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Note

Generalized Knapsack problems and fixed degree separations[1]

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One weaker form of the $P \neq NP$ problem consists of fixing the degree of the polynomial bounds for deterministic and nondeterministic time. The problem obtained in that way, to determine whether

$$\text{DTIME}(\text{O}(n^d)) = \text{NTIME}(\text{O}(n^d))$$

has been solved in the Boolean setting only for the special case $d = 1$. In fact, it is shown in [13] (see also [1]) that nondeterministic linear time is strictly more powerful than deterministic linear time.

In the context of real Turing machines the question of whether nondeterministic time is more powerful than deterministic time for fixed degree polynomial time bounds also arises naturally. As a first result one can remark that the quoted result for the Boolean situation still holds in the real case since the Knapsack problem can be solved in linear nondeterministic time while it is known that it has an $\Omega(n^2)$ lower bound for deterministic time (see [2]).

This paper shows that in this context a stronger result holds; namely, that for each $d \geqslant 1$ the classes $\text{NTIME}(\text{O}(n^d))$ and $\text{co-NTIME}(\text{O}(n^d))$ are different and therefore that the inclusion $\text{DTIME}(\text{O}(n^d)) \subset \text{NTIME}(\text{O}(n^d))$ is strict.

1. Some facts from linear geometry

In the following, let $k \in \mathbb{N}$, $k \geqslant 2$ and $K = \{0, 1, \ldots, k-1\}$.

Extending the notations used in threshold logic (see [12]) we shall say that a set $T \subseteq K^n$ is a *threshold subset* of K^n when there exists a linear function of the form

$$h = a_1 x_1 + \cdots + a_n x_n - 1 \quad a_1, \ldots a_n \in \mathbb{R}$$

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such that for any $x \in T$, $h(x) > 0$ and for any $x \in K^n - T$, $h(x) < 0$. In this case we shall also say the h *defines* the threshold subset T.

Note that for a given threshold set T there exist uncountably many choices of $a_1, \dots a_n$ satisfying the condition above. The following easy proposition characterizes the set of hyperplanes that separate a given threshold set.

Proposition 1. *Let* $\mathscr{H}_n = \{(x_1,\dots,x_n) \in \mathbb{R} : \forall b_1,\dots,b_n \in K\ \sum_{i=1}^n x_i b_i \neq 1\}$. *Two points* $a, b \in \mathbb{R}^n$ *define the same threshold set if and only if they belong to the same connected component of* $\mathscr{H}_n$.

Proof. The points a and b belong to the same connected component of $\mathscr{H}_n$ iff $\forall b_1,\dots,b_n \in K$ they lie on the same side of the hyperplane given by the equation $\sum_{i=1}^n x_i b_i = 1$. But this is equivalent to say that the functions

$$\sum_{i=1}^{n} a_i Y_i - 1 \quad \text{and} \quad \sum_{i=1}^{n} b_i Y_i - 1$$

define the same threshold subset of K^n. □

A trivial consequence of the preceding proposition is the following corollary.

Corollary 1. *For any threshold subset* T *of* K^n *the set of points* $(a_1,\dots,a_n) \in \mathbb{R}$ *s.t. the function* $\sum_{i=1}^n a_i Y_i - 1$ *defines* T *is an open set of* $\mathscr{H}_n$.

Lemma 1. *Let* H *be an affine subspace of* $\mathbb{R}^n$ *of dimension* $n-1$ *containing* K^{n-1} *and* T *be a threshold subset of* K^{n-1} *in* H. *Also, let* S *be a finite set of points in* $\mathbb{R}^n - H$. *Then, there exists an* $(n-2)$*-dimensional affine subspace* L *of* H *defining* T *such that every* $(n-1)$*-dimensional affine subspace of* $\mathbb{R}^n$ *passing through* L *contains at most one point of* S.

Proof. Let us consider a fixed injection

$$\mathbb{R}^n \hookrightarrow \mathbb{P}(\mathbb{R}^n)$$

of the affine space of dimension n in its projective closure. For any affine subspace A of $\mathbb{R}^n$ we denote by $\tilde{A}$ its closure in $\mathbb{P}(\mathbb{R}^n)$.

Now, for any two different points $x, y \in S$ we consider the intersection of the projective line defined by x and y with $\tilde{H}$. This defines a finite set of points F in $\tilde{H}$.

