\mathbf{T}^{2(m+n)})$, there exists a $h \in \mathcal{V}$ such that h is stably accessible and stably ergodic.

COROLLARY A. *For f any C^∞, symplectic Anosov diffeomorphism, the map $f \times g_\lambda$ can be C^∞ approximated arbitrarily well by a symplectic, stably ergodic diffeomorphism if λ is sufficiently close to zero.*

Let f and g be symplectic diffeomorphisms of tori. The product $f \times g$ is not ergodic if g is not ergodic. The proof of Theorem A can be slightly adapted to show that, regardless of what properties g has, f can be chosen so that $f \times g$ can be approximated arbitrarily well by a stably ergodic diffeomorphism.

THEOREM B. *Let $g : \mathbf{T}^{2m} \to \mathbf{T}^{2m}$ be a C^r symplectic diffeomorphism, $r \geq 2$. For any $n \geq 1$, there exists a C^r symplectic Anosov diffeomorphism $f : \mathbf{T}^{2n} \to \mathbf{T}^{2n}$ such that $f \times g : \mathbf{T}^{2(m+n)} \to \mathbf{T}^{2(m+n)}$ can be C^r approximated arbitrarily well by $h : \mathbf{T}^{2(m+n)} \to \mathbf{T}^{2(m+n)}$, where*

- *h is a stably accessible, stably ergodic symplectic diffeomorphism,*
- *h preserves $\{(0,0)\} \times \mathbf{T}^{2m}$ and*

$$h|_{\{(0,0)\}\times\mathbf{T}^{2m}} = \mathrm{Id} \times g.$$

We remark that the word 'symplectic' may be replaced by the phrase 'volume-preserving' in Theorems A and B.

0.2. *An algebraic example.* Let G be a connected Lie group and let B be a closed subgroup of G such that G/B is compact. For $g \in G$ denote by $L_g : G/B \to G/B$ the left translation $L_g(aB) = gaB$. Let $A : G \to G$ be an automorphism such that $A(B) = B$; then A induces a diffeomorphism $A : G/B \to G/B$. An *affine diffeomorphism* of G/B is a map of the form

$$L_g \circ A : G/B \to G/B.$$

Suppose that the Haar measure on G projects to a finite measure ν on G/B, invariant under left translations and under the action of A. Then the affine diffeomorphism $L_g \circ A$ preserves ν, and the ergodic properties we discuss below are with respect to ν. Let $\mathfrak{g}$ be the Lie algebra of G and let $f = L_g \circ A : G/B \to G/B$ be an affine diffeomorphism. Then f induces the Lie algebra automorphism $l(f) : \mathfrak{g} \to \mathfrak{g}$:

$$l(f) = \mathrm{ad}(g) \circ T_e(A),$$

where $\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of g on $\mathfrak{g}$. Then $\mathfrak{g}$ splits into generalized eigenspaces for $l(f)$:

$$\mathfrak{g} = \mathfrak{g}^u \oplus \mathfrak{g}^c \oplus \mathfrak{g}^s,$$

where the eigenvalues of $l(f)$ have modulus >1, $=1$ and <1, on $\mathfrak{g}^u$, $\mathfrak{g}^c$ and $\mathfrak{g}^s$, respectively. The Lie subalgebra $\mathfrak{h}$ generated by $\mathfrak{g}^u \oplus \mathfrak{g}^s$ is an ideal in $\mathfrak{g}$. The following is proved in [**PS2**].

PROPOSITION 0.1. *The affine diffeomorphism $f = L_g \circ A$ of G/B is partially hyperbolic and has the stable accessibility property if and only if $\mathfrak{h} \oplus \mathfrak{b} = \mathfrak{g}$, where $\mathfrak{b}$ is the Lie subalgebra $\mathfrak{b} = T_e B$.*

In view of the main theorem in [**PS2**] (Theorem 1.1 below), this has a corollary as follows.

COROLLARY 0.2. *The affine diffeomorphism $f = L_g \circ A$ of G/B is stably ergodic if $l(f)$ has at least one eigenvalue of modulus different than one, and $\mathfrak{h} \oplus \mathfrak{b} = \mathfrak{g}$.*

This corollary, combined with the results in Brezin and Shub [**BS**], completely classifies the stably ergodic affine transformations of simple Lie groups. Nonetheless, there are very basic examples which are not covered by this corollary, and for which stable ergodicity is not known to hold. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}.$$

The matrix A induces a volume-preserving diffeomorphism of the 4-torus $f_A : \mathbf{T}^4 \to \mathbf{T}^4$. The map $l(f_A) = A$ has eigenvalues $\{\exp \pm 2\pi\alpha, \lambda^{\pm 1}\}$, where α is irrational and $\lambda > 1$; since none of these are roots of one, it is easy to see that f_A is ergodic.

For this example, $G = \mathbf{R}^4$, $B = \mathbf{Z}^4$, and the hyperbolic subalgebra $\mathfrak{h}$ is two dimensional, as is $\mathfrak{h} \oplus \mathfrak{b}$, and so Corollary 0.2 does not apply: f_A does not have the accessibility property. In this paper, we prove the following.

THEOREM C. *f_A can be approximated (in the C^∞ topology) arbitrarily well by a stably accessible, stably ergodic diffeomorphism.*

It is still an open question whether f_A itself is stably ergodic. The same techniques that prove Theorem C also show the following.

THEOREM D. *Any ergodic automorphism of $\mathbf{T}^n$ that induces an isometry on the center space $\mathfrak{g}^c$ can be C^∞ approximated arbitrarily well by a stably accessible, stably ergodic diffeomorphism.*

The techniques of this paper do not, however, extend to all ergodic, partially hyperbolic affine transformations.

1. *Preliminaries*

Recently, Pugh and Shub proved the following theorem [**PS2**].

THEOREM 1.1. *If $f \in \mathrm{Diff}^2_m(M)$ is a center bunched, partially hyperbolic, stably dynamically coherent diffeomorphism that is stably accessible, then f is stably ergodic.*

In the proofs of Theorems A and C below, we rely on this result to show stable ergodicity.

A partially hyperbolic diffeomorphism f is *center bunched* if, for every $p \in M$, the quantity

$$\mu_c = \|T_p f|_{E^c_f}\| / m(T_p f|_{E^c_f})$$

is sufficiently close to one. The property is C^1-open, and immediately satisfied when $\mu_c = 1$. The details can be found in [**PS2**, §4].

For the diffeomorphisms $f \times g$, in the statement of Theorems A and B, and f_A, in the statement of Theorem C, μ_c is either one or can be made arbitrarily close to one, and so both examples are center bunched.

A partially hyperbolic diffeomorphism f is *dynamically coherent* if the distributions E^c_f, $E^c_f \oplus E^u_f$ and $E^c_f \oplus E^s_f$ are integrable, and everywhere tangent to foliations $\mathcal{W}^c_f$, $\mathcal{W}^{cu}_f$ and $\mathcal{W}^{cs}_f$, called the *center*, *center-unstable* and *center-stable* foliations, respectively. If f is dynamically coherent and $\mathcal{W}^c_f$ is a C^1 foliation, then by [**PS1**, Proposition 2.3], f is *stably* dynamically coherent, i.e. C^1-small perturbations of f are also dynamically coherent. This is the case for $f \times g$ and f_A in Theorems A and C, as we verify in the next section. The heart of the matter in proving Theorems A and C is to produce perturbations of $f \times g$ and f_A that are stably accessible, and this is the focus of the following sections.

2. *Proof of Theorem A*

To simplify the notation of the proof, assume that $m = n = 1$ and that $r = \infty$. Let $T(\mathbf{T}^2) = E^u_f \oplus E^s_f$ be the Anosov splitting for f. Write $\mathbf{T}^4 = T_1 \times T_2$, where $T_i = \mathbf{T}^2$, and let $\pi_i : T_1 \times T_2 \to T_i$ be the projection onto the ith $\mathbf{T}^2$ factor, for $i = 1, 2$. In what follows, the metric d on $\mathbf{T}^n$ is

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{i \in \{1, \ldots, n\}} d_0(x_i, y_i),$$

where d_0 is the standard metric on the circle. We use the letters 'u' and 'v' to refer to points in $\mathbf{T}^4$; 'x', 'y' and 'p' to refer to points in T_1; and 'q' and 'r' to refer to points in T_2. The symbols 'z' and 'w' are reserved for the $\mathbf{T}^1$-coordinates of points in $\mathbf{T}^4$.

