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The distribution of the maximum condition number on great circles through a fixed 2×2 real matrix[☆]

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Abstract

If $P_A(\chi)$ denotes the probability that the maximum condition number along a great circle passing through a matrix A in the unit sphere in the space of 2×2 matrices is less than χ, then $P_A(\chi)$ always attains its maximum at the normalized identity matrix. This result is the first nontrivial case of a linear algebra version of a conjecture formulated in Shub and Smale (M. Shub and S. Smale, Theoretical Computer Science 113 (1994) 141–164) for homotopies of systems of homogeneous equations. The Hopf fibration is used to relate the probability $P_A(\chi)$ to the area of an 'ellipse' on a sphere in $\mathbb{R}^3$. © 1999 Elsevier Science Inc. All rights reserved.

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Endow the space $\mathscr{M}(n,m) = \mathscr{M}_{\mathbb{R}(\mathbb{C})}(n,m)$ of $n \times m$ real (complex) matrices, where $m \geqslant n$, with the inner (respectively Hermitian) product $(A, B) = \text{trace}(B^* A)$, where B^* denotes the adjoint of B. Let $S = S_{n,m} \subset \mathscr{M}(n,m)$ denote the unit sphere

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in this space. For every surjective $A \in S$ we define the condition number $\kappa(A)$ of A as

$$\kappa(A) := \left\| A^*(AA^*)^{-1} \right\|, \tag{1}$$

the operator norm with respect to the usual norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ of the Morse–Penrose inverse of A. If A is not surjective, we set $\kappa(A) := \infty$. If $n = m$, then $\kappa(A) = \left\| A^{-1} \right\|$ if A is invertible. For any $A \in S$ the set of great circles passing through A in S is parametrized by the set of lines in the tangent space

$$T_A S = \{B \in M(n, m) : (A, B) = 0\}$$

to the sphere S at A. Given a line L in $T_A S$, we let G_L denote the great circle in S tangent to L at A and set

$$C_A(L) := \max_{C \in G_L} \kappa(C).$$

Using the uniform probability distribution μ_A on the projective space $\mathbb{P}(T_A S)$ we can induce the cumulative probability function $P_A : \mathbb{R} \to [0, 1]$

$$P_A(\chi) := \{\text{probability } C_A < \chi\} = \mu_A(\{L \in \mathbb{P}(T_A S) : C_A(L) < \chi\}). \tag{2}$$

We shall say that a matrix $A \in S$ is "best" if $P_A(\chi) \geqslant P_B(\chi)$ for all $B \in S$ and $\chi \in \mathbb{R}$.

Let $\mathbf{1}_n$ denote the $n \times n$ identity matrix and I_n denote the $n \times n$ matrix $I_n := \frac{1}{\sqrt{n}} \mathbf{1}_n$. Let $I_{n,m}$ denote the $n \times m$ block matrix $I_{n,m} := (I_n\ \mathbf{0})$, where $\mathbf{0}$ denotes the $n \times (m - n)$ zero matrix.

Conjecture. $I_{n,m}$ is best.

This conjecture is a linear algebra version of a problem appearing in [3]. In that paper continuation methods are used to lift homotopies of systems of homogeneous equations which proceed along great circles in the space of systems to curves in the space of solutions. It is conjectured there that the homogenization of I_n is a good starting point for such homotopies. If it were in fact the best starting point, then the conjecture would be proven. Here we prove our conjecture in the first nontrivial case, namely 2×2 real matrices.

Theorem 1. *I_2 is best in $S \subset \mathscr{M}_{\mathbb{R}}(2, 2)$.*

A natural approach to proving the conjecture would be to exploit the symmetries of the system by reducing the problem to the lowest dimensional setting possible. The condition number κ is invariant under the action of the product of the orthogonal groups $\mathrm{O}(n) \times \mathrm{O}(m)$ on $S \subset \mathscr{M}_{\mathbb{R}}(n, m)$ by left and right multiplication, i.e.

$$(O_1, O_2) \cdot A = O_1 A O_2.$$

The quotient of $\mathscr{M}_{\mathbb{R}}(n, m)$ with respect to this action is the set

$$\left\{\sigma \in \mathbb{R}^n : \sigma_1 \geqslant \cdots \geqslant \sigma_n \geqslant 0\right\}.$$

The quotient $S/(\mathrm{O}(n) \times \mathrm{O}(m))$ can be implemented by assigning to $A \in S$ the vector $\sigma(A)$ of the singular values of A. Note that if $A \in S$, then $\|\sigma(A)\| = 1$. In particular, if $m = n = 2$, then the quotient can be identified with the quarter circle $\left\{(\cos\theta, \sin\theta) : 0 \leqslant \theta \leqslant \frac{\pi}{2}\right\}$. However, it does not seem to be particularly easy to study the images of great circles using this decomposition. We find that a partial reduction of the problem, leading to a system of intermediate dimension retaining some symmetry, yields a formulation of the problem that is amenable to a simple geometric analysis in the case $m = n = 2$.

