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STABLY ERGODIC SKEW PRODUCTS

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In [PS] it is conjectured that among the volume preserving C^2 diffeomorphisms of a closed manifold M which have some hyperbolicity, the ergodic ones contain an open and dense set. In this paper we prove an analogous statement for skew products of Anosov diffeomorphisms of tori and circle rotations. Thus this paper may be seen as an example of the phenomenon conjectured in [PS]. The corresponding theorem for skew products of Anosov diffeomorphisms and translations of arbitrary compact groups, is an interesting open problem.

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Let $\mathbb{S}^1$ denote the multiplicative group of complex numbers of modulus one. Let $\mathbb{T}^n$ denote the n-dimensional torus, *i.e.* the product of S^1 with itself n times. The group operation on $\mathbb{T}^n$ is componentwise multiplication. The character group of $\mathbb{T}^n$ is isomorphic to $\mathbb{Z}^n$. For $\bar{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{T}^n$, $\bar{r}(x) = x_1^{r_1} \cdots x_n^{r_n}$. The only measure we consider is Haar measure on the torus. Let $L^2(\mathbb{T}^n, \mathbb{C})$ be the L^2 functions from $\mathbb{T}^n$ to the Complex numbers. The group characters of $\mathbb{T}^n$ form a basis over $\mathbb{C}$ for $L^2(\mathbb{T}^n, \mathbb{C})$. This means for $h \in L^2(\mathbb{T}^n, \mathbb{C})$, $h = \sum a_{\bar{r}}\bar{r}$ with $a_{\bar{r}} \in \mathbb{C}$ and the sum is over all $\bar{r} \in \mathbb{Z}^n$.

Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a measure preserving, Anosov diffeomorphism of the n-dimensional torus. Let $\text{Höl}(\mathbb{T}^n, \mathbb{S}^1)$ be the set of Hölder continuous functions from $\mathbb{T}^n$ to $\mathbb{S}^1$. For $\varphi \in \text{Höl}(\mathbb{T}^n, \mathbb{S}^1)$ define the skew product $f_\varphi : \mathbb{T}^n \times \mathbb{S}^1 \to \mathbb{T}^n \times \mathbb{S}^1$ by $f_\varphi(x, y) = (f(x), y\varphi(x))$. The map f_φ preserves Haar measure on $\mathbb{T}^n \times \mathbb{S}^1$.

Lemma. *Let f be a measure preserving, Anosov diffeomorphism of $\mathbb{T}^n$. Let φ be in $\text{Höl}(\mathbb{T}^n, \mathbb{S}^1)$. If f_φ is not ergodic then φ is cohomologous to a constant function, $exp(2\pi i \frac{k}{r})$, for some $k, r \in \mathbb{Z}$.*

Proof. If f_φ is not ergodic there is a nonconstant function $h \in L^2(\mathbb{T}^n, \mathbb{C})$ with $h = h \circ f_\varphi$. Suppose there is such a function. We express h in terms of the characters of $\mathbb{S}^1$,

$$h(x, y) = \sum_{r \in \mathbb{Z}} h_r(x)\bar{r}(y),$$

where each h_r is in $L^2(\mathbb{T}^n, \mathbb{C})$. Then

$$h \circ f_\varphi(x, y) = \sum_{r \in \mathbb{Z}} h_r(f(x))\bar{r}(y\varphi(x)).$$

If $h = h \circ f_\varphi$ we have

$$h_r(x) = h_r(f(x))\bar{r}(\varphi(x))$$

for each $r \in \mathbb{Z}$ and $(x, y) \in \mathbb{T}^n \times \mathbb{S}^1$. The function h_0 is constant since $h_0(x) = h_0(f(x))$ and f is ergodic. If h is a nonconstant function there is an $r \neq 0$ in $\mathbb{Z}$ with $h_r \neq 0$. For this r we have

$$|h_r(x)| = |h_r(f(x))|$$

for every $x \in \mathbb{T}^n$ since $|\bar{r}(\varphi(x))| = 1$. The map f is ergodic so h_r has a constant nonzero modulus and we can assume it is one. We use this r to see that $\bar{r} \circ \varphi$ is cohomologous to the constant function 1 by

$$\bar{r} \circ \varphi(x) = (\varphi(x))^r = h_r(x)(h_r \circ f(x))^{-1}$$

with $h_r \in L^2(\mathbb{T}^n, \mathbb{S}^1)$. Since f is Anosov we can apply a theorem of Livšic [L] and assume h_r is Hölder.

We want to conclude that φ is cohomologous to $exp(2\pi i \frac{k}{r})$ for some $k \in \mathbb{Z}$. To do this we produce a g in $\text{Höl}(\mathbb{T}^n, \mathbb{S}^1)$ with $(g)^r = h_r$. Here, $(g)^r$ denotes the r^{th} power of g. The fundamental group of $\mathbb{T}^n$ is isomorphic to $\mathbb{Z}^n$ and the fundamental group of $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$. We take the maps φ and h_r induce on the fundamental groups and get the equation

$$r\varphi_\sharp = (h_r)_\sharp - (h_r)_\sharp f_\sharp = (h_r)_\sharp (I - f_\sharp).$$

Since f is Anosov $(I - f_\sharp)$ is invertible and we can solve for a map $g_\sharp$ on the fundamental groups,

$$g_\sharp = \frac{1}{r}(h_r)_\sharp = \varphi_\sharp (I - f_\sharp)^{-1}.$$

By standard covering space arguments we can conclude there is a g in $\text{Höl}(\mathbb{T}^n, \mathbb{S}^1)$ with $(g)^r = h_r$. This allows us to take roots in the cocycle equation and conclude

$$\varphi(x) = exp(2\pi i \frac{k}{r}) g(x)(g(f(x))^{-1}.$$

So φ is cohomologous to a constant function $exp(2\pi i \frac{k}{r})$. □

Theorem. *Let f be a measure preserving, Anosov diffeomorphism of $\mathbb{T}^n$. The set of functions in $\text{Höl}(\mathbb{T}^n, \mathbb{S}^1)$ for which the skew product, $f_\varphi : \mathbb{T}^n \times \mathbb{S}^1 \to \mathbb{T}^n \times \mathbb{S}^1$, is ergodic contains an open dense set (in the C^0 topology).*

Proof. This is an immediate consequence of the previous Lemma. If a function φ is cohomologous to a constant function, $exp(2\pi i \frac{k}{r})$, and $x \in \mathbb{T}^n$ is a point of period p under f then

$$\prod_{j=0}^{p-1} \varphi(f^j(x)) = exp(2\pi i \frac{pk}{r}).$$

Pick two points, x and y, in $\mathbb{T}^n$, each of period p under f and lying in different orbits. The set of functions φ in $\text{Höl}(\mathbb{T}^n, \mathbb{S}^1)$ satisfying

$$\prod_{j=0}^{p-1} \varphi(f^j(x)) \neq \prod_{j=0}^{p-1} \varphi(f^j(y))$$

contains an open dense set. □

References

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