STABLE ERGODICITY IN HOMOGENEOUS SPACES

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Our goal in this paper is to classify stably ergodic translations and affine maps on homogeneous spaces. We will assume that our spaces are of the form G/B where G is a connected Lie group and B is a closed subgroup which, in addition, is admissible in a certain technical sense (see below).

For $g \in G$ let L_g denote left translation by g i.e. $L_g(h) = gh$ for all $h \in G$. Then L_g induces a map on G/B which we call L_g as well. Given an automorphism A of G and $g \in G$ we call $L_gA : G \to G$ an affine diffeomorphism of G, we also denote this map by gA. If $A(B) = B$ then we continue to denote the induced map on G/B by L_gA or gA and call it an affine diffeomorphism of G/B. For our discussion of ergodicity, we will assume that the Haar measure on G induces a finite measure on G/B which is invariant under left translation and that $A : G/B \to G/B$ is measure preserving.

A measure preserving diffeomorphism is called ergodic if the only measurable invariant sets have measure zero or one. We say that an affine diffeomorphism aA of G/B is stably ergodic under perturbations by left translations if there is a neighborhood U of a in G such that $a'A$ is ergodic for every $a' \in U$.

Given an affine diffeomorphism $aA : G \to G$, aA induces an automorphism of the Lie Algebra $\mathfrak{g}$ of G by $ad(g)DA(e)$ where e is the identity of G. In particular, $ad(g) \cdot DA(e)$ is a linear map. Let $\mathfrak{g}^s$ and $\mathfrak{g}^u$ be the generalized eigenspaces of $\mathfrak{g}$ corresponding to the contracting and expanding eignevalues of $ad(g) \cdot DA(e)$. Let $\mathcal{L} \subset \mathfrak{g}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^s$ and $\mathfrak{g}^u$. Then it is not hard to see (P-S) that $\mathcal{L}$ is an ideal in $\mathfrak{g}$ which is $ad(g) \cdot DA(e)$ invariant. As an ideal $\mathcal{L}$ is tangent to the connected normal subgroup which we denote by H and call the hyperbolically generated subgroup of G. It is now easy to state our main theorem.

Main Theorem: If an affine diffeomorphism is stably ergodic under perturbations by left translations then $\overline{HB} = G$ where H is the hyperbolically generated subgroup of G.

Remark (1). If $\overline{HB} = G$ then the affine diffeomorphism is ergodic, this is essentially Hopf's proof of the ergodicity of the geodesic flow. See [P-S] where generalizations are proven in the C^2 category and a version of the Main Theorem was conjectured and proven for left translations on $SL(n, \mathbb{R})$. One of the themes of [P-S] is that the same phenomenon which produces chaotic behavior (i.e. some hyperbolicity) may also guarantee

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robust statistics in the guise of stable ergodicity, and in fact may be necessary for it. The main theorem establishes the necessity for affine diffeomorphism of homogeneoous spaces.

(2) Our proof relies heavily on [B-M] where all the hard work of the theorem is carried out. By concentrating on stable ergodicity many of the subtleties of [B-M] disappear.

(3) That $\overline{HB} = G$ is the same as the action of H on G/B being ergodic, which in this setting is the same as the essential accessibility property of [P-S].

(4) We expect that in our main theorem that stable egodicity is actually equivalent to the condition that $\overline{HB} = G$. In the next two propositons we state some special cases of the theorem in which this is actually the case.

Proposition 1. *Let G be a connected nilpotent Lie group and Γ a uniform discrete subgroup of G. Then the affine diffeomorphism $Lg \cdot A$ of G/Γ is stably ergodic among left translations of G if and only if $\overline{H\Gamma} = G$.*

Proposition 2. *Let G be a connected semi-simple Lie group and Γ a lattice in G.*

a) If G has no compact factors, then the affine diffeomorphism $Lg \cdot A$ of G/Γ is stably ergodic among left translations of G if and only if $H = G$.

b) If G has compact factors then the affine diffeomorphism $Lg \cdot A$ of G/Γ is stably ergodic among left translations of G if and only if $\overline{H\Gamma} = G$.