The map $f \times g$ is partially hyperbolic: the unstable, center and stable manifolds through the point $(x, q) \in T_1 \times T_2$ are, respectively,

$$\mathcal{W}^u_{f \times g}(x, q) = \{(y, q) : y \in \mathcal{W}^u_f(x)\},$$
$$\mathcal{W}^c_{f \times g}(x, q) = \{(x, r) : r \in T_2\},$$

and

$$\mathcal{W}^s_{f \times g}(x, q) = \{(y, q) : y \in \mathcal{W}^s_f(x)\}.$$

Clearly $\mathcal{W}^c_{f \times g}$ is a C^1 foliation, and so it is stably dynamically coherent. Furthermore, $f \times g$ is center bunched if g is chosen sufficiently C^1-close to G. Assume that g is so chosen. There exists an $\epsilon_0 > 0$ such that if $h : \mathbf{T}^4 \to \mathbf{T}^4$ is a diffeomorphism with $d_{C^1}(h, f \times g) < \epsilon_0$, then h is partially hyperbolic, center bunched and dynamically coherent (see the discussion in §1). Fix this ϵ_0.

In order to apply Theorem 1.1 to prove Theorem A, it now suffices to show that $f \times g$ can be approximated arbitrarily C^∞ well by a symplectic, stably accessible diffeomorphism.

We are starting with a product diffeomorphism, for which the unstable and stable foliations are jointly integrable. In this situation, any $\mathcal{W}^u_{f \times g}$, $\mathcal{W}^s_{f \times g}$ path lies in a leaf of the foliation of $\mathbf{T}^4$ by horizontal, codimension 2 tori $T_1 \times \{q\}$. We shall perturb $f \times g$

into the previously inaccessible vertical direction, which will have the effect of lifting the unstable and stable foliations out of the horizontal direction. This perturbation is most easily accomplished in the vertical $\{x\} \times T_2$ tori that lie over neighborhoods of heteroclinic orbits for f in T_1. This argument mimics those found in [**B**]. We begin with some lemmas describing the vector fields which will produce the vertical lift.

LEMMA 2.1. *For all $\alpha, \beta \in (0, 1/2)$, there exist flows $Z_t = Z_t^{\alpha,\beta}$ and $W_t = W_t^{\alpha,\beta}$ on $\mathbf{T}^4$ with the properties:*

(1) *Z_t and W_t are C^∞ and symplectic;*

(2) *$Z_t(\{(0,0)\} \times T_2) = \{(0,0)\} \times T_2$, and $W_t(\{(0,0)\} \times T_2) = \{(0,0)\} \times T_2$;*

(3) *if $|w_2| < \beta$, then*

$$Z_t(0, 0, z_2, w_2) = (0, 0, z_2 + t, w_2),$$

and if $|z_2| < \beta$, then

$$W_t(0, 0, z_2, w_2) = (0, 0, z_2, w_2 + t);$$

(4) *if $z_1^2 + w_1^2 > 4\alpha^2$, then $Z_t(z_1, w_1, z_2, w_2) = W_t(z_1, w_1, z_2, w_2) = (z_1, w_1, z_2, w_2)$, for all t.*

We prove Lemma 2.1 at the end of this section.

For $\mathcal{F}$ a foliation of a Riemannian manifold M and U a neighborhood of a point $v \in M$, denote by $\mathcal{F}(v)$ the leaf of $\mathcal{F}$ containing v and by $\mathcal{F}_U(v)$ the connected component of v (in the leaf topology) of $\mathcal{F}(v) \cap U$. For $\rho > 0$, let $\mathcal{F}_\rho(p) = \mathcal{F}_{B_\rho(v)}(v)$.

Since the foliations $\mathcal{W}_f^u$ and $\mathcal{W}_f^s$ of T_1 are uniformly transverse, for every $\rho_0 > 0$ sufficiently small, there exists $\delta_0 > 0$ such that for $x, y \in T_1$ with $d(x, y) \leq \delta_0$, the set $\mathcal{W}_{f,\rho_0}^u(x) \cap \mathcal{W}_{f,\rho_0}^s(y)$ contains a single point, which we denote by $[x, y]$. Fix such $\rho_0 \leq 1/1000$ and $\delta_0 \leq \rho_0$. We introduce notation for the orbit of a set under f: for $C \subseteq T_1$, let

$$\mathcal{O}(C) = \bigcup_{j \in \mathbf{Z}} f^j(C).$$

For $q = (z_2, w_2) \in T_2$, and $\beta > 0$, define $H_\beta(q)$ and $V_\beta(q)$, the *horizontal and vertical strips* in T_2 of radius β, by

$$H_\beta(q) = \{(z_2', w_2',) \in T_2 : |w_2 - w_2'| < \beta\},$$

and

$$V_\beta(q) = \{(z_2', w_2',) \in T_2 : |z_2 - z_2'| < \beta\}.$$

LEMMA 2.2. *Let $p_0, p_1 \in T_1$ be periodic points for f with $d(p_0, p_1) \leq \delta_0$, let $x_1 = [p_0, p_1]$ and $y_1 = [p_1, p_0]$. Let $r_1 \in T_2$ and $\beta \in (0, 1/2)$ be given.*

Let $U \subset T_1$ be a neighborhood of $f^{-1}(x_1)$ and let $h : \mathbf{T}^4 \to \mathbf{T}^4$ be a symplectic diffeomorphism such that:

(A) *h agrees with $f \times g$ on $\mathcal{O}(\{p_0, p_1, x_1, y_1\}) \times T_2$;*

(B) *$d_{C^\infty}(h, f \times g) < \epsilon_0$, where ϵ_0 is defined above;*

(C) *$U \cap (\mathcal{O}(\{p_0, p_1, x_1, y_1\}) \setminus \{f^{-1}(x_1)\}) = \emptyset$.*

Then there exist $\alpha_0 > 0$, $C > 0$ *and* $T > 0$ *such that, for* $|t_0| \leq T$, *if* $X_t = Z_t^{\alpha_0,\beta}$ *or* $X_t = W_t^{\alpha_0,\beta}$ *is a flow given by Lemma 2.1 and* $k = k_{t_0} : \mathbf{T}^4 \to \mathbf{T}^4$ *is the diffeomorphism*

$$k_{t_0}(p, q) = (x_1, r_1) + X_{t_0}(p - x_1, q - r_1),$$

then $k \circ h$ *is symplectic and partially hyperbolic, and*

(1) $k \circ h$ *coincides with* h *outside of* $U \times T_2$*;*

(2) $d_{C^\infty}(h, k \circ h) \leq C|t_0|$*;*

(3) *for any* $q \in T_2$,

$$\mathcal{W}^u_{k\circ h,\rho_0}(p_0, q) \cap (\{x_1\} \times T_2) = (x_1, r_1) + X_{t_0}(0, q - r_1)$$
$$= \begin{cases} (x_1, q + (t_0, 0)) & \text{if } X_t = Z_t \text{ and } q \in H_\beta(r_1) \\ (x_1, q + (0, t_0)) & \text{if } X_t = W_t \text{ and } q \in V_\beta(r_1); \end{cases}$$

(4) *for all* $q \in T_2$,

$$\mathcal{W}^s_{k\circ h,\rho_0}(p_0, q) \cap (\{y_1\} \times T_2) = (y_1, q),$$
$$\mathcal{W}^s_{k\circ h,\rho_0}(p_1, q) \cap (\{x_1\} \times T_2) = (x_1, q)$$

and

$$\mathcal{W}^u_{k\circ h,\rho_0}(p_1, q) \cap (\{y_1\} \times T_2) = (y_1, q).$$

Proof of Lemma 2.2. Pick α_0 so that $B_{2\alpha_0}(x_1) \subseteq f(U)$ and fix $\beta > 0$. Let $X_t = Z_t^{\alpha_0,\beta}$ or $W_t^{\alpha_0,\beta}$ and let k_t be defined as in the statement of the lemma. Clearly property (1) holds. Since X_t is a C^∞ flow and $\mathbf{T}^4$ is compact, there exists a $C > 0$ such that property (2) holds for all $t_0 \in \mathbf{R}$.

Choose $T < \min\{1/4, \epsilon_0/C\}$ small enough so that $\mathcal{W}^u_{k\circ h,\rho_0}(p_0, q)$ intersects $\{x_1\} \times T_2$ in at most one point. We now show that $v = (x_1, r_1) + X_{t_0}(0, q - r_1)$ is in this intersection.

