Test Complexity of Generic Polynomials

PETER BÜRGISSE AND THOMAS LICKTEIG

Institut für Informatik V, Universität Bonn, Römerstrasse 164, D-5300 Bonn, Germany

AND

MICHAEL SHUB

IBM Thomas J. Watson Research Center, Department of Mathematical Sciences, P.O. Box 218, Yorktown Heights, New York 10598

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We investigate the complexity of algebraic decision trees deciding membership in a hypersurface \( X \subset \mathbb{C}^m \). We prove an optimal lower bound on the number of additions, subtractions, and comparisons and an asymptotically optimal lower bound on the number of multiplications, divisions, and comparisons that are needed to decide membership in a generic hypersurface \( X \subset \mathbb{C}^m \). Over the reals, where in addition to equality branching also \( \leq \)-branching is allowed, we prove an analogous statement for irreducible "generic" hypersurfaces \( X \subset \mathbb{R}^m \). In the case \( m = 1 \) we give also a lower bound for finite subsets \( X \subset \mathbb{R} \).

I. INTRODUCTION

Given a polynomial \( f: \mathbb{C}^m \rightarrow \mathbb{C} \) we may check whether \( f(\xi) = 0 \) by evaluating \( f \) at \( \xi \). But testing for zero may be easier than evaluating. We show in this paper that this is not the case if \( f \) is sufficiently general.

The complexity of evaluating polynomials has been extensively studied in the last decades, starting with Ostrowski, Motzkin, Belaga, Pan, Winograd, and Strassen. (See Borodin and Munro, 1975, von zur Gathen, 1988, and Strassen, 1984, 1990, for a review of this work.) Recently, interest has begun to focus on the complexity of testing for zero. Strassen (1981) contains a lower bound on decision complexity in terms of the degree of an algebraic set. (For an application see Schuster, 1980.) Ben-Or (1983) proves lower bounds on the decision complexity of semialgebraic sets in
terms of the number of connected components, generalizing previous results by Dobkin and Lipton (1978) and Steele and Yao (1982). In turn Yao (1989) extends these results back to the discrete setting. Recio and Pardo (1987) give lower bounds based on the width of a semialgebraic set continuing work by Rabin (1972) and Jaromczyk (1981). Montaña, Pardo, and Recio (1990) replace the number of connected components by intersection numbers and improve Ben-Or (1983) in various cases of connected semialgebraic sets. In Lickteig (1990) lower bounds on decision complexity are proved via differential methods from algebraic complexity theory (Baur and Strassen, 1983; Strassen, 1973) and approximative complexity (see the references in Lickteig, 1990) which for instance allow lower bounds for various decision problems from linear algebra to be given in terms of the complexity of the two fundamental problems of solving a system of linear equations and matrix multiplication.

In this paper we investigate the decision complexity of a generic hypersurface \( X \subset \mathbb{C}^m \) employing the dimension or transcendence degree bound. This technique for proving lower bounds on the complexity of rational functions was introduced by Motzkin (1955), Belaga (1961), Paterson and Stockmeyer (1971), Reingold and Stocks (1972), and later was improved by Baur and Rabin (1982). They show in particular that for a polynomial \( f \in \mathbb{C}[x_1, \ldots, x_m] \) of degree \( d \) with algebraically independent coefficients over \( \mathbb{Q} \) one has

\[
L_+(f) \geq \binom{d + m}{m} - 1, \tag{1}
\]

\[
L_+(f) \geq \left\lfloor \left( \frac{d + m}{m} \right) - 1 \right\rfloor /2, \tag{2}
\]

where \( L_+(f), L_+(f) \) denote the additive, respectively the multiplicative complexity of \( f \).

We briefly recall the definitions. We study straight line programs that compute rational functions in \( \mathbb{C}(x_1, \ldots, x_m) \) from some input of the form \((\xi_1, \ldots, \xi_n; x_1, \ldots, x_m)\), where \( \xi \in \mathbb{C}^n, n \in \mathbb{N} \), using operations from \( \Omega := \mathbb{Q} \sqcup \{+, -, \times, /\} \). Here \( \mathbb{C}(x_1, \ldots, x_m) \) is considered as a \( \mathbb{Q} \)-algebra and \( \lambda \in \mathbb{Q} \) stands for scalar multiplication by \( \lambda \). Allowing arbitrarily many constants \( \xi_i \in \mathbb{C} \) is sometimes referred to as “coefficient preparation.”