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§1.

In this section, we consider three special cases which illustrate the Main Theorem and which are necessary for our proof of it; left translation on simple Lie groups, and affine diffeomorphisms on tori and compact semi-simple groups.

First we begin with a simple generalization of the Jacobian-Morozov Theorem.

Proposition 1.1 (Invariant Jacobson-Morozov). Let G be a connected, semi-simple Lie group, let ν be a non-zero unipotent element of G, and let σ a semi-simple automorphism of G that leaves ν fixed: $\sigma(\nu) = \nu$. Then there is a nilpotent element y in the Lie algebra g of G so that if x denotes $\log \nu$, and h denotes $[x, y]$, then

- x, y, and h generate a 3-dimensional Lie subalgebra of g isomorphic to sl_2, and
- σ centralizes that subalgebra.

To prove the proposition we use a simple lemma from Linear Algebra.

Lemma 1.1 Let V be a finite dimensional real or complex vector space and suppose

i) $A, L : V \to V$ are linear maps
ii) $AL = LA$
iii) $A(x) = x$
iv) $L(y) = x$ for $0 \neq x, y \in V$
v) A is semi-simple.

Then there is a $y' \in V$ such that

$$A(y') = y' \quad \text{and} \quad L(y') = x.$$

Proof of Lemma Let $V = V_1 \oplus V_2$ where V_1 is the $+1$ eigenspace of A, which is non trivial by iii), and V_2 is the invariant complement. Let $y = y_1 + y_2$ with respect to this direct sum decomposition. As $AL = LA, L(V_1) \subset V_1$ and $L(V_2) \subset V_2$. Since $x \in V_1$, $L(y_1) = x$ and $A(y_1) = y_1$.

Now we prove the proposition.

Let us also use σ to denote the automorphism of $\mathfrak{g}$ induced by our original σ. Because σ fixes ν, the lifted σ fixes x in $\mathfrak{g}$. Since σ is an automorphism of $\mathfrak{g}$ σ commutes with $ad(x)$. Now apply the lemma with $\sigma = A$ and $L = ad(x)^2$ to Bourbaki's argument to find y' with $\sigma(y') = y'$ and $[x,[x,y']] = 2x$. Setting $h = -2[x,y']$, $\sigma(h) = h$ and we are done.

Theorem 1.1. Let G be a connected simple Lie group and let $g \in G$. Then either

(1) The hyperbolically generated subgroup H for L_g equals G, or
(2) g may be arbitrarily closely approximated by elements g' of G such that $Ad(g')$ has finite order.

Proof. The proof for $SL(n,\mathbb{R})$ may be found in [P-S]. We shall show that the special case $n = 2$, together with the generalized Jacobson-Morozov lemma, implies theorem 1.1. We shall use $PSL(2,\mathbb{R})$ to denote the quotient of $SL(n,\mathbb{R})$ by its center.

Let us assume that $H \neq G$. We must show, then, that g may be arbitrarily closely approximated by elements g' of G such that $Ad(g')$ has finite order.

The element g factors into a product $s \cdot u$ in which s is semi-simple, u is unipotent, and s commutes with u. Because H is a connected normal subgroup of G, and G is simple, $H \neq G$ implies $H = \{e\}$ and hence that the eigenvalues of $Ad(s)$ must be of unit modulus. Thus the powers of $Ad(s)$ have compact closure, K.

We shall construct sequences u_n and s_n in G such that $Ad(u_n)$ and $Ad(s_n)$ are all of finite order, the s_n's all commute with the u_n's, $s_n \to s$, and $u_n \to u$. Since $s_n \cdot u_n \to s \cdot u$ and each $s_n \cdot u_n$ has finite order, that will prove the theorem.