The unstable manifold $\mathcal{W}^u_{k\circ h}(u)$ for $k \circ h$ through the point u is the set of points v such that

$$\lim_{n\to\infty} d((k \circ h)^{-n}(u), (k \circ h)^{-n}(v)) = 0.$$

One easily calculates that

$$k \circ h(f^{-1}(x_1), g^{-1}(q)) = k \circ f \times g(f^{-1}(x_1), g^{-1}(q)) = k(x_1, q) = v,$$

and so

$$k \circ h^{-1}(v) = (f^{-1}(x_1), g^{-1}(q)).$$

Hypotheses (B) and (C) imply that $k\circ h$ agrees with $f \times g$ on the set $\{x_1, f^{-2}(x_1), \ldots\} \times T_2$. Thus, for $n \geq 1$,

$$k \circ h^{-n}(v) = (f^{-n}(x_1), g^{-n}(q)),$$

and

$$\begin{aligned}
&\lim_{n\to\infty} d((k \circ h)^{-n}(v), (k \circ h)^{-n}((p_0, q))) \\
&\quad= \lim_{n\to\infty} d((f^{-n}(x_1), g^{-n}(q)), (f^{-n}(p_0), g^{-n}(q))) \\
&\quad= \lim_{n\to\infty} d(f^{-n}(x_1), f^{-n}(p_0)) \\
&\quad= 0,
\end{aligned}$$

since $x_1 \in \mathcal{W}^u_f(p_0)$. Thus $v \in \mathcal{W}^u_{k\circ h}(p_0, q)$. This proves property (3). Property (4) is proved similarly. □

Returning to the proof of Theorem A, let $\beta = 1/4$ and let $N = 25$. Let $\mathcal{V} = \{V_1, \ldots, V_N\}$ be a cover of T_2 by $\beta/2$-balls, with centers $q_1, \ldots, q_N$: $V_i = B_{\beta/2}(q_i)$. Since f is a transitive Anosov diffeomorphism, periodic points for f are dense in T_1. Let $p_{00}, p_{01}, p_{02}, p_{10}, \ldots, p_{N0}, p_{N1}, p_{N2}$ be distinct periodic points of f such that:

(1) $d(p_{i0}, p_{ij}) < \delta_0$ for $i \in \{1, \ldots N\}$, $j \in \{1, 2\}$;

(2) for every $i, j \in \{1, \ldots, N\}$, with $i \neq j$,

$$B_{2\rho_0}(p_{i0}) \cap B_{2\rho_0}(p_{j0}) = \emptyset.$$

This second property is easily satisfied, since $\rho_0 < 1/1000$.

We now describe the remainder of the construction. For each $i \in \{1, \ldots, N\}$, we perturb $f \times g$ inside the neighborhood $B_{2\rho_0}(p_{i0}) \times T_2$, applying Lemma 2.2 twice. We arrange that the unstable and stable foliations (for the perturbed system) are stably accessible inside $B_{\rho_0}(p_{i0}) \times V_i$. We show that this 'vertical accessibility' can be achieved via an arbitrarily small perturbation of $f \times g$.

Finally, we show that, if the perturbation of $f \times g$ is small enough, then any two sets $B_{\rho_0}(p_{i0}) \times V_i$ and $B_{\rho_0}(p_{j0}) \times V_j$ can be connected by stable and unstable manifolds. This 'horizontal accessibility', combined with vertical accessibility implies stable accessibility for the perturbed system.

We summarize these arguments in two lemmas.

LEMMA 2.3. (Vertical accessibility) *For every $\epsilon > 0$ sufficiently small, there exists $\epsilon_1 > 0$ and a partially hyperbolic symplectic diffeomorphism $h : \mathbf{T}^4 \to \mathbf{T}^4$ with the following properties:*

(1) $d_{C^\infty}(h, f \times g) < \epsilon$,

(2) *if $h' : \mathbf{T}^4 \to \mathbf{T}^4$ is a diffeomorphism with $d_{C^1}(h, h') < \epsilon_1$, then for $i \in \{1, \ldots, N\}$ and $u, v \in B_{\rho_0}(p_{i0}) \times V_i$ there is a $\mathcal{W}^u_{h'}, \mathcal{W}^s_{h'}$-path $\gamma : [0, 1] \to \mathbf{T}^4$ such that*

$$\gamma(0) = u \quad \textit{and} \quad \gamma(1) = v.$$

LEMMA 2.4. (Horizontal accessibility) *There exists $\epsilon_2 > 0$ such that, if $h : \mathbf{T}^4 \to \mathbf{T}^4$ is a diffeomorphism with $d_{C^1}(h, f \times g) < \epsilon_2$, then h is partially hyperbolic, and*

(1) *for every $v \in \mathbf{T}^4$ there exists an $i \in \{1, \ldots, N\}$ and a $\mathcal{W}^u_h, \mathcal{W}^s_h$-path $\gamma_1 : [0, 1] \to \mathbf{T}^4$ such that*

$$\gamma_1(0) = v \quad \textit{and} \quad \gamma_1(1) \in B_{\rho_0}(p_{i0}) \times V_i;$$

(2) *for every $i, j \in \{1, \ldots, N\}$, if $V_i \cap V_j \neq \emptyset$, then there exists a $\mathcal{W}^u_h$, $\mathcal{W}^s_h$-path $\gamma_2 : [0, 1] \to \mathbf{T}^4$ such that*

$$\gamma_2(0) \in B_{\rho_0}(p_{i0}) \times V_i \quad \textit{and} \quad \gamma_2(1) \in B_{\rho_0}(p_{j0}) \times V_j.$$

Using these lemmas we complete the proof of Theorem A: given $\epsilon > 0$, we shall construct a stably accessible symplectic diffeomorphism $h : \mathbf{T}^4 \to \mathbf{T}^4$ such that $d_{C^\infty}(h, f \times g) < \epsilon$. Let ϵ_2 be given by Lemma 2.4. Let h be a diffeomorphism given by Lemma 2.3, chosen so that

$$d_{C^\infty}(h, f \times g) < \min\{\epsilon, \epsilon_2/2, \epsilon_0/2\}.$$

Then h is partially hyperbolic, center bunched, and dynamically coherent.

We now show that h is stably accessible. Let ϵ_1 be given by Lemma 2.3; we may assume that $\epsilon_1 < \min\{\epsilon, \epsilon_2/2, \epsilon_0/2\}$. Let h' be any diffeomorphism with $d_{C^1}(h, h') < \epsilon_1$ and let $u, v \in \mathbf{T}^4$. Then $d_{C^1}(f \times g, h') < \epsilon_1 + d_{C^1}(f \times g, h) < \min\{\epsilon_2, \epsilon_0\}$. By Lemma 2.4, there exist $i, i' \in \{1, \ldots, N\}$ such that u can be connected to a point $r \in B_{\rho_0}(p_{i0}) \times V_i$ and v to $s \in B_{\rho_0}(p_{i'0}) \times V_{i'}$, along $\mathcal{W}^u_{h'}$, $\mathcal{W}^s_{h'}$-paths.

Since T_2 is connected there exists an I-tuple $(i = i_1, \ldots, i_I = i') \in \{1, \ldots, N\}^I$ such that $V_{i_j} \cap V_{i_{j+1}} \neq \emptyset$, for $j \in \{1, \ldots, I-1\}$. By Lemma 2.4, for each $j \in \{1, \ldots, I-1\}$, there exists a $\mathcal{W}^u_{h'}$, $\mathcal{W}^s_{h'}$-path connecting a point $s_j \in B_{\rho_0}(p_{i_j 0}) \times V_{i_j}$ to a point $r_{j+1} \in B_{\rho_0}(p_{i_{j+1}0}) \times V_{i_{j+1}}$. Finally, by Lemma 2.3, there are $\mathcal{W}^u_{h'}$, $\mathcal{W}^s_{h'}$-paths from r to s_1, from r_N to s, and from r_j to s_j, for each $j \in \{2, \ldots, I-1\}$. Concatenating these paths gives a $\mathcal{W}^u_{h'}$, $\mathcal{W}^s_{h'}$-path from u to v. Since any two points u and v can be connected by a $\mathcal{W}^u_{h'}$, $\mathcal{W}^s_{h'}$-path, h' is accessible, and h is stably accessible. By Theorem 1.1, h is stably ergodic.

This completes the proof of Theorem A. We now prove Lemmas 2.3 and 2.4. □

Proof of Lemma 2.3. We omit a proof of the following proposition. In slightly less general form a proof appears in [**KK**] or [**BPW**]. The construction originally appears, in the context of skew products, in [**B**].

PROPOSITION 2.5. *Let M be a compact Riemannian manifold and let $h : M \to M$ be a partially hyperbolic, dynamically coherent diffeomorphism. Then there exist constants $\rho_1 > 0$ and $\rho_2 > 0$ such that: if $\gamma : [0,1] \to M$ is a 4-legged $\mathcal{W}^u_h$, $\mathcal{W}^s_h$-path satisfying:*

(A) $\gamma(1) \in \mathcal{W}^c_h(\gamma(0))$;

(B) $\operatorname{diam}(\gamma[0,1]) \le \rho_1$;

then there exist 4-legged $\mathcal{W}^u_h$, $\mathcal{W}^s_h$-paths $\gamma_t : [0,1] \to M$ satisfying:

(1) γ_0 *is the constant path* $\gamma(0)$;

(2) $\gamma_1 = \gamma$;

(3) $\gamma_t(0) = \gamma(0)$ *for all* $t \in [0,1]$;

(4) $\gamma_t(1) \in \mathcal{W}^c_h(\gamma_t(0))$ *for all* $t \in [0,1]$;

(5) $\operatorname{diam}(\gamma_t[0,1]) \le \rho_2$.