A second approach uses only one of the orthogonal groups at a time. For example, the quotient with respect to the left action of $\mathrm{O}(m)$ can be implemented using the QR decomposition. This is how we proceed for $\mathscr{M}_{\mathbb{R}}(2,2)$; however, we use the Hopf fibration, which we now describe, rather than the QR decomposition. Some convenient geometric properties of the Hopf fibration allow a more straightforward analysis of the condition numbers on great circles than seems possible using the QR factorization.

Define $m : \mathbb{C}^2 \to \mathscr{M}_{\mathbb{R}}(2,2)$ by associating each column of the matrix with a complex number, i.e.,

$$m(z,w) := \begin{pmatrix} \mathrm{re}\,(z) & \mathrm{re}\,(w) \\ \mathrm{im}\,(z) & \mathrm{im}\,(w) \end{pmatrix}.$$

The map m takes the unit sphere $S^3 = \left\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\right\}$ in $\mathbb{C}^2$ to the sphere $S \subset \mathscr{M}_{\mathbb{R}}(2,2)$.

The condition number $\kappa(A)$ of an invertible matrix A satisfies $\kappa(A) = \left\|A^{-1}\right\| = \lambda_{\min}^{-1/2}$, where $\lambda_{\min}$ is the minimum eigenvalue of A^*A. If $A = m(z,w) \in S$, then $\det m(z,w) = \mathrm{im}\,(\bar{z}\,w)$; hence the eigenvalues of A^*A are

$$\tfrac{1}{2}\left(1 \pm \sqrt{1 - 4(\mathrm{im}\,(\bar{z}\,w))^2}\right)$$

and

$$\kappa(m(z,w)) = \sqrt{\frac{2}{1 - \sqrt{1 - 4\,(\mathrm{im}\,(\bar{z}\,w))^2}}}. \tag{3}$$

Eq. (3) suggests the use of the Hopf fibration

$$h(z,w) := \bar{z}(z,w) = (|z|^2, \bar{z}\,w)$$

from S^3 to $S^2_{1/2} \subset \mathbb{R} \times \mathbb{C}$, the sphere of radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, 0\right)$ in $\mathbb{R} \times \mathbb{C}$, in implementing the quotient with respect to the left action of $SO(2)$ on S. For any point $(z,w) \in S^3$, the fiber $h^{-1}(h(z,w))$ over $h(z,w)$ is the great circle tangent to $(\mathrm{i}\,z, \mathrm{i}\,w)$ at (z,w). These fibers are the orbits with respect to the diagonal circle action on S^3, $e^{\mathrm{i}\,\theta} \cdot (z,w) = (\mathrm{e}^{\mathrm{i}\,\theta}\,z, \mathrm{e}^{\mathrm{i}\,\theta}\,w)$, corresponding to the action of $SO(2)$ on $S \subset \mathscr{M}_{\mathbb{R}}(2,2)$ by left multiplication.

Remark. If $z \neq 0$, then, using the natural embedding of $\mathbb{R} \times \mathbb{C}$ in $\mathbb{C}^2$,

$$m(h(z,w)) = \begin{pmatrix} |z|^2 & z \cdot w \\ 0 & \det m(z,w) \end{pmatrix}$$

$$= |z| \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sgn} \det m(z,w) \end{pmatrix} R(z,w),$$

where $R(z,w)$ denotes the upper triangular factor of the QR factorization of $m(z,w)$. In particular, $m(h(m^{-1}(I_2))) = m\left(\frac{1}{2}, \frac{\mathrm{i}}{2}\right) = \frac{1}{2}\mathbf{1}_2$.