By the extended Jacobson-Morozov proposition, u is contained in a connected subgroup X of G that is isomorphic to a covering group of $PSL(2,\mathbb{R})$ and is centralized by $\{g \in G : Ad(g) \in K\}$. Since the theorem is true for $SL(2,\mathbb{R})$, we can approximate u in X by elements u_n whose images in $PSL(n,\mathbb{R})$ are of finite order. As any finite dimensional representation of the universal cover of $SL(2,\mathbb{R})$ factors through $SL(2,\mathbb{R})$ (see [F-H] p. 143), each $Ad(u_n)$ has finite order. It remains to approximate s. Because K is compact, we can find a sequence s_n in G so that $Ad(s_n)$ is in K, $Ad(s_n)$ converges to $Ad(s)$, and each $Ad(s_n)$ has finite order. The map Ad from G to $Aut(\mathfrak{g})$ is open, so we can choose s_n convergent to s itself. Finally, the s_n's commute with the u_n's, because K centralizes X.

Corollary 1.1 Let G be a simple Lie group and B a closed subgroup such that G/B has a finite left invariant volume. Let $g \in G$. Then either

1) the hyperbolically generated group H for L_g equals G, or

2) g may be arbitrarily closely approximated by g' such that $L_{g'}$ is not ergodic and in fact has a C^∞ invariant function.

For proof of 2) see [B-M] Thm. 5.5 p. 599.

Remark. A continuous function gives an interval in $\mathbb{R}$ with the identity map as a quotient for $L_{g'}$.

Now we consider abelian groups. We represent the Torus T^n as $\mathbb{R}^n/\mathbb{Z}^n$. Affine maps are of the form $a + A$ where $a \in \mathbb{R}^n$ and $A \in GL(n, \mathbb{Z})$. The subgroup H of $\mathbb{R}^n$ is the direct sum of the contracting and expanding subspaces of A.

Theorem 1.2. Let $a + A$ be an affine automorphism of $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

1) The following are equivalent

a) A has no roots of unity as eigenvalue
b) $a + A$ is stably ergodic
c) $\overline{H\mathbb{Z}^n} = \mathbb{R}^n$ (in fact $\overline{H^s\mathbb{Z}^n} = \mathbb{R}^n$)

2) If A has a root of unity as eigenvalue then there is a perturbation a' of a and a positive integer k such that $(a' + A)^k$ has a non-trivial periodic quotient with quotient space a torus.

Proof. 1) a) That A is ergodic iff A has no root of unity is a standard fact that is easily proven using Fourier series.

b) and 2) If one is not an eigenvalue of A, then $a + A$ has a fixed point and $a + A$ is conjugate to A by a translation. So if A is ergodic $a + A$ is ergodic for all a.

On the other hand, if 1 is an eigenvalue of A, A is conjugate in $GL(n, \mathbb{Z})$ to $\begin{pmatrix} I & 0 \\ * & B \end{pmatrix}$ and if the component of a in the basis corresponding to the I are rationally dependent then $(a + A)$ has a non-trivial torus quotient with a periodic map on it.

If A has roots of unity as eigenvalues, but not 1, then $a + A$ is conjugate to A by a translation and $A^k = \begin{pmatrix} I & 0 \\ * & B \end{pmatrix}$ for some k. So we have proven 2) and the equivalence of 1a) and 1b). As for 1c), its equivalence to 1a) is a simple consequence of the primary decomposition theorem, which implies that A has a root of unity ζ as an eigenvalue iff it is conjugate in $GL(n, \mathbb{Q})$ to a matrix of the form $\begin{pmatrix} U & 0 \\ 0 & B \end{pmatrix}$ in which (1) all the eigenvalues of U are roots of unity, and (2) ζ is not an eigenvalue of B. See [Parry].

Now we turn to the compact case. First we need a standard fact about the affine diffeomorphisms of semi-simple Lie groups.

Let $I(G) \subset \mathrm{Aut}(G)$ be the group of inner automorphisms of G.

Proposition 1.2. *If G is a semi-simple Lie group then*

1) $I(G)$ is the connected component of the identity in $Aut(G)$. It has finite index in $Aut(G)$.

2) $G \times I(G)$ is the connected component of the identity in $G \times Aut(G)$ the group of affine diffeomorphisms of G. It has finite index in $G \times Aut(G)$.

Proof. $\mathrm{Aut}(\mathfrak{g})$ is an algebraic group so it has finitely many connected components and $I(G)$ is the identity component, see [Varadarajan] Theorem 3.10.8. This proves 1). 2) follows directly from 1.