The homotopy given in Proposition 2.5 is not unique, but the family of such homotopies is compact. For such a homotopy, the function $t \mapsto \gamma_t(1)$ defines a curve in $\mathcal{W}^c_h(\gamma(0))$ from $\gamma(0)$ to $\gamma(1)$.

For $f \times g$, condition (A) is the same as $\pi_1(\gamma(0)) = \pi_1(\gamma(1))$. The foliations $\mathcal{W}^u_{f\times g}$ and $\mathcal{W}^s_{f\times g}$ are jointly integrable, so if γ is a 4-legged $\mathcal{W}^u_{f\times g}$, $\mathcal{W}^s_{f\times g}$-path satisfying $\pi_1(\gamma(0)) = \pi_1(\gamma(1)) = p_0$, then $\gamma(0) = \gamma(1)$, and for any such homotopy, $\gamma_t(1) = \gamma(0)$, for all t. The foliations $\mathcal{W}^u_h$ and $\mathcal{W}^s_h$ depend continuously on h, there exists $\epsilon_3 > 0$ such that, if $d_{C^1}(h, f \times g) < \epsilon_3$, then for any homotopy γ_t of the type described above, $\operatorname{diam}(\gamma_{[0,1]}(1)) < 1/100$. Fix this ϵ_3.

For $d(h, f \times g) < \epsilon_0$ and $v \in \mathbf{T}^4$, there are families of paths $\gamma^u = \gamma^u_h(v, \cdot) : \mathbf{R} \to \mathbf{T}^4$ and $\gamma^s = \gamma^s_h(v, \cdot) : \mathbf{R} \to \mathbf{T}^4$ such that $\gamma^a(v, 0) = v$ and $\gamma^a(v, t) \in \mathcal{W}^a_h(v)$, for $t \in \mathbf{R}$ and $a = u, s$. These families can be chosen to depend continuously on v and on h in the C^1 topology. (We remark here that when $\dim(T_1) = 2n > 2$, we must instead choose families of paths $\gamma^u_1, \ldots, \gamma^u_n$ and $\gamma^s_1, \ldots, \gamma^s_n$ spanning $\mathcal{W}^u_h(v)$ and $\mathcal{W}^s_h(v)$, respectively.)

For $v \in \mathbf{T}^4$, and $(s_1, s_2) \in \mathbf{R}^2$, let $\omega_h(v, s_1, s_2) = \gamma^s(\gamma^u(v, s_1), s_2)$. The map $\pi_1 \circ \omega_{f\times g}(v, \cdot, \cdot)$ is a local homeomorphism; a degree argument similar to that in [**BW**, §5], shows that there is an $\epsilon_4 > 0$ and a number $J > 0$, such that for $d_{C^1}(h, f \times g) < \epsilon_3$, and $v \in \mathbf{T}^4$,

$$\pi_1(\omega_h(\{v\} \times [-J, J]^2)) \supseteq B_{2\rho_0}(\pi_1(v)). \qquad (*)$$

and

$$\pi_2(\omega_h(\{v\} \times [-J, J]^2)) \subseteq B_{1/100}(\pi_2(v)). \qquad (**)$$

Step 1: construction of h. We may assume that $\epsilon < \max\{\epsilon_0/2, \epsilon_3/2, \epsilon_4/2\}$. Let $N = 25$ as before. For $i \in \{1, \ldots, N\}$ and $j \in \{1, 2\}$, let $x_{ij} = [p_{i0}, p_{ij}]$ and $y_{ij} = [p_{ij}, p_{i0}]$. The set

$$\mathcal{B} = \bigcup_{n=1}^{N} \mathcal{O}(\{p_{n0}, p_{n1}, p_{n2}, x_{n1}, x_{n2}, y_{n1}, y_{n2}\})$$

is compact and its set of limit points is precisely $\bigcup_{n=1}^{N} \mathcal{O}(\{p_{n0}, p_{n1}, p_{n2}\})$. For $i \in \{1, \ldots, N\}$ and $j \in \{1, 2\}$, choose a neighborhood U_{ij} of $f^{-1}(x_{ij})$ in T_1 such that:

- $U_{ij} \cap \mathcal{B} \setminus \{f^{-1}(x_{ij})\} = \emptyset$;
- $U_{ij} \cap U_{lm} = \emptyset$, if $(l, m) \neq (i, j)$.

For $i \in \{1, \ldots, K\}$, and $j = 1, 2$, let

$$k^{(ij)}_{\alpha,t}(p, q) = \begin{cases} (x_{i1}, q_i) + Z_t^{\alpha,1/4}(p - x_{i1}, q - q_i) & \text{if } j = 1 \\ (x_{i2}, q_i) + W_t^{\alpha,1/4}(p - x_{i2}, q - q_i) & \text{if } j = 2. \end{cases}$$

Let

$$h_{\alpha,t}(p, q) = k^{(11)}_{\alpha,t} \circ k^{(12)}_{\alpha,t} \circ \cdots \circ k^{(1K)}_{\alpha,t} \circ k^{(2K)}_{\alpha,t} \circ f \times g.$$

Since the neighborhoods U_{ij} are disjoint and $U_{ij} \cap \mathcal{O}(\{p_{i0}, p_{ij}, x_{ij}, y_{ij}\}) \setminus \{f^{-1}(x_{ij})\} \subseteq U_{ij} \cap \mathcal{B} \setminus \{f^{-1}(x_{ij})\} = \emptyset$, we may apply Lemma 2.2 independently in each of these neighborhoods to conclude that there exist constants $\alpha_1 > 0$ and $t_1 = \beta/(K+2) < 1/100$, for some integer $K > 0$, such that the map $h = h_{\alpha_1,t_1}$ is symplectic and has the following properties:

(1) $d_{C^\infty}(h, f \times g) < \epsilon$;

(2) for $i \in \{1, \ldots, N\}$, and for all $q \in T_2$,

$$\mathcal{W}^u_{h,\rho_0}(p_{i0}, q) \cap (\{x_{ij}\} \times T_2) = \begin{cases} (x_{i1}, q + (t_1, 0)) & \text{if } j = 1 \text{ and } q \in H_\beta(q_i) \\ (x_{i2}, q + (0, t_1)) & \text{if } j = 2 \text{ and } q \in V_\beta(q_i); \end{cases}$$

(3) for $i \in \{1, \ldots, N\}$, $j \in \{1, 2\}$, and for all $q \in T_2$,

$$\mathcal{W}^s_{h,\rho_0}(p_{i0}, q) \cap (\{y_{ij}\} \times T_2) = (y_{ij}, q),$$
$$\mathcal{W}^s_{h,\rho_0}(p_{ij}, q) \cap (\{x_{ij}\} \times T_2) = (x_{ij}, q)$$

and

$$\mathcal{W}^u_{h,\rho_0}(p_{ij}, q) \cap (\{y_{ij}\} \times T_2) = (y_{ij}, q).$$

Step 2: Stable vertical accessibility. We now show that for each i and for h' sufficiently near h, any two points in $B_{\rho_0}(p_{i0}) \times V_i$ can be connected by a $\mathcal{W}^u_{h'}, \mathcal{W}^s_{h'}$-path.

Fix i, and to simplify notation, let $p_0 = p_{i0}$, $p_1 = p_{i1}$ and $p_2 = p_{i2}$. It follows from Step 1(3) that for each $q \in H_\beta(q_i)$, there is a 4-legged $\mathcal{W}^u_h, \mathcal{W}^s_h$-path $\zeta(q, \cdot)$ such that:

- $\zeta(q, 0) = (p_0, q)$,
- $\zeta(q, 1/4) = (x_1, q + (t_1, 0))$,
- $\zeta(q, 1/2) = (p_1, q + (t_1, 0))$,
- $\zeta(q, 3/4) = (y_1, q + (t_1, 0))$,
- $\zeta(q, 1) = (p_0, q + (t_1, 0))$.

These paths can be chosen to depend continuously on $q \in B_\beta(q_i)$. Similarly, if $q \in V_\beta(q_i)$, then there is a continuous family of 4-legged $\mathcal{W}^u_h, \mathcal{W}^s_h$-paths $\{\eta(q, \cdot) \mid q \in V_\beta(q_i)\}$ such that:

- $\eta(q, 0) = (p_0, q)$,
- $\eta(q, 1/4) = (x_2, q + (0, t_1))$,
- $\eta(q, 1/2) = (p_2, q + (0, t_1))$,
- $\eta(q, 3/4) = (y_2, q + (0, t_1))$,
- $\eta(q, 1) = (p_0, q + (0, t_1))$.