Each vector $\mathbf{v}$ in the tangent space $T_{(z,w)}S^3$ determines a great circle $G_{(z,w)}(\mathbf{v})$ in S^3; we now show that the image of a great circle under the Hopf fibration h is a circle passing through $h(z,w)$ in $S^2_{1/2}$. In fact, we shall find it convenient to work with the shifted and scaled Hopf $\tilde{h}(z,w) := 2\,h(z,w) - (1,0)$, which maps the unit sphere S^3 in $\mathbb{C}^2$ into the unit sphere $S^2 \subset \mathbb{R} \times \mathbb{C}$ centered at the origin. Since $\tilde{h}$ is related to h by a dilation and a translation, it is clear that if the image of a great circle under $\tilde{h}$ is a circle, then its image under h is also a circle.

Given points x and $y \in S^2$, let $C(x,y)$ denote the circle

$$C(x,y) := \left\{ z \in S^2 : d(x,z) = d(x,y) \right\}$$

centered at x and passing through y. Here $d(x,y)$ denotes the Riemannian distance between x and y in S^2.

Proposition 1. *For any $(z,w) \in S^3$, there is a linear isometry $O_{(z,w)} : T_{(z,w)}S^3 \to \mathbb{R} \times \mathbb{C}$ mapping the open hemisphere*

$$H_{(z,w)} := \left\{ \mathbf{v} \in T_{(z,w)}S^3 : |\mathbf{v}| = 1 \quad \textit{and} \quad \mathbf{v} \cdot (\mathrm{i}(z,w)) > 0 \right\}$$

diffeomorphically onto the open hemisphere

$$\tilde{H}_{(z,w)} := \left\{ y \in S^2 : y \cdot \tilde{h}(z,w) > 0 \right\}$$

in such a way that for any $\mathbf{v} \in H_{(z,w)}$, $\tilde{h}$ maps the great circle $G_{(z,w)}(\mathbf{v})$ tangent to $\mathbf{v}$ at (z,w) onto the circle $C(O_{(z,w)}\mathbf{v}, \tilde{h}(z,w))$.

To simplify our calculations in the proof of Proposition 1, we make use of the isometries of S^3 and S^2. The group $U(2)$ acts transitively by isometries on $S^3 \subset \mathbb{C}^2$, while $SO(3)$ acts transitively by isometries on S^2. Here the action of $SO(3)$ on $\mathbb{R} \times \mathbb{C}$ is that induced from the usual action of $SO(3)$ on $\mathbb{R}^3$ by requiring that the isomorphism $(r,z) \mapsto (r, \operatorname{re}(z), \operatorname{im}(z))$ be equivariant. Once the images under the modified Hopf map of great circles passing through a single point have been determined, transitivity of the $U(2)$ action and equivariance of the Hopf map determine the images of all other great circles. For convenience, we choose the point $(1,0) \in S^3$, with tangent space

$$T_{(1,0)}S^3 = \{(u\,\mathrm{i}, v) : u \in \mathbb{R} \text{ and } v \in \mathbb{C}\}.$$

Lemma 1. *For any $u \in \mathbb{R}$ and $v \in \mathbb{C}$, the modified Hopf map $\tilde{h}$ maps the great circle tangent to $(u\,\mathrm{i}, v)$ at $(1,0)$ to the circle $C((u, -\mathrm{i}\,v), (1,0))$ in S^2.*

Proof. Let $(u\,\mathrm{i}, v)$ be an element of the unit sphere in $T_{(1,0)}S^3$. The great circle tangent to $(u\,\mathrm{i}, v)$ at $(1,0)$ is equal to

$$\left\{\pm\frac{(1,0)+t(u\,\mathrm{i}, v)}{\|(1,0)+t(u\,\mathrm{i}, v)\|} : t \in \mathbb{R}\right\} \cup \{\pm(u\,\mathrm{i}, v)\}.$$

The Hopf map h (respectively $\tilde{h}$) can be expressed as the composition of the map $h_1(z,w) := \left(1, \frac{w}{z}\right)$ with stereographic projection onto $S^2_{1/2}$ (respectively S^2). The map $t \mapsto h_1\left((1,0)+t(u\,\mathrm{i}, v)\right)$ is a fractional linear transformation mapping the real line into a circle in $\{1\} \times \mathbb{C}$; inclusion of the point $\left(1, \frac{v}{u\,\mathrm{i}}\right) = h_1(u\,\mathrm{i}, v)$ completes the circle. (If $u = 0$, the circle has infinite radius.) Since stereographic projection maps circles to circles, the images of great circles under the Hopf map are circles.