Theorem 1.3. Let aA be an affine automorphism of G/B where G is a compact semi-simple group and B is a closed proper subgroup. Then aA admits a non-trivial periodic quotient (on a quotient space of G/B by a torus action) and hence aA is not ergodic.

Proof. Because $(aA)^k = x_k A^k$ for some $x_k \in G$, we can choose k such that $(aA)^k = x_k Ad(b)$ for some $b \in G$. Since B is mapped to itself by A, it follows that b normalizes B, and hence that the closed subgroup T_b generated by b acts by right translation on G/B. Let T_x be the closed subgroup generated by $x_k b$. Then T_x acts on the *left* on G/B. Since the left and right actions commute, the product $T_x \times T_b$ acts on G/B. It is easy to see that $(aA)^k$ leaves the orbits of this product invariant.

If the action of the abelian group $T_x \times T_b$ were transitive, G/B would be a solvmanifold, and hence (see [B-M]) G solvable, which it is not. The $T_x \times T_b$ action thus defines the action required by the proposition.

Next we state a corollary which will be useful later.

Corollary 1.2. Let $aA, a'A' : G/B \to G'/B'$ be affine automorphisms. Suppose that there is a surjective homomorphism $\rho : G \to G'$ such that $a'A'\rho = \rho aA$ and $B' = \overline{\rho(B)}$. Then aA ergodic and G' compact semi-simple implies $B' = G'$.

Proof. If $B' \neq G'$ then $a'A'$ cannot be ergodic so neither is aA.

§2.

In this section we gather together some of the lemmas and techniques we will need for a proof of our Main Theorem and make a reduction of the Main Theorem to Proposition 2.4.

In order to proceed to the general case of the main theorem, we first recall the definition and a few of the properties of admissible subgroup from [B-M].

Definition 2.1. The closed subgroup $B \subset G$ is *admissible* if 1) G/B has a finite volume.

2) There exists a closed solvable subgroup A of G which contains the radical of G, is normalized by B and such that $A \cdot B$ is closed.

We remark that condition 2) in the definition is automatically satisfied if G is solvable or semi-simple or if B is discrete, see [B-M].

Proposition 2.1. *Let $\rho : G_1 \to G_2$ be a continuous surjective homomorphism from the Lie group G_1 to G_2.*

1) If $B_2 \subset G_2$ in an admissible subgroup of G_2 then $B_1 = \rho^{-1}(B_2)$ is an admissible subgroup of G_2.

2) If $B_1 \subset G_1$ is an admissible subgroup of G_1 then $B_2 = \overline{\rho(B_1)}$ is an admissible subgroup of G_2.

Proof. 1) B_2 is a closed subgroup of G_2 and G_2/B_2 can be identified with G_1/B_1 and is hence of finite volume. ρ^{-1} of the radical of R_1 of G_1 contains the radical R_2 of G_2. So if $A_1 \supset R_1$ is the solvable group of the definition of admissible, then the radical $\hat{R}_2$ of $\rho^{-1}(A_1)$ is a solvable subgroup of G_2 containing R_2. Moreover, since $\rho^{-1}(A_1)$ is normalized by $B_2 = \rho^{-1}(B_1)$ so is $\hat{R}_2$. $\hat{R}_2$ is closed and normal in $\rho^{-1}(A_1)$ and maps onto A_1. So $\hat{R}_2 \cdot \rho^{-1}(B_1) = \rho^{-1}(A_1 B_1)$ which is closed.

2) See [B-M].

Our main technique to show that an affine diffeomorphism f is not ergodic is to produce a non-trivial periodic quotient of a power of f.