Let $\{\zeta_t(q, \cdot) \mid q \in H_\beta(q_i), t \in [0, 1]\}$ and $\{\eta_t(q, \cdot) \mid q \in V_\beta(q_i), t \in [0, 1]\}$ be homotopies through 4-legged $\mathcal{W}^u_h, \mathcal{W}^s_h$-paths given by Proposition 2.5. These families of homotopies can be chosen to be continuous in q, t and on h in the C^1 topology. The maps $\phi_\zeta(q, t) = \zeta_t(q, 1)$ and $\phi_\eta(q, t) = \eta_t(q, 1)$ are paths in $\{p_0\} \times T_2$ from (p_0, q) to $\zeta_1(q, 1)$ and $\eta_1(q, 1)$, respectively. The diameter of each of these paths is bounded by $1/100$, since $d_{C^1}(h, f \times g) < \epsilon < \epsilon_3$. Concatenating m of these paths $\phi_1, \phi_2, \ldots, \phi_m$ with $\phi_i(1) = \phi_{i+1}(0)$ gives a path $\phi_1 \cdots \phi_m : [0, m] \to \mathbf{T}^4$ such that, for every $t \in [0, m]$,

- $\phi_1 \cdots \phi_m(t) \in \{p_0\} \times T_2$,
- $\phi_1 \cdots \phi_m(t)$ is the endpoint of a $\leq 4m$-legged $\mathcal{W}^u_h, \mathcal{W}^s_h$-path of diameter $\leq m\rho_2$.

Recall that $K = \beta/t_1 - 2$ is a positive integer. By concatenating the paths $\phi_\zeta(q_i + m(t_1, 0), t)$, for $m = 0, \ldots, K - 1$ and $\phi_\zeta(q_i + m(t_1, 0), 1 - t)$, for $m = -1, \ldots, -K$, we extend the map $\phi_\zeta(q_i, \cdot)$ to a map $\phi_\zeta(q_i, \cdot) : [-K, K] \to \mathbf{T}^4$ such that, for all $t \in [-K, K]$:

- $\phi_\zeta(q_i, t) \in \{p_0\} \times T_2$;
- $\phi_\zeta(q_i, t)$ is the endpoint of a $\leq 4K$-legged $\mathcal{W}^u_h, \mathcal{W}^s_h$-path of diameter $\leq K\rho_2$;
- for every $m \in \mathbf{Z} \cap [-K, K]$,

$$\phi_\zeta(q_i, m) = (p_0, q_i + (mt_1, 0));$$

- for every $s \in [-K, K]$,

$$d(\phi_\zeta(q_i, s), \phi_\zeta(q_i, \lfloor s \rfloor)) \leq 1/100.$$

Similarly for $q \in V_\beta(q_i)$, extend $\phi_\eta(q, \cdot)$ to $\phi_\eta(q, \cdot) : [-K, K] \to \mathbf{T}^4$ such that, for all $t \in [-K, K]$:

- $\phi_\eta(q, t) \in \{p_0\} \times T_2$;
- $\phi_\eta(q, t)$ is the endpoint of a $\leq 4K$-legged $\mathcal{W}^u_h, \mathcal{W}^s_h$-path of diameter $\leq K\rho_2$;
- for every $m \in \mathbf{Z} \cap [-K, K]$,

$$\phi_\eta(q, m) = (p_0, q + (0, mt_1));$$

- for every $s \in [-K, K]$,

$$d(\phi_\eta(q, s), \phi_\zeta(q, \lfloor s \rfloor)) \leq 1/100.$$

Next, consider the map

$$\Phi : [-K, K]^2 \to \mathbf{T}^4$$

given by

$$\Phi(s_1, s_2) = \phi_\eta(\pi_2 \circ \phi_\zeta(q_i, s_1), s_2).$$

Notice that Φ is well defined, since $\pi_2 \circ \phi_\zeta(q_i, s_1) \in V_\beta(q_i)$, for $s_1 \in [-K, K]$. For all $(s_1, s_2) \in [-K, K]^2$:

- $\Phi(s_1, s_2) \in \{p_0\} \times T_2$;
- $\Phi(s_1, s_2)$ is the endpoint of a $\leq 8K$-legged $\mathcal{W}_h^u, \mathcal{W}_h^s$-path of diameter $\leq 2K\rho_2$;
- for every $(m_1, m_2) \in \mathbf{Z}^2 \cap [-K, K]^2$,

$$\Phi(m_1, m_2) = (p_0, (m_1 t_1, m_2 t_1));$$

- $d(\Phi(s_1, s_2), \Phi(\lfloor s_1 \rfloor, \lfloor s_1 \rfloor)) \leq 1/50$.

For h' sufficiently C^1-close to h, we similarly construct a map $\Phi_{h'} : [-K, K]^2 \to \mathbf{T}^4$ so that every $\Phi_{h'}(s_1, s_2)$ is the endpoint of a $\leq 8K$-legged $\mathcal{W}_{h'}^u, \mathcal{W}_{h'}^s$-path of diameter $\leq 2K\rho_2$ and $\lim_{h' \to h} \Phi_{h'} = \Phi$ uniformly. Choose $\epsilon_5 \in (0, \epsilon)$ such that if $d_{C^1}(h, h') < \epsilon_5$, then

$$\pi_1(\Phi_{h'}([-K, K]^2)) \subseteq B_{\rho_0}(p_0). \tag{***}$$

The image of $\partial([-K, K]^2)$ under Φ is very thin: it is contained in a $1/50$-neighborhood of $\{p_0\} \times \partial B_\beta(q_i)$. Recall that $V_i = B_{\beta/2}(q_i)$ and $\beta = 1/4$. Another degree argument shows that there exists $\epsilon_{i1} \in (0, \epsilon_5)$ such that, if $d_{C^1}(h, h') < \epsilon_{i1}$, then

$$\pi_2(\Phi_{h'}([-K, K]^2)) \supseteq V_i. \tag{****}$$

Now let $\Psi_{h'} : [-K, K]^2 \times [-J, J]^2 \to \mathbf{T}^4$ be defined by

$$\Psi_{h'}(s_1, s_2, s_3, s_4) = \omega_{h'}(\Phi_{h'}(s_1, s_2), (s_3, s_4)).$$

Then $\Psi_{h'}(s_1, s_2, s_3, s_4)$ is the endpoint of a $\leq 8K + 2$-legged $\mathcal{W}_{h'}^u, \mathcal{W}_{h'}^s$-path from (p_0, q_i). By $(*)$–$(****)$ we have that, for $d_{C^1}(h, h') < \epsilon_{i1}$,

$$\Psi_{h'}([-K, K]^2 \times [-J, J]^2) \supseteq B_{\rho_0}(p_0) \times V_i.$$

Thus, for $d_{C^1}(h, h') < \epsilon_{i1}$, every point in $B_{\rho_0}(p_0) \times V_i$ can be accessed from (p_0, q_i) along a $\mathcal{W}_{h'}^u, \mathcal{W}_{h'}^u$-path, and so any two points in $B_{\rho_0}(p_0) \times V_i$ can be connected by a $\mathcal{W}_{h'}^u$, $\mathcal{W}_{h'}^u$-path. Setting $\epsilon_1 = \min_i \epsilon_{i1}$ completes the proof. □

Proof of Lemma 2.4. The first assertion in Lemma 2.4 holds for the unperturbed system $f \times g$; the foliations $\mathcal{W}_{f \times g}^u$ and $\mathcal{W}_{f \times g}^s$ project under π_1 to the stably accessible foliations $\mathcal{W}_f^u$ and $\mathcal{W}_f^s$ (stably accessible because f is a transitive Anosov diffeomorphism). Under π_2, the foliations $\mathcal{W}_{f \times g}^u$ and $\mathcal{W}_{f \times g}^s$ project to the trivial foliations by points. Given a point $(p, q) \in \mathbf{T}^4$, there exists an $i \in \{1, \ldots, N\}$ such that $q \in V_i$, and there is a $\mathcal{W}_f^u, \mathcal{W}_f^s$-path $\gamma : [0, 1] \to T_1$ from p to p_{i0}. This lifts to a $\mathcal{W}_{f \times g}^u, \mathcal{W}_{f \times g}^s$-path $\tilde{\gamma} : [0, 1] \to \mathbf{T}^4$

from (p, q) to $(p_{i0}, q) \in B_{\rho_0}(p_{i0}) \times V_i$. Clearly property (1) is open under C^1-small perturbations of $f \times g$.