We now compute the center point and radius of the image of a great circle under the modified Hopf map $\tilde{h}$. The image of the parametrization $\gamma_{(u,v)} : S^1 \to S^2$,

$$\gamma_{(u,v)}(\theta) := \cos\theta\,(1,0) + \sin\theta\,(u\,\mathrm{i}, v),$$

of the great circle tangent to $(u\,\mathrm{i}, v)$ at $(1,0)$ is

$$\tilde{h}(\gamma_{(u,v)}(\theta)) = u(u, -\mathrm{i}\,v) + \tau(u, v, \theta),$$

where

$$\tau(u,v,\theta) := \left(|v|^2 \cos(2\theta), (\sin(2\theta) + \mathrm{i}\,u\,\cos(2\theta))v\right).$$

Since $|\tau(u,v,\theta)| = |v|$ for all (u,v,θ), it follows that the curve $\tilde{h} \circ \gamma_{(u,v)}$ lies on the sphere of radius $|v|$ centered at $u(u, -\mathrm{i}\,v)$. If $u \neq 0$, the intersection of this sphere with the unit sphere S^2 is a circle; if $u = 0$, the curve is the great circle in $\operatorname{span}\{(1,0),(0,v)\}$. In both cases, the circle is the Riemannian circle in S^2 centered at $(u, -\mathrm{i}\,v)$ and containing the point $\tilde{h}(1,0) = (1,0)$, namely the circle $C((u, -\mathrm{i}\,v), (1,0))$. $\square$

We now use the equivariance of the Hopf map to obtain an explicit formula for the linear isometry $O_{(z,w)}$.

Lemma 2. *Given $(z,w) \in S^3$, set*

$$Q_{(z,w)} := \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

and

$$q_{(z,w)} := \begin{pmatrix} |z|^2 - |w|^2 & -2\,\mathrm{re}\,(z\,w) & -2\,\mathrm{im}\,(z\,w) \\ 2\,\mathrm{re}\,(\bar{z}\,w) & \mathrm{re}\left(z^2 - w^2\right) & \mathrm{im}\left(z^2 - w^2\right) \\ 2\,\mathrm{im}\,(\bar{z}\,w) & -\mathrm{im}\left(z^2 + w^2\right) & \mathrm{re}\left(z^2 + w^2\right) \end{pmatrix}.$$

Then $Q_{(z,w)} \in SU(2)$, $q_{(z,w)} \in SO(3)$, *and* $\tilde{h}(Q_{(z,w)}(\zeta,\omega)) = q_{(z,w)}\tilde{h}(\zeta,\omega)$ *for all* $(\zeta,\omega) \in S^3$.

Proof. $Q_{(z,w)} \in SU(2)$ follows immediately from $|z|^2 + |w|^2 = 1$. To see that $q_{(z,w)} \in SO(3)$, let ϕ, ψ, and θ denote angles satisfying $z = \mathrm{e}^{\frac{\mathrm{i}(\phi+\psi)}{2}} \cos\frac{\theta}{2}$ and $w = \mathrm{e}^{\frac{\mathrm{i}(\phi-\psi)}{2}} \sin\frac{\theta}{2}$. Substituting these expressions into $q_{(z,w)}$ and regrouping terms yields

$$q_{(z,w)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}.$$

Thus $q_{(z,w)}$ is a product of rotation matrices.
Given (z,w) and $(\zeta,\omega) \in S^3$, let $(\eta,\xi) := \tilde{h}(\zeta,\omega) = \left(2\,|\zeta|^2 - 1, 2\,\bar{\zeta}\,\omega\right)$. Then

$$\begin{aligned} \tilde{h}(Q_{(z,w)}(\zeta,\omega)) &= \tilde{h}(z\,\zeta - \bar{w}\,\omega, w\,\zeta + \bar{z}\,\omega) \\ &= \left(2\,|z\,\zeta - \bar{w}\,\omega|^2 - 1, 2\,(z\,\zeta - \bar{w}\,\omega)(w\,\zeta + \bar{z}\,\omega)\right) \\ &= \big(\,|z|^2\,(1+\eta) - 2\,(z\,w)\cdot\xi + |w|^2\,(1-\eta) \\ &\quad - (|z|^2 + |w|^2), 2\,\eta\,\bar{z}\,w + \bar{z}^2\xi - w^2\bar{\xi}\big) \\ &= \big((|z|^2 - |w|^2)\eta - 2\,(z\,w)\cdot\xi, 2\,\eta\,\bar{z}\,w \\ &\quad + (z^2 - w^2)\cdot\xi + \mathrm{i}(\mathrm{i}(z^2 + w^2))\cdot\xi\big) \\ &= q_{(z,w)}(\eta,\xi). \qquad \square \end{aligned}$$