Definition 2.2. The affine diffeomorphism f *admits a non-trivial periodic quotient* if there is a continuous surjective map $\rho : G/B \to X$ onto a metric space X, a map $g : X \to X$, a point $x_0 \in X$, and positive integers k, ℓ such that

$$g^k = Id_X, \quad \rho f^\ell = g\rho \ \text{ and } \ \mu(\rho^{-1}(x_0)) = 0.$$

Proposition 2.2. *If the affine diffeomorphisms f admits a non-trivial periodic quotient then f is not ergodic.*

Proof. Suppose x_0, k, ℓ etc. are as above. Let U_i be a nested family of open sets for $i \in \mathbb{N}$ such that $\bigcap_{i\in\mathbb{N}} U_i = x_0$. Then $\mu(\rho^{-1}(U_i)) > 0$ for all i but $\mu(\rho^{-1}(U_i))$ tends to zero. Take i_0 large enough such that $\mu(\rho^{-1}(U_{i_0})) < \frac{1}{k\ell}$ then $\bigcup_{j=0}^{k\ell-1} f^j(\rho^{-1}(U_{i_0}))$ has measure < 1 and is an invariant set for f, so f is not ergodic.

In practice our space X is itself a non-trivial homogeneous space or the non-trivial quotient of a homogeneous space by a compact Lie group. In either case the spaces are metrizable and ρ^{-1} of any point has measure zero.

Now we prove a proposition and lemma useful in the reduction of the Main Theorem to proposition.

Proposition 2.3. *Let $gA : G \to G$ be an affine diffeomorphism and let H be the hyperbolically generated subgroup of G. Then $A(H) = H$.*

Proof. The Lie algebra $\mathfrak{h}$ of H is invariant for the Lie algebra automorphism $ad(g)DA(e)$. Therefore H is invariant for the Lie group automorphism $h \to gA(h)g^{-1}$ but that says $H = gA(H)g^{-1}$ and as H is normal $H = g^{-1}Hg = A(H)$.

Lemma 2.1 Let G_1 be a closed normal subgroup of G, let gA be an affine diffeomorphism of G such that $A(G_1) = G_1$, and let ρ be the natural projection $G \to G/G_1$. If H is the hyperbolically generated subgroup of gA in G, and H_1 is the hyperbolically generated subgroup of the induced map $\rho(g)A : G/G_1 \to G/G_1$, then $H_1 = \rho(H)$.

Proof. The derivative of ρ maps the Lie algebra of G onto the Lie algebra G/G_1 and

$$D\rho(e)ad(g)DAe = ad(\rho(g))DA(e)D\rho(e).$$

Thus the hyperbolic subspace of $ad(\rho(g))DA(e)$ is contained in the image of the hyperbolic subspace of $ad(g)DA(e)$ and similarly for the algebras generated.

Note: That the lemma implies that if H is trivial, so is H_1.

Proposition 2.4. *Let $aA : G/B \to G/B$ be an ergodic affine diffeomorphism with $H = e$. Then there is an admissible subgroup $B' \supset B$ and arbitrarily small perturbations a' of a such that $a'A : G/B' \to G/B'$ admits a non-trivial periodic quotient and hence $a'A$ is not ergodic.*

Now we show their proposition implies the Main Theorem.

Proof of the Main Theorem. $\bar{H}$ is a closed normal subgroup of G and $\overline{HB}/\bar{H}$ is an admissible subgroup of $G/\bar{H}$. aA acts on $G/\bar{H}/\overline{HB}/\bar{H}$ and we may assume the action is ergodic. There is no hyperbolic subspace for $ad(a)DA_e$ on $G/\bar{H}$, so we apply proposition 2.4 to produce a $B_1 \supset \overline{HB}/\bar{H}$ $B_1 \subset G/\bar{H}$ and $a_1 \in G/\bar{H}$ arbitrarily close to $a\bar{H}_1$ such that $a_1A : G/B_1 \to G/B_1$ admits a periodic quotient. Now pull back a_1 to a perturbation a' of a and let B' be the inverse image of B_1. B' is admissible and contains B. The lack of ergodicity follows from proposition 2.2.

§3.

In this section we prove proposition 2.4. We begin by reducing the proof to two special cases where G is assumed solvable or G is assumed semi-simple which cases we now assume proven. They are proven below.