The same argument shows that if $d_{C^1}(h, f \times g)$ is sufficiently small and $V_i \cap V_j \neq \emptyset$, then there is a $\mathcal{W}^u_h$, $\mathcal{W}^s_h$-path from some point in $B_{\rho_0}(p_{i0}) \times V_i$ to some point in $B_{\rho_0}(p_{j0}) \times V_j$. □

Proof of Lemma 2.1. We construct W_t; the construction of Z_t is similar. Let $g, h : \mathbf{R} \to \mathbf{R}$ be C^∞ functions ($g = g_\beta$, $h = h_\alpha$) with the following properties:

- $h(x) = 0$, for $|x| > 4\alpha^2$,
- $h(x) = 1$, for $|x| < \alpha^2$,
- for all x, $0 \leq h(x) \leq 1$,

and

- $g(x) = x$, for $|x| < \beta$,
- $g(x) = 0$, for $2\beta < |x| \leq 1/2$,
- for all x, $|g'(x)| < 1$,
- $g(x + n) = g(x)$, for all $n \in \mathbf{Z}$ and $x \in \mathbf{R}$.

Define $H : [-1/2, 1/2]^2 \times \mathbf{R}^2 \to \mathbf{R}$ by

$$H(z_1, w_1, z_2, w_2) = -g(z_2)h(z_1^2 + w_1^2).$$

Then H generates a C^∞ Hamiltonian vector field $\widetilde{W}$ on $[-1/2, 1/2]^2 \times \mathbf{R}^2$:

$$\widetilde{W}(z_1, w_1, z_2, w_2) = -2w_1 g(z_2)h'(z_1^2 + w_1^2)\frac{\partial}{\partial z_1} + 2z_1 g(z_2)h'(z_1^2 + w_1^2)\frac{\partial}{\partial w_1} + g'(z_2)h(z_1^2 + w_1^2)\frac{\partial}{\partial w_2}.$$

Since $\widetilde{W}$ is Hamiltonian, it preserves the symplectic form $dz_1 \wedge dw_1 + dz_2 \wedge dw_2$, and $\widetilde{W}$ has the following additional properties.

(1) If $z_1^2 + w_1^2 < \alpha^2$, then $\widetilde{W}(z_1, w_1, z_2, w_2) = g'(z_2)\partial/\partial w_2$. In particular, if $|z_2| < \beta$, then $\widetilde{W}(0, 0, z_2, w_2) = \partial/\partial w_2$.

(2) $\widetilde{W}(z_1, w_1, z_2 + n_1, w_2 + n_2) = \widetilde{W}(z_1, w_1, z_2, w_2)$, for all $(z_1, w_1, z_2, w_2) \in [-1/2, 1/2]^2 \times \mathbf{R}^2$ and $m_1, m_2, \in \mathbf{Z}$.

(3) If $z_1^2 + w_1^2 > 4\alpha^2$, then $\widetilde{W} = 0$ (in particular, $\widetilde{W}|_{\partial[-1/2,1/2]^2} = 0$).

Properties (2) and (3) imply that $\widetilde{W}$ extends to a C^∞ vector field on $\mathbf{R}^4$ satisfying

$$\widetilde{W}(z_1 + m_1, w_1 + m_2, z_2 + n_1, w_2 + n_2) = \widetilde{W}(z_1, w_1, z_2, w_2),$$

for all $(z_1, w_1, z_2, w_2) \in \mathbf{R}^4$ and $(m_1, m_2, n_1, n_2) \in \mathbf{Z}^4$.

Thus $\widetilde{W}$ defines a symplectic vector field W on $\mathbf{T}^4$ by $W = p_* \tilde{X}$, where $p : \mathbf{R}^4 \to \mathbf{T}^4$ is the canonical projection. The map p is symplectic, and so W is symplectic. Let W_t be the flow generated by W. Then W_t has the desired properties. □

3. *Proof of Theorem C*

The following is standard; a proof can be found in [**P**].

LEMMA 3.1. (Properties of toral automorphisms). *Let f be any automorphism of $\mathbf{T}^n$. Then*

(1) *periodic points of f are dense in $\mathbf{T}^n$.*

Furthermore, if f is ergodic and partially hyperbolic, then:

(2) *every leaf of $\mathcal{W}^u_f$ and of $\mathcal{W}^s_f$ is a dense, Euclidean submanifold of $\mathbf{T}^n$;*

(3) *the distributions E^c_f, $E^u_f \oplus E^s_f$, $E^u_f \oplus E^c_f$, and $E^c_f \oplus E^s_f$ are integrable and tangent to C^∞ foliations $\mathcal{W}^c_f$, $\mathcal{W}^{us}_f$, $\mathcal{W}^{cu}_f$ and $\mathcal{W}^{cs}_f$, respectively;*

(4) *every leaf of $\mathcal{W}^c_f$ is a dense, even-dimensional Euclidean submanifold of $\mathbf{T}^n$. The eigenvalues of $Tf|_{E^c_f}$ lie on the unit circle in $\mathbf{C}$, but are not roots of one;*

(5) *the $\mathcal{W}^u_f$-holonomy maps between $\mathcal{W}^{cs}_f$-leaves and the $\mathcal{W}^s_f$-holonomy maps between $\mathcal{W}^{cu}_f$-leaves are Euclidean isometries.*

The proof of Theorem C now proceeds much along the lines of the proof of Theorem A. Instead of periodic tori, we work with periodic $\mathcal{W}^c_f$-leaves. The density of $\mathcal{W}^u_f$-leaves simplifies the argument. We will use the following lemma, which we prove at the end of this section.

LEMMA 3.2. *For any two open sets B and C in $\mathbf{R}^4$ with $\overline{B} \subseteq C$, there exists a C^∞, volume-preserving flow Y_t on $\mathbf{R}^4$ and $T > 0$ such that:*

(1) *if $(z_1, z_2, z_3, z_4) \in \mathbf{R}^4 \setminus \overline{C}$, then for all $t \in \mathbf{R}$,*

$$Y_t((z_1, z_2, z_3, z_4)) = (z_1, z_2, z_3, z_4);$$

(2) *if $(z_1, z_2, z_3, z_4) \in \overline{B}$, then for all $|t| \le T$,*

$$Y_t((z_1, z_2, z_3, z_4)) = (z_1, z_2, z_3, z_4 + t).$$

Let $f = f_A$, and let p be a fixed point for f. Let $\beta_0 = 1/8$. In this proof, the metric on $\mathbf{T}^4$ is the standard Euclidean metric:

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(\sum_{i=1}^n (d_0(x_i, y_i))^2 \right)^{1/2}.$$

If $\mathcal{F}$ is a foliation of $\mathbf{T}^4$, and C and W are any subsets of $\mathbf{T}^4$, then let

$$\mathcal{F}_W(C) = \bigcup_{z \in C} \mathcal{F}_W(z),$$

and let

$$\mathcal{O}(C) = \bigcup_{n \in \mathbf{Z}} f^n(C).$$

For $r > 0$, let $\mathcal{F}_r(C) = \mathcal{F}_{N_r(C)}(C)$, where $N_r(C) = \bigcup_{z \in C} B_r(z)$.

We will work in a neighborhood U_0 of p defined by

$$U_0 = \mathcal{W}^s_{f,\beta_0}(\mathcal{W}^u_{f,\beta_0}(\mathcal{W}^c_{f,\beta_0}(p))).$$

By Lemma 3.1, any pair of the three foliations $\mathcal{W}^u_f$, $\mathcal{W}^c_f$ and $\mathcal{W}^s_f$ are jointly integrable and meet at a constant angle. If all of these angles were $\pi/2$, then the volume of U_0 would be $(2\beta_0)^2 \cdot \pi\beta_0^2 = 4\pi\beta_0^4$. Let ν be the actual volume of U_0, and let $\rho_0 = (\nu/(4\pi\beta_0^2))^{1/2}$. Define a neighborhood D_0 of the origin in $\mathbf{R}^4$ by

$$D_0 = \{(z_1, z_2, z_3, z_4) : |z_1| < \beta_0, |z_2| < \beta_0, z_3^2 + z_4^2 < \rho_0^2\}.$$

Then $\mathrm{vol}(D_0) = \nu$, and there is a C^∞, volume-preserving diffeomorphism $\psi : D_0 \to U_0$ such that:

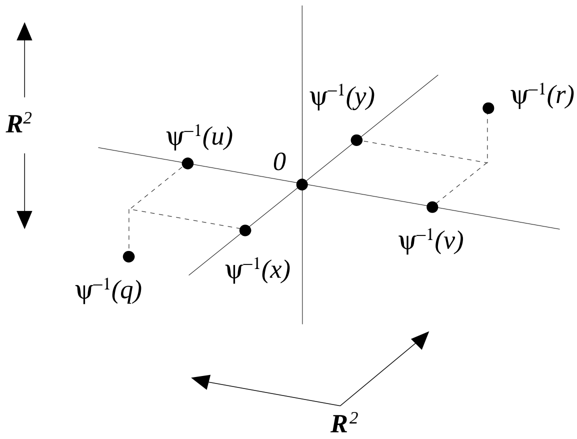

FIGURE 1. $\psi^{-1}\{p, q, r, x, y, u, v\}$.