Proof of Proposition 1. $Q_{(z,w)} \in SU(2)$ maps $T_{(1,0)}S^3 \approx \mathrm{i}\,\mathbb{R} \times \mathbb{C}$ isometrically onto $T_{(z,w)}S^3$. Thus if we define the isometry $O_{(z,w)} : T_{(z,w)}S^3 \to \mathbb{R} \times \mathbb{C}$ by

$$O_{(z,w)}\mathbf{v} := -q_{(z,w)}\left(\mathrm{i}\,Q_{(z,w)}^{-1}\mathbf{v}\right),$$

where $\mathrm{i}\,T_{(1,0)}S^3 \subset \mathbb{C}^2$ is identified with $\mathbb{R} \times \mathbb{C}$, then Lemmas 1 and 2 imply that

$$\begin{aligned} \tilde{h}\left(G_{(z,w)}(\mathbf{v})\right) &= \tilde{h}\left(Q_{(z,w)}G_{(1,0)}\left(Q_{(z,w)}^{-1}\mathbf{v}\right)\right) \\ &= q_{(z,w)}\tilde{h}\left(G_{(1,0)}\left(Q_{(z,w)}^{-1}\mathbf{v}\right)\right) \\ &= q_{(z,w)}C\left(\mathrm{i}\,Q_{(z,w)}^{-1}\mathbf{v}, (1,0)\right) \\ &= C\left(O_{(z,w)}\mathbf{v}, \tilde{h}(z,w)\right). \qquad \square \end{aligned}$$

Eq. (3) shows that the condition number of $m(z,w)$ depends only on the ‘height’ of $h(z,w)$, i.e., the distance from $h(z,w)$ to the plane $\{(r,x) : r, x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{C}$. If we let ν denote the ‘north pole’, i.e. $\nu := (0,i) = \tilde{h}(m^{-1}(I_2))$, then $\det m(z,w) = \frac{1}{4}\cos d(\nu, \tilde{h}(z,w))$ and

$$\kappa(m(z,w)) = k(d(\nu, \tilde{h}(z,w))), \quad \text{where } k(r) := \sqrt{\frac{2}{1-\sin r}},$$

for any $(z,w) \in S^3$. Hence

$$C_{m(z,w)}(\mathrm{span}(\mathbf{v})) = \max_{y \in C(O_{(z,w)}\mathbf{v},(z,w))} k(d(\nu, y)). \tag{4}$$

If $\tilde{h}(z,w) \in \tilde{H}_\nu$, then either $C(O_{(z,w)}\mathbf{v}, (z,w)) \subset \tilde{H}_\nu$ or $C_{m(z,w)}(\mathrm{span}(\mathbf{v})) = k(\frac{\pi}{2}) = \infty$. Thus, since the set of lines in $T_{(z,w)}S^3$ orthogonal to $\mathrm{i}(z,w)$ has measure zero and $O_{(z,w)}$ is area preserving, (2) and (4) imply that if $\tilde{h}(z,w) \in \tilde{H}_\nu$, then

$$\begin{aligned} P_{m(z,w)}(\chi) &= \mu_{m(z,w)}\left(\left\{L \in \mathbb{P}(T_{m(z,w)}S) : C_{m(z,w)}(L) < \chi\right\}\right) \\ &= \mu_{m(z,w)}\Bigg(\Bigg\{\mathrm{span}(\mathbf{v}) : \mathbf{v} \in H_{(z,w)} \text{ and} \\ &\qquad \max_{y \in C(O_{(z,w)}\mathbf{v},(z,w))} d(\nu, y) < k^{-1}(\chi)\Bigg\}\Bigg). \end{aligned}$$

Given $r > 0$ and points x and $y \in S^2$, let

$$D(x,r) := \left\{z \in S^2 : d(x,z) \leqslant r\right\}$$

and

$$E(x,y,r) := \left\{z \in S^2 : d(z,x) + d(z,y) \leqslant r\right\}$$

denote the ‘disc’ of radius r centered at x and ‘ellipse’ of size r with ‘foci’ x and y. Then, for any $z \in S^2$, $C(z,y) \subset D(x,r)$ if and only if $z \in E(x,y,r)$. In particular, if $\tilde{h}(z,w) \in \tilde{H}_\nu$ and $\mathbf{v} \in H_{(z,w)}$, then

$$C(O_{(z,w)}\mathbf{v}, \tilde{h}(z,w)) \subset D(\nu, r) \quad \Longleftrightarrow \quad O_{(z,w)}\mathbf{v} \in E(\nu, \tilde{h}(z,w), r).$$