Proof of Proposition 2.4 As in [B-M], we let R be the radical of G then G/R is semi-simple and $\overline{BR}/R$ is admissible in G/R, let $\rho : G \to G/R$ be the natural map. $Ad(a)A$ is an automorphism of G and hence fixes R and $\rho(a)A$ defines an affine diffeomorphism of $G/R/\overline{BR}/R$ with trivial hyperbolically generated subgroup. As before, using the semi-simple case of the proposition we may perturb $\rho(a)$ to a_1 and lift back to a perturbation a' of a_1 and find $aB_1 \supset \overline{BR}/R$ so that $a_1A : G/R/B_1 \to G/R/B_1$ and hence $a'A : G/\rho^{-1}(B_1) \to G/\rho^{-1}(B_1)$ have non-trivial periodic quotients.

Now we turn to the case that $G/\overline{BR}$ might be a point. In that case, there is a normal close connected subgroup N of G such that $N \subset B$, G/N is a solvable group and B/N is admissible. (Theorem 4.9 of [B-M].)

Now we proceed as above, apply the theorem in the solvable case and lift back to G.

In fact the argument shows that it is sufficient to prove proposition when G is semi-simple or solvable and B does not contain a connected closed normal subgroup of G. Also, if we let $\hat{G}$ be the universal covering group of G and $\rho : \hat{G} \to G$ the covering map, then proving proposition 2.4 for $\hat{G}$ and $\rho^{-1}(B)$ proves it as well for G, B since any automorphism of G preserving B lifts to an automorphism of $\hat{G}$ preserving $\rho^{-1}(B)$ and ρ is open so perturbations of left translations Lg in $\hat{G}$ are inverse images of perturbations of $L\rho(g)$.

Now we consider the semi-simple case of Proposition 2.4, and we assume G simply connected. First we record some facts about the automorphisms of G which preserve B and affine diffeomorphism of G/B.

Proposition 3.1. *Let G be a simply compact semi-simple group with no compact factors and B an admissible subgroup of G which contains no connected non-trivial normal subgroup of G. Then the group $Aut(G, B)$ of automorphisms of G which preserve B is* *a discrete subgroup of $Aut(G)$. Moreover, the group of inner automorphisms given by the normalizer of B, $N(B)$, is of finite index in $Aut(G,B)$.*

Proof. That B is discrete follows from the Borel density theorem (see the version in [Zimmer]). Moreover, any automorphism of G which is the identity on B is the identity automorpism of G. This proves that the automorphism group $Aut(G,B)$ is discrete. Now $Aut(G)$ is an algebraic group and the inner automorphisms are the identity component of G which has finitely many components since algebraic groups have only finitely many components, and we are done.

Corollary 3.1. Let G, B be as above and aA be an affine automorphism of G with $A(B) = B$. Let U be a neighborhood of $a \in G$ then for any $k \in \mathbb{N}$ the set of affine automorphism $a'A(a') \cdots A^{k-1}(a')A^k = (a'A)^k$ for $a' \in U$ contains a neighborhood of $aA(a) \cdots A^{k-1}(a)A^k = (aA)^k$ in the group of affine automorphisms of G which define maps of G/B.

Proof. The affine automorphisms of G whose automorphism belong to $Aut(G,B)$ are $G \times Aut(G,B)$ and the second factor of this Lie group is discrete. Raising to the kth power is open in a Lie group and we are done.

We are now ready to prove Proposition 2.4 for some semi-simple Lie groups.

Proposition 3.2. *Let $aA : G/B \to G/B$ be an ergodic affine automorphism with G a simply connected semi-simple Lie group with no compact factors, B an admissible subgroup of G containing no non-trivial normal subgroup of G and $H = e$. Then there is an admissible subgroup $B' \supset B$ and arbitrarily small perturbations a' of a such that $a'A : G/B' \to G/B'$ admits a non-trivial periodic quotient hence is not ergodic.*