The minimum number of nonscalar multiplications and divisions sufficient to compute \( f \) by an \( \Omega \)-straight line program from some input \((\xi_1, \ldots, \xi_n; x_1, \ldots, x_m)\) is called the multiplicative complexity \( L_+(f) \). By counting only additions and subtractions we get the additive complexity \( L_+(f) \). The bound in (1) is obviously sharp.

Let \( X \subset \mathbb{C}^n \) be a hypersurface, \( X = X_1 \cup \cdots \cup X_t \) its decomposition into irreducible components, and
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Xi = Zeroset(f_i), f_i ∈ C[x_1, . . . , x_m] irreducible,

deg f_i − d_i (i = 1, . . . , t).

(For definitions and results from classical algebraic geometry see, e.g., Shafarevich, 1974.) We call X Q-generic of degree format (d_1, . . . , d_t) if the polynomials f_1, . . . , f_t may be chosen such that all their coefficients are algebraically independent over Q. If T is an algebraic decision tree using the operation symbols in Ω, the relation symbol "=", and as constants a finite subset of C we denote by C_+(T) the maximum number of additions, subtractions, and comparisons occurring in a path from the root to a leaf of T. By minimizing C_+(T) over all such decision trees deciding membership in X we obtain the additive branching decision complexity C_+(X) of X. The multiplicative branching decision complexity C_*(X) is defined analogously.

Our main result is

**Theorem 1.** Let X ⊂ C^m be a Q-generic hypersurface of degree format (d_1, . . . , d_t) ∈ N^t. Then we have for the additive branching decision complexity C_+(X) of X

\[ C_+(X) = 2 \left( \sum_{i=1}^{t} \frac{(d_i + m)}{m} - 1 \right), \]

and the multiplicative branching decision complexity C_*(X) of X satisfies

\[ C_*(X) \geq \frac{1}{2} \sum_{i=1}^{t} \left( \frac{(d_i + m)}{m} - 1 \right). \]

We further show that the lower bound in (4) is asymptotically sharp as \( \min_{1 \leq i \leq t} d_i \to \infty \), keeping m, t fixed.

In the special case of zero-dimensional hypersurfaces we get

**Corollary 1.** Let X be a finite subset of C with t elements that are algebraically independent over Q. Then

\[ C_+(X) = t, \]

\[ t/2 \leq C_*(X) \leq t/2 + 3. \]

We compare this with the situation over the reals where we also allow ≤-comparisons to be performed. For a subset X ⊂ R^m the additive and multiplicative decision complexities C_+,≤(X) and C_*,≤(X) are defined in an obvious way.
Theorem 2. Let $X \subset \mathbb{R}^m$ be an irreducible hypersurface. Assume that

$$X = \text{Zeroset}(f), \quad f \in \mathbb{R}[x_1, \ldots, x_m] \text{ irreducible}, \quad \deg f = d$$

and that the coefficients of $f$ are algebraically independent over $\mathbb{Q}$. Then

$$C_{+,\leq}(X) = \left( \frac{d + m}{m} \right) - 1,$$

$$C_{*,\leq}(X) \geq \left( \left( \frac{d + m}{m} \right) - 1 \right)/2.$$  

(5)

(6)

So for an irreducible "generic" hypersurface $X \subset \mathbb{R}^m$ the lower bounds (3), (4) remain valid for the complexities $C_{+,\leq}(X)$ and $C_{*,\leq}(X)$. However, if the hypersurface $X$ has several irreducible components, the situation may drastically change. We demonstrate this fact in the case where $X$ is zero-dimensional.

Proposition 1. For every finite subset $X \subset \mathbb{R}$ with $t$ elements the following hold $(\log = \log_2)$:

$$\frac{1}{2} \sqrt{\log t} \leq C_{+,\leq}(X) \leq \left\lceil \log t \right\rceil + 1,$$

$$\log(t/3)/\log 6 \leq C_{*,\leq}(X) \leq \left\lceil \log t \right\rceil + 1.$$  

(7)

(8)

If the elements of $X$ are algebraically independent over $\mathbb{Q}$ we even have

$$\log t - \log \log(t + 1) \leq C_{+,\leq}(X).$$

2. Transcendence Degree Bounds

For $u_1, \ldots, u_r \in \mathbb{C}[x_1, \ldots, x_m]$ let us denote by $T(u_1, \ldots, u_r)$ the subfield of $\mathbb{C}$ generated by the coefficients of the polynomials $u_i$. The following lemma is based on the fact that prime factorization in $\mathbb{C}[x_1, \ldots, x_m]$ is unique up to scaling.