Thus if $\tilde{h}(z,w) \in \tilde{H}_\nu$ and $\chi \geqslant \sqrt{2}$, then

$$P_{m(z,w)}(\chi) = \frac{\text{area } E(\nu, \tilde{h}(z,w), k^{-1}(\chi))}{\text{area } \tilde{H}_\nu}. \tag{5}$$

(If $\chi < \sqrt{2}$, then $P_{m(z,w)}(\chi) = 0$, since $\sqrt{2}$ is the minimum condition number.)

Eq. (5) and the following proposition shows that I_2 is best among matrices in $m\big(\tilde{h}^{-1}\big(\tilde{H}_\nu\big)\big)$.

Proposition 2. Area $E(\nu, x, r) \leqslant$ area $E(\nu, \nu, r)$ *for any* $x \in \tilde{H}_\nu$ *and* $r \in \left[0, \frac{\pi}{2}\right]$.

Proof. Let $R_x \in SO(3)$ denote the rotation preserving the great circle through ν and x and taking the midpoint of the arc between ν and x to the point ν. It suffices to show that

$$R_x E(\nu, x, r) = E(R_x\nu, R_x x, r) \subset E(\nu, \nu, r)$$

and hence

$$\text{area } E(\nu, x, r) = \text{area } (R_x E(\nu, x, r)) \leqslant \text{area } E(\nu, \nu, r).$$

In the following lemma we prove something more general. $\square$

Lemma 3. *If x, x', y, and y' are points on a great circle G in S^2 such that $d(x, x') = d(y, y')$ and $d(x', y') \leqslant d(x, y)$, then $E(x, y, r) \subset E(x', y', r)$ for all $r \in \left[0, \frac{\pi}{2}\right]$.*

Proof. To establish the inclusion $E(x, y, r) \subset E(x', y', r)$, we must show that

$$d(z, x) + d(z, y) \geqslant d(z, x') + d(z, y')$$

for all $z \in S^2$ such that $\max\{d(x, z), d(y, z)\} \leqslant \frac{\pi}{2}$. For any $z \in S^2$ and $x \in H_z$, $d(x, z) = \cos^{-1}(x \cdot z)$, the angle between x and z. Let x_0 denote the closest point to z on G and let $x : S^1 \to S^2$ denote the arc length parametrization of G with $x(0) = x_0$. Define $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ by

$$f(\theta) := d(x(\theta), z) = \cos^{-1}(x_0 \cdot z \cos\theta).$$

If z is orthogonal to the plane containing G, then $f(\theta) \equiv \frac{\pi}{2}$; if $z \in G$, then $f(\theta) = \theta$. In all other cases

$$f''(\theta) = \frac{\left(1 - (x_0 \cdot z)^2\right) x_0 \cdot z \cos\theta}{\left(1 - (x_0 \cdot z \cos\theta)^2\right)^{3/2}} > 0$$

for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus f is convex.

Let $\theta_j \in S^1$, $j = 1, \ldots, 4$ denote the values such that $x = x(\theta_1)$, $x' = x(\theta_2)$, $y' = x(\theta_3)$, and $y = x(\theta_4)$. The conditions $d(x', y') \leqslant d(x, y)$ and $d(x, x') = d(y, y')$ imply that x' and y' lie 'between' x and y on G; specifically,

$$\theta_2 = \lambda\theta_1 + (1 - \lambda)\theta_4 \quad \text{and} \quad \theta_3 = (1 - \lambda)\theta_1 + \lambda\theta_4$$

for some $\lambda \in [0, 1]$. Thus convexity of f implies that

$$\begin{aligned} f(\theta_2) + f(\theta_3) &= f(\lambda\theta_1 + (1 - \lambda)\theta_4) + f((1 - \lambda)\theta_1 + \lambda\theta_4) \\ &\leqslant f(\theta_1) + f(\theta_4). \qquad \square \end{aligned}$$