Proof. By proposition 3.1 and corollary 3.1 we assume that there is a k such that $(aA)^k = b\ Ad(c)$ so that in particular the product decomposition of $G = G_1 \times \cdots \times G_k$ into simple groups is preserved; write $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$. Let B' be the normalizer of B, $B' = N(B)$. Then by the following lemma $c \in N(B)$ and $b\ Ad(c)$ and L_{bc} left multiplication by bc induce the same map on $G/N(B)$. By Theorem 1.1 bc may be arbitrarily closely approximate by an element $b'c$ with $ad(b'c)$ of finite order in $Aut(\mathfrak{g})$ so by corollary 3.1 there exists a' arbitrarily close to a such that $ad((a'A)^k) = ad(b'c)$ is of finite order in $Aut(G)$ and hence $a'A$ is not ergodic on $G/N(B)$ and admits a non-trivial periodic quotient. Now to finish the proof we need only see that $G \neq N(B)$. If G were $N(B)$ then G/B would be a finite volume semi-simple Lie group and hence compact. As we have assumed that G has no compact factors we are done.

Lemma 3.1 If $h\ Ad(a)$ defines a map on G/B then $a \in N(B)$ the normalizer of B, and $h\ Ad(a)$ and L_{ha} induce the same map on $G/N(B)$.

Proof. Since $h\ Ad(a)$ induces a map on G/B, given $x \in G$ and $b \in B$ there exists $b' \in B$ such that

$$haxba^{-1} = haxa^{-1}b' \quad \text{or} \quad aba^{-1} = b' \quad \text{thus} \quad a \in N(B).$$

To see the second assertion note that for all $x \in G$ hax is equivalent to $haxa^{-1}$ mod $N(B)$ since $a^{-1} \in N(B)$.

Now we prove Proposition 2.4 for the general semi-simple case.

Proposition 3.3. *Let $aA : G/B \to G/B$ be an ergodic affine diffeomorphism with G a simply connected semi-simple Lie group, B an admissible subgroup of G containing no non-trivial connected normal subgroup of G and $H = e$. Then there is an admissible subgroup $B' \supset B$ and arbitrarily small perturbations a' of a such that $a'A : G/B' \to G/B'$ admits a non-trivial periodic quotient and is not ergodic.*

Proof. As in [B-M] $G = C \times F$ with C the maximal normal compact subgroup and F a product of non-compact simple groups. The projection of B into F by dividing by C is a discrete group. If F is not trivial, we apply the previous proposition and use the same perturbation a' of a. If F is trivial, Theorem 1.3 finishes the proof.

Now we turn to the solvable case and finish the proof of the main theorem. The general solvable case follows from the toral case by the Mostow structure theorem.

Proposition 3.4. *Let $aA : G/B \to G/B$ be an ergodic affine automorphism with G a solvable Lie group and B an admissible subgroup which contains no non-trivial, connected, normal subgroup of G. Then if $H = e$ there is an arbitrarily small perturbation a' of a such that $a'A : G/B \to G/B$ admits a non-trivial periodic quotient on a torus of positive dimension and hence is not ergodic.*

Proof. By the Mostow structure theorem [EMS vol. 20] p. 167 we have the exact sequence

$$N/B \cap N \to G/B \to G/BN$$

where N is the Nil radical of G and hence is preserved by A and G/BN is a torus. Thus, BN is preserved by A and aA has $aA : G/BN \to G/BN$ is a quotient. Now apply Theorem 1.2. If G/BN is trivial so that G is nilpotent then G/B is a nilmanifold and our hypotheses assume that B is discrete. Once again we may fiber G/B over a non-trivial torus.

This finishes the proof of the propositions and the Main Theorem.

Finally we turn to the proofs of the two propositions following the Main Theorem. They are already established in one direction. First proposition 1.

Proof. By [Parry] gA is ergodic on G/Γ if and only if it is on the maximal torus quotient. Thus $\overline{H\Gamma} = G$ implies the same on the torus quotient. This implies that the quotient automorphism on the torus is ergodic and we have seen in section 2 that this last implies the stable ergodicity on the toral quotient, and hence on G/Γ.

Now Proposition 2.

Proof. a) $H = G$ is an open condition and by [M] or even [P-S] $H = G$ implies the ergodicity of gA. b)Write $G = C \times F$ as in Proposition 3.4. The condition $\overline{H\Gamma} = G$

implies by that H contains F and by corollary 1.2 that $\overline{F\Gamma} = G$ so this is again an open condition.

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