If we denote by $\tilde{\mathscr{H}}_{n-1}$ the space of all the $(n-2)$-dimensional projective subspaces of $\tilde{H}$ we have that $\mathscr{H}_{n-1}$ is an open subset of $\tilde{\mathscr{H}}_{n-1}$. Moreover, for each point p in F, the condition $p \notin l, l \in \tilde{\mathscr{H}}_{n-1}$ defines a Zariski open set in $\tilde{\mathscr{H}}_{n-1}$. Thus, the requirement that no point of F belongs to l also defines a Zariski open set W in $\tilde{\mathscr{H}}_{n-1}$ (see Chapter 1 of [7] for first notions of algebraic geometry). Since Zariski open sets are dense, the intersection of W with the subset of $\mathscr{H}_{n-1}$ containing those $(n-2)$-dimensional affine subspaces of H that define T is nonempty. Any such subspace L in this intersection will fulfill our statement. □

Remark 1. In the preceding lemma, the set S can be supposed to be countable and the result still holds. This is due to the fact that in a complete metric space a countable intersection of open dense sets is still dense or, arguing differently, to the fact that the resulting set W is the complement of a countably union of sets with measure zero and then this complement has measure zero itself. However, since we shall use only the level of generality stated in the lemma, we confine the result for infinite countable sets to this remark.

Theorem 1. *For any $k \geqslant 2$ and any $n \geqslant 1$ there are at least $k^{n^2/4}$ threshold subsets of K^n.*

Proof. The statement is clearly true for $n = 1$ since in this case we have exactly k threshold subsets in K.

Let us then suppose $n \geqslant 2$ and let H be the affine subspace of $\mathbb{R}^n$ defined by the equation $X_n = 0$. There is a natural inclusion $K^{n-1} \subset H$ and, by induction hypothesis, the number of threshold subsets of K^{n-1} is greater than $k^{(n-1)^2/4}$. For each one of these threshold subsets, let us consider a $(n-2)$-dimensional affine subspace L satisfying the statement of Lemma 1. A hyperplane that rotates around the axis L will meet the points of K^n satisfying $X_n \neq 0$ one at a time. Therefore, we can find $k^n - k^{n-1}$ hyperplanes through L defining different threshold subsets of K^n, and consequently, the total number of threshold subsets of K^n must be greater than

$$\begin{aligned} k^{(n-1)^2/4}(k^n - k^{n-1}) &= k^{(n-1)^2/4}k^{n-1}(k-1) \\ &\geqslant k^{(n^2+2n-2)/4} \\ &\geqslant k^{n^2/4}. \qquad \square \end{aligned}$$

2. Generalized Knapsack problems

The Knapsack problem over the reals is defined to be the set

$$\mathrm{KP} = \{x \in \mathbb{R}^\infty : |x| = n \text{ and } \exists S \subseteq \{1,\ldots,n\} \text{ with } \sum_{j \in S} x_j = 1\}$$

where, we recall from [3], $\mathbb{R}^\infty$ is the direct sum of countably many copies of $\mathbb{R}$ and for any $x \in \mathbb{R}^\infty$, $|x|$ denotes the size of x i.e. the largest $i \in \mathbb{N}$ such that $x_i \neq 0$. It is a set that has been studied under several points of view. Thus, lower bounds for deterministic sequential time and parallel time as well as for its topological complexity have been given in [2], [11] and [3] respectively. Also, a polynomial upper bound for its nonuniform deterministic complexity has been shown in [10].

We shall call *generalized Knapsack problem of order d* the following set:

$$\mathrm{KP}_d = \{x \in \mathbb{R}^\infty : |x| = n \text{ and } \exists 0 \leqslant b_1,\ldots,b_n < 2^{n^d} \text{ with } \sum_{j=1}^{n} b_j x_j = 1\}.$$

Note that the usual Knapsack problem is the generalized Knapsack problem of order 0.

The following theorem is our main reason to introduce the generalized Knapsack problem.

Theorem 2. *For any $d \geqslant 0$ a lower bound of $\Omega(n^{d+2})$ holds for the non-deterministic time necessary to solve the complement of the set* KP_d.