- $\psi(0, 0, 0, 0) = p$;
- $\mathcal{W}^u_{f,U_0}(\psi(z_1, z_2, z_3, z_4)) = \psi(\{(w_1, z_2, z_3, z_4) : |w_1| < \beta_0\})$;
- $\mathcal{W}^s_{f,U_0}(\psi(z_1, z_2, z_3, z_4)) = \psi(\{(z_1, w_2, z_3, z_4) : |w_2| < \beta_0\})$;
- $\mathcal{W}^c_{f,U_0}(\psi(z_1, z_2, z_3, z_4)) = \psi(\{(z_1, z_2, s_1, s_2) : s_1^2 + s_2^2 < \rho_0^2\})$;
- restricted to the line segments $\{(0, z_2, z_3, z_4) + (w_1, 0, 0, 0) : |w_1| < \beta_0\}$ and $\{(z_1, 0, z_3, z_4) + (0, w_2, 0, 0) : |w_2| < \beta_0\}$, ψ is an isometry;
- restricted to the disks $\{(z_1, z_2, 0, 0) + (0, 0, s_1, s_2) : s_1^2 + s_2^2 < \rho_0^2\}$, ψ is a dilation about $\psi(z_1, z_2, 0, 0)$ by β_0/ρ_0.

Since $\beta_0 < 1/2$, for every $x, y \in U_0$, the sets $\mathcal{W}^u_{f,U_0}(x) \cap \mathcal{W}^{cs}_{f,U_0}(y)$ and $\mathcal{W}^{cu}_{f,U_0}(x) \cap \mathcal{W}^s_{f,U_0}(y)$ each contain precisely one point; let

$$[x, y]_1 = \mathcal{W}^u_{f,U_0}(x) \cap \mathcal{W}^{cs}_{f,U_0}(y),$$

and let

$$[x, y]_2 = \mathcal{W}^{cu}_{f,U_0}(x) \cap \mathcal{W}^s_{f,U_0}(y).$$

Let $D_1 = \{(z_1/2, z_2/2, z_3/2, z_4/2) : (z_1, z_2, z_3, z_4) \in D_0\}$ and let $U_1 = \psi(D_1)$.

Since, by Lemma 3.1, periodic points of f are dense in $\mathbf{T}^4$, there exist distinct periodic points $q, r \in U_1$ such that, $p \notin \{q, r\}$ and

$$\max\{d([p, q]_1, [p, q]_2), d([p, r]_1, [p, r]_2)\} < \beta_0/100. \qquad (*)$$

Let $x = [p, q]_1$, $y = [p, r]_1$, $u = [q, p]_1$ and $v = [r, p]_1$. Note that $x, y, u, v \in U_1$. For $b \in \{q, r, x, y, u, v\}$, let b_i be the ith coordinate of $\psi^{-1}(b)$, so that

$$\psi^{-1}(b) = (b_1, b_2, b_3, b_4).$$

LEMMA 3.3. *There exist real numbers $\beta_1, \rho_1 > 0$ such that, for $b \in \{x, y\}$, if*

$$C_b = \{\psi^{-1}(b) + (z_1, z_2, z_3, z_4) : |z_1| < 2\beta_1, |z_2| < 2\beta_1, z_3^2 + z_4^2 < 4\rho_1^2\},$$

and $V_b = \psi(C_b)$, then:

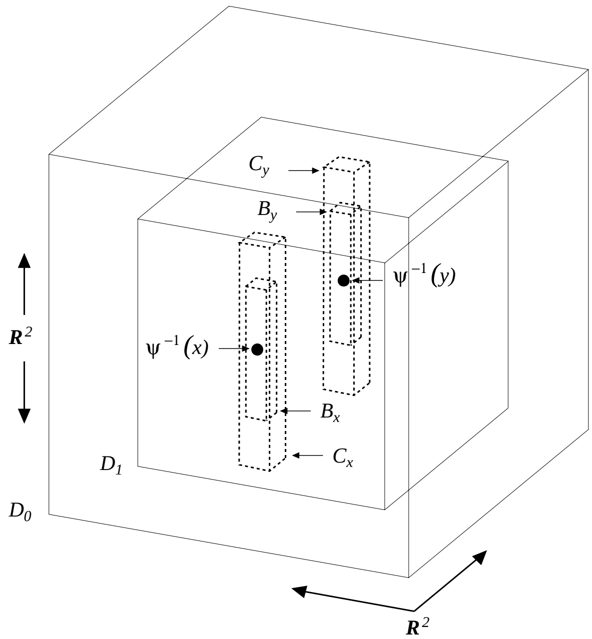

FIGURE 2. The neighborhoods B_x, B_y, C_x and C_y.

(1) *$C_b \subseteq D_1$ (which implies $V_b \subseteq U_1$);*
(2) *$V_x \cap V_y = \emptyset$;*
(3) *$f^{-1}(V_b) \cap (\mathcal{O}(\mathcal{W}^c_{f,U_1}(\{p, q, r, x, y, u, v\})) \setminus \mathcal{W}^c_{f,U_1}(f^{-1}(b))) = \emptyset$.*

Proof of Lemma 3.3. Clearly, if β_1 and ρ_1 are chosen sufficiently small, properties (1) and (2) will be satisfied.

For $b \in \{p, x, y, u, v\}$,

$$\mathcal{W}^c_{f,U_1}(b) = \mathcal{W}^c_{f,\beta_0/2}(b),$$

and for $b \in \{q, r\}$, it follows from $(*)$ that

$$\mathcal{W}^c_{f,U_1}(b) \subseteq \mathcal{W}^c_{f,\beta_0/2+\beta_0/100}(b).$$

As a consequence of Proposition 3.1, for any $\beta < 1/2$ and any $z \in \mathbf{T}^4$,

$$f(\mathcal{W}^c_{f,\beta}(z)) = \mathcal{W}^c_{f,\beta}(f(z)).$$

The α-limit set of $\{x, y\}$ and ω-limit set of $\{u, v\}$ are both $\mathcal{O}(\{p\})$; the α-limit set of $\{u, v\}$ and ω-limit set of $\{x, y\}$ are both contained in $\mathcal{O}(\mathcal{W}^c_{\beta_0/100}(\{q, r\}))$. Thus, the set of limit points of $\mathcal{O}(\mathcal{W}^c_{f,U_1}(\{p, q, r, x, y, u, v\}))$ is contained in the set:

$$\mathcal{O}(\mathcal{W}^c_{f,\beta_2}(\{p, q, r\})),$$

where $\beta_2 = \beta_1/2 + \beta_1/50$. It is therefore possible to choose ρ_1 and β_1 so that property (3) is also satisfied. □

Let ρ_1 and β_1 be given by Lemma 3.3, let C_x, C_y, V_x and V_y be defined as in the lemma, and for $b \in \{x, y\}$, let

$$B_b = \{\psi^{-1}(b) + (z_1, z_2, z_3, z_4) : |z_1| < \beta_1, |z_2| < \beta_1, z_3^2 + z_4^2 < \rho_1^2\}.$$

By Lemma 3.2, there are C^∞ , volume-preserving flows X_t and Y_t on D_0 such that, for all t sufficiently small,

- $X_t = id$ outside $\overline{C}_x$;
- if $(z_1, z_2, z_3, z_4) \in \overline{B}_x$, then

$$X_t((z_1, z_2, z_3, z_4)) = (z_1, z_2, z_3 + t, z_4);$$

- $Y_t = id$ outside $\overline{C}_y$;
- if $(z_1, z_2, z_3, z_4) \in \overline{B}_y$, then

$$Y_t((z_1, z_2, z_3, z_4)) = (z_1, z_2, z_3, z_4 + t).$$

For $|t|$ sufficiently small, we define the volume-preserving, C^∞ diffeomorphism $g_t : \mathbf{T}^4 \to \mathbf{T}^4$ by

$$g_t = (\psi \circ (Y_t \circ X_t)) \circ f.$$

LEMMA 3.4. *If $|t|$ is sufficiently small, then $g = g_t$ is partially hyperbolic, and*

(1) *if $z_3^2 + z_4^2 < \rho_1^2$ and $b \in \{x, y\}$, then*

$$\psi^{-1}(\mathcal{W}^u_{g,U_0}(\psi(0, 0, z_3, z_4)) \cap \mathcal{W}^c_{g,U_0}(b)) = \begin{cases} (x_1, x_2, z_3 + t, z_4) & \text{if } b = x \\ (y_1, y_2, z_3, z_4 + t) & \text{if } b = y; \end{cases}$$

(2) *if $z_3^2 + z_4^2 < \rho_1^2$ and $b \in \{u, v\}$, then*

$$\psi^{-1}(\mathcal{W}^s_{g,U_0}(\psi(0, 0, z_3, z_4)) \cap \mathcal{W}^c_{g,U_0}(b)) = (b_1, b_2, z_3, z_4);$$