Combining (5) and Proposition 2 yields the inequality

$$P_{m(z,w)}(\chi) = \frac{\text{area } E(\nu, \tilde{h}(z, w), k^{-1}(\chi))}{\text{area } \tilde{H}_\nu} \leqslant \frac{\text{area } E(\nu, \nu, k^{-1}(\chi))}{\text{area } \tilde{H}_\nu} = P_{I_2}(\chi)$$

for all $(z, w) \in S^3$ with image $\tilde{h}(z, w) \in \tilde{H}_\nu$ and all $\chi \geqslant \sqrt{2}$. Entirely analogous arguments show that the same estimate holds for all $(z, w) \in S^3$ with image $\tilde{h}(z, w) \in \tilde{H}_{-\nu}$. Since $\tilde{h}(z, w) \in S^2 \backslash (\tilde{H}_\nu \cup \tilde{H}_{-\nu})$ if and only if $m(z, w)$ is singular, and hence $P_{m(z,w)}(\chi) = 0$ for all finite χ, this completes the proof of Theorem 1.

Finally, we compute the probability that all matrices on a great circle in S passing through the 'best' matrix I_2 have condition number less than χ.

Proposition 3. *If* $\chi \geqslant \sqrt{2}$, *then* $P_{I_2}(\chi) = 1 - \sqrt{\frac{1}{2} + \chi^{-2}\sqrt{\chi^2 - 1}}$.

Proof. Eq. (5) implies that

$$P_{I_2}(k(r)) = \frac{\text{area } E(\nu, \nu, r)}{\text{area } \tilde{H}_\nu} = \frac{\text{area } D\left(\nu, \frac{r}{2}\right)}{\text{area } D\left(\nu, \frac{\pi}{2}\right)} = 1 - \cos\frac{r}{2}$$

if $0 \leqslant r \leqslant \frac{\pi}{2}$. Combining this with the identity $k^{-1}(\chi) = \sin^{-1}\left(1 - 2\chi^{-2}\right)$ yields the formula for P_{I_2}. $\square$

Note that the probability that all matrices along a great circle through I_2 are surjective, i.e., have finite condition number, is $1 - \frac{1}{\sqrt{2}} \approx 0.293$.

Remark. The condition number (1) is the restriction to the unit sphere $S \subset \mathscr{M}(n, m)$ of the condition number

$$\kappa(A) = \begin{cases} \|A\|_{\mathrm{F}} \left\|A^*(A^*A)^{-1}\right\| & A \text{ surjective,} \\ \infty & \text{otherwise,} \end{cases}$$

used in [1,2]. Here $\| \|_{\mathrm{F}}$ denotes the Frobenius norm, $\|A\|_{\mathrm{F}}^2 = (A, A) = \mathrm{trace}(A^*A)$ and $\| \|$ denotes the operator norm with respect to the standard norm on $\mathbb{R}^n$ or $\mathbb{C}^n$. Edelman [2] gives a precise formula for the probability that an arbitrary matrix A has condition number less than χ. For 2×2 real matrices, this probability takes the form

$$\{\text{probability } \kappa(A) < \chi\} = 1 - 2\chi^{-2}\sqrt{\chi^2 - 1}.$$

If we work with the condition number $\tilde{\kappa}(A) = \|A\| \left\|A^{-1}\right\|$ on $\mathscr{M}(n, n)$, rather than κ, then

$$\tilde{\kappa}(m(z, w)) = \frac{1 + \sqrt{1 - 4\,(\mathrm{im}\,(z, w))^2}}{2\,|\mathrm{im}\,(\bar{z}\,w)|}.$$

This definition of condition number leads to the expressions

$$\tilde{k}(r) = \frac{1 + \sin r}{|\cos r|}, \quad \tilde{k}^{-1}(\chi) = \cos^{-1}\left(\frac{2\,\xi}{1 + \xi^2}\right),$$

and hence

$$\tilde{P}_{I_2}(\chi) = 1 - \sqrt{\frac{1}{2} + \frac{\chi}{1 + \chi^2}},$$

corresponding to the functions k and P_{I_2} determined by κ. Note that $\tilde{P}_{I_2}(\tilde{\kappa}(I_2)) = P_{I_2}(\kappa(I_2)) = 0$, since $\tilde{\kappa}(I_n) = 1$. In fact, $0 < P_{I_2}(\sqrt{2}\,\chi) - \tilde{P}_{I_2}(\chi) < 0.0411$ for all $\chi > 1$ and the difference approaches zero as χ tends to ∞.

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