Lemma 1. Let $u_1, \ldots, u_r \in \mathbb{C}[x_1, \ldots, x_m]$ be polynomials different from zero. Then

$$\text{trdeg}_Q T(u_1 \cdots u_r) \geq \text{trdeg}_Q T(u_1, \ldots, u_r) - (r - 1).$$

Proof. By Kronecker's trick we may assume w.l.o.g. that $m = 1$ (replace $x_1, \ldots, x_m$ by $x_1, x_1^N, \ldots, x_1^{N^{m-1}}$ with $N > \deg(u_1 \cdots u_r)$). Let $u_1, \ldots, u_r$ be monic. Using the fact that the splitting field of $u_i$ is a
finite algebraic extension of \( T(u_i) \) we see that
\[
\text{trdeg}_Q T(u_1 \cdots u_r) = \text{trdeg}_Q T(u_1, \ldots, u_r).
\]

Taking into account that for \( \alpha_i \in \mathbb{C} \)
\[
\text{trdeg}_Q T(u_i) \geq \text{trdeg}_Q T(\alpha_i u_i) - 1
\]
the lemma follows. ■

The proof of Theorem 1 rests on an observation in Lickteig (1990) stating that the bound (2) also holds when one is allowed to perform any \( Q \)-rational operation of arity two at unit cost. So we not only focus on \( \Omega \)-straight line programs computing a rational function \( f \in \mathbb{F}(x_1, \ldots, x_m) \) but also consider straight line programs where a basic computation step is any \( Q \)-rational operation of arity less than or equal to a given natural number \( a \), i.e., given by a rational function in \( \mathbb{Q}(t_1, \ldots, t_a) \). For \( a \in \mathbb{N} \) we put \( \Omega_{(a)} := \{ \text{Q-rational operations of arity } \leq a \} \). We define the complexity \( L_{(a)}(f) \) of \( f \) with respect to arity \( a \) as the minimum number of nonlinear \( Q \)-rational operations of arity less than or equal to \( a \) sufficient to compute \( f \) from some input of the form \( (\xi_1, \ldots, \xi_n; x_1, \ldots, x_m) \) where \( n \in \mathbb{N}, \; \xi_i \in \mathbb{C} \) by an \( \Omega_{(a)} \)-straight line program. (Compare Schnorr, 1981; Ben-Or, 1983; Lickteig, 1990.) Obviously
\[
L_{(2)}(f) \leq L_*(f).
\]

For example, a rational function \( f = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \) can be computed from \( (x_1, \ldots, x_m) \) with only \( m - 1 \) rational operations of arity \( \leq 2 \), namely with operations of the form \( (t_1, t_2) \mapsto t_1^{e_1} t_2^{e_2} \) \( (\mu_i \in \mathbb{Z}) \), no matter how big the \( |e_i| \) are. So
\[
L_{(2)}(x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}) \leq m - 1.
\]

The next theorem will be our main tool.

**Theorem 3.** (Motzkin, 1955; Belaga, 1961; Reingold and Stocks, 1972; Baur and Rabin, 1982; Lickteig, 1990). Let \( a \in \mathbb{N}, \; a \geq 1, \; \text{and} \; f \in \mathbb{C}(x_1, \ldots, x_m) \). Then there exist \( \alpha, \dot{\alpha} \in \mathbb{C}, \; u, \dot{u} \in \mathbb{C}[x_1, \ldots, x_m], \; v, \dot{v} \in \mathbb{C}[x_1, \ldots, x_m] \) satisfying
\[
f = \dot{\alpha} \frac{\dot{u}}{\dot{v}}, \quad \text{trdeg}_Q T(\dot{u}, \dot{v}) \leq L_*(f),
\]
\[
f = \alpha + \frac{u}{v}, \quad \text{trdeg}_Q T(u, v) \leq aL_{(a)}(f).
\]
Proof. We prove the following statement by induction on $s$: If $\beta = (\beta_1, \ldots, \beta_s)$ is an $\Omega$-straight line program with $s$ instructions executable on the input $(\xi_1, \ldots, \xi_n; x_1, \ldots, x_m), \xi \in \mathbb{C}^n$ with result sequence

$$(b_{-n-m+1}, \ldots, b_0, b_1, \ldots, b_s),$$

where $b_{-n-m+1} = \xi_1, \ldots, b_0 = x_m$, then there are

$$\bar{u}_i, \bar{v}_i \in \mathbb{C}[x_1, \ldots, x_m], \bar{\alpha}_i \in \mathbb{C} \quad (i = 1, \ldots, s)$$

satisfying the two conditions

$$b_i = \bar{\alpha}_i \bar{u}_i / \bar{v}