Proof. Let M be a nondeterministic machine solving the complement of KP_d that we shall denote by C. Then, for each $n \in \mathbb{N}$ and each $x \in \mathbb{R}^n$ we have that $x \in C$ if and only if there exists $y \in \mathbb{R}^{t(n)}$ such that M accepts $(x, y) \in \mathbb{R}^{n+t(n)}$ ($t(n)$ is a bound for the running time of M, see [3]).

In other words, if we denote by C_n the subset of elements in C having size n and by S_n the subset of elements in $\mathbb{R}^{n+t(n)}$ accepted by M, we have that C_n is the image of S_n under the projection

$$\pi_n : \mathbb{R}^{n+t(n)} \longrightarrow \mathbb{R}^n.$$

Since π_n is continuous we deduce that the number of connected components of S_n must be greater than that of C_n. Now, because of Proposition 1, the latter coincides with the number of threshold subsets of K^n for $k = 2^{n^d}$ and because of Theorem 1, this is greater than $k^{n^2/4}$ and therefore greater than

$$(2^{n^d})^{n^2/4} = 2^{n^{d+2}/4}.$$

According to [2], a lower bound of $\Omega(\log \#\text{c.c.}(S_n))$ holds then for the running time of M where $\#$ c.c.(S_n) is the number of connected components of S_n and therefore the desired lower bound follows. □

Let us recall that by co-NTIME$(t(n))$ we denote the class of sets whose complements can be solved in nondeterministic time $t(n)$.

Corollary 2. *For every $d \in \mathbb{N}$, $d \geqslant 1$ the classes* NTIME$(O(n^d))$ *and* co-NTIME$(O(n^d))$ *are different.*

Proof. Because of the preceding theorem for any $d \geqslant 0$ the set KP_d does not belong to co-NTIME$(O(n^{d+1}))$ whilst the following algorithm shows that it belongs to NTIME$(O(n^{d+1}))$:

> **input**$(x_1, \ldots, x_n)$
> **for** $j \in \{1, \ldots, n\}$ and $i \in \{0, \ldots, n^d - 1\}$ **do**
>   guess $b_{i,j} \in \{0, 1\}$
> **od**
> compute $s := \sum_{i,j} b_{i,j} 2^i x_j$
> ACCEPT iff $s = 1$ □

Since deterministic time is closed under complements the following result is straightforward.

Corollary 3. *For every $d \in \mathbb{N}$, $d \geqslant 1$ we have the strict inclusion*

$$\mathrm{DTIME}(\mathrm{O}(n^d)) \subset \mathrm{NTIME}(\mathrm{O}(n^d)).$$

Remark 2. In the context of computations with real numbers two kinds of nondeterminism are possible. On the one hand, the nondeterminism as it was introduced in [3] where at any step of the computation an arbitrary real number can be guessed. On the other hand, a weaker form of nondeterminism was introduced in [5] where the guesses are restricted to belong to the set $\{0, 1\}$. The relation between the computational power of this last "digital" nondeterminism and the "full" one first quoted are more or less well understood depending on the kind of real Turing machines allowed. Thus, for additive real Turing machines (that do not perform multiplications or divisions) it was shown in [9] and in [4] that both nondeterminisms generate the same polynomial hierarchy the coincidence being level by level. In particular, it follows that the classes $\mathrm{NP}_{\mathrm{add}}$ and $\mathrm{DNP}_{\mathrm{add}}$ coincide (we use the letters DNP for denoting Digital Nondeterministic Polynomial time). On the other hand, for the weak model introduced in [8] (where the use of multiplications and divisions is allowed but restricted in number) it was shown in [6] that the classes $\mathrm{DNP}_{\mathrm{W}}$ and NP_{W} are actually different.

For the standard real Turing machine model, the proof given for Corollary 3 actually shows that $\mathrm{DTIME}(\mathrm{O}(n^d))$ is strictly included in $\mathrm{DNTIME}(\mathrm{O}(n^d))$ (DNTIME stands for Digital Nondeterministic time). It remains open however, whether fixed degree separations hold for digital and full nondeterministic time in this context.

On the other hand, Corollary 3 remains valid in the additive and weak contexts where the separation between P and DNP are still open.

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Addendum. While this note was in print we learned that a result similar to Theorem 1 had already been shown by F. Meyer auf der Heide in "Lower bounds for solving linear diophantine equations on random access machines", *J. ACM* **32**(4) (1985) 929–937.