(3) *if $z_3^2 + z_4^2 < \rho_1^2$ and $b \in \{x, u\}$, then*

$$\psi^{-1}(\mathcal{W}^s_{g,U_0}(\psi(q_1, q_2, z_3, z_4)) \cap \mathcal{W}^c_{g,U_0}(b)) = (b_1, b_2, z_3, z_4);$$

(4) *if $z_3^2 + z_4^2 < \rho_1^2$ and $b \in \{y, v\}$, then*

$$\psi^{-1}(\mathcal{W}^s_{g,U_0}(\psi(r_1, r_2, z_3, z_4)) \cap \mathcal{W}^c_{g,U_0}(b)) = (b_1, b_2, z_3, z_4).$$

Proof of Lemma 3.4. Suppose $z_3^2 + z_4^2 < \rho_1^2$, and let $z = \psi(x_1, x_2, z_3 + t, z_4)$. If $|t|$ is sufficiently small, then $\mathcal{W}^u_{g,U_0}(\psi(0, 0, z_3, z_4)) \cap \mathcal{W}^c_{g,U_0}(x)$ contains at most one point. We show that z is contained in this intersection. As in the proof of Theorem A, Lemma 2.2, one verifies that

$$\lim_{n\to\infty} d(g_t^{-n}(z), g_t^{-n}(x)) = 0.$$

The details are left to the reader. This proves property (1), when $b = x$. The other properties are proved similarly. □

LEMMA 3.5. *For $|t|$ sufficiently small, g_t has the stable accessibility property.*

Proof of Lemma 3.5. Let $\beta_2 = \min\{\rho_1\beta_0/\rho_0, \beta_1\}/4$. By Lemma 3.1, every leaf of the unstable foliation $\mathcal{W}^u_f$ is dense in $\mathbf{T}^4$. Thus, there exists $\epsilon_0 > 0$, such that, if $h : \mathbf{T}^4 \to \mathbf{T}^4$ is any diffeomorphism satisfying $d_{C^1}(f, h) < \epsilon_0$, then h is partially hyperbolic, and, for every $z \in \mathbf{T}^4$, $\mathcal{W}^u_h(z)$ is $\beta_2/4$-dense in $\mathbf{T}^4$. Fix this ϵ_0.

It follows from Lemma 3.4, that for $|t|$ sufficiently small, if $z_3^2 + z_4^2 < \rho_1^2$, then there is a 4-legged $\mathcal{W}^u_{g_t}, \mathcal{W}^s_{g_t}$-path $\zeta((z_3, z_4), \cdot)$ such that:

- $\zeta((z_3, z_4), 0) = \psi(0, 0, z_3, z_4)$,
- $\zeta((z_3, z_4), 1/4) = \psi(x_1, x_2, z_3 + t, z_4)$,
- $\zeta((z_3, z_4), 1/2) = \psi(q_1, q_2, z_3 + t, z_4)$,
- $\zeta((z_3, z_4), 3/4) = \psi(u_1, u_2, z_3 + t, z_4)$,
- $\zeta((z_3, z_4), 1) = \psi(0, 0, z_3 + t, z_4)$.

These paths can be chosen to depend continuously on z_3, z_4. Similarly there is a continuous family of 4-legged $\mathcal{W}^u_{g_t}, \mathcal{W}^s_{g_t}$-paths $\{\eta((z_3, z_4), \cdot) \mid z_3^2 + z_4^2 < \rho_1^2\}$ such that:

- $\eta((z_3, z_4), 0) = \psi(0, 0, z_3, z_4)$,
- $\eta((z_3, z_4), 1/4) = \psi(y_1, y_2, z_3, z_4 + t)$,
- $\eta((z_3, z_4), 1/2) = \psi(r_1, r_2, z_3, z_4 + t)$,
- $\eta((z_3, z_4), 3/4) = \psi(v_1, v_2, z_3, z_4 + t)$,
- $\eta((z_3, z_4), 1) = \psi(0, 0, z_3, z_4 + t)$.

As in the proof of Theorem A, Lemma 2.3, it follows that for $|t|$ sufficiently small ($< t_0$, for some $t_0 > 0$), there exists $\epsilon_t > 0$, such that if $d_{C^1}(g_t, h) < \epsilon_t$, then any two points in $B_{\beta_2}(p)$ can be connected by a $\mathcal{W}^u_h, \mathcal{W}^s_h$-path.

Choose t, with $|t| < t_0$, so that $d_{C^1}(f, g_t) < \epsilon_0$. Pick ϵ_t as in the previous paragraph, and suppose that $d_{C^1}(g_t, h) < \min\{\epsilon_t, \epsilon_0/2\}$. Then $d_{C^1}(g_t, f) < \epsilon_0/2 + \epsilon_0/2 < \epsilon_0$, and so for any two points $z, w \in \mathbf{T}^4$,

$$\mathcal{W}^u_h(z) \cap B_{\beta_2}(p) \neq \emptyset,$$

and

$$\mathcal{W}^u_h(w) \cap B_{\beta_2}(p) \neq \emptyset.$$

Thus z and w can be connected to points $z_1, w_1 \in B_{\beta_2}(p)$, respectively, along pieces of unstable manifold for h. Since $d_{C^1}(g_t, h) < \epsilon_t$, the points z_1 and w_1 can be connected by a $\mathcal{W}^u_h, \mathcal{W}^s_h$-path. Concatenating these paths gives a $\mathcal{W}^u_h, \mathcal{W}^s_h$-path from z to w, and so g_t is stably accessible. □

To finish the proof of Theorem C, let $\epsilon > 0$ be given. Since $d_{C^\infty}(g_t, f) \to 0$ as $t \to 0$, there exists a $t > 0$ such that:

- $d_{C^\infty}(g_t, f) < \epsilon$,
- g_t is partially hyperbolic and center bunched (these properties are C^1-open),
- g_t is dynamically coherent (since f is normally hyperbolic, this property is C^1-open; see §1),
- g_t is stably accessible (by Lemma 3.5).

By Theorem 1.1, g_t is stably ergodic. □

Proof of Lemma 3.2. Pick an open set D with $\overline{B} \subseteq D \subseteq \overline{D} \subseteq C$, and let T be the distance from $\overline{B}$ to the boundary of D.

Let $H : \mathbf{R}^4 \to \mathbf{R}$ be a C^∞ function satisfying:

- $H(z_1, z_2, z_3, z_4) = z_3$ if $(z_1, z_2, z_3, z_4) \in \overline{D}$,
- $H(z_1, z_2, z_3, z_4) = 20$ if $(z_1, z_2, z_3, z_4) \in \mathbf{R}^4 \setminus \overline{C}$.

Let

$$Y(z_1, z_2, z_3, z_4) = -\frac{\partial H}{\partial z_2}\frac{\partial}{\partial z_1} + \frac{\partial H}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial H}{\partial z_4}\frac{\partial}{\partial z_3} + \frac{\partial H}{\partial z_3}\frac{\partial}{\partial z_4}$$

be the Hamiltonian vector field generated by H, and let Y_t be the flow generated by H. Then Y_t has the desired properties. □

REFERENCES

[B] M. Brin. Topological transitivity of a class of dynamical systems and frame flows on manifolds with negative curvature. *Func. Anal. Appl.* **9**(1) (1975), 9–19.

[BPW] K. Burns, C. Pugh and A. Wilkinson. Stable ergodicity and Anosov flows. *Topology* **39** (2000), 149–159.

[BS] J. Brezin and M. Shub. Stable ergodicity in homogeneous spaces. *Bol. Soc. Brasil. Mat.* **28**(2) (1997), 197–210.

[BW] K. Burns and A. Wilkinson. Stable ergodicity of skew products. *Ann. Sci. de l'Ecole Norm. Sup.* **32** (1999), 859–889.

[CS] C. Q. Cheng and Y. S. Sun. Existence of invariant tori in three-dimensional measure-preserving mappings. *Celestial Mech. Dynam. Astronom.* **47**(3) (1989/90), 275–292.

[KK] A. Katok and A. Kononenko. Cocycles' stability for partially hyperbolic systems. *Math. Res. Lett.* **3**(2) (1996), 191–210.

[P] W. Parry. Dynamical systems on nilmanifolds. *Bull. London Math. Soc.* **2** (1970), 37–40.

[PS1] C. Pugh and M. Shub. Stably ergodic dynamical systems and partial hyperbolicity. *J. Complexity* **13**(1) (1997), 125–179.

[PS2] C. Pugh and M. Shub. Stable ergodicity and Julienne quasiconformality. *J. Eur. Math. Soc.* to appear.

[X] Z. Xia. Existence of invariant tori in volume-preserving diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **12**(3) (1992), 621–631.

[Y] J.-C. Yoccoz. Travaux de Herman sur les tores invariants. *(Séminaire Bourbaki, Vol. 1991/92). Astérisque* **4**(206) (1992), 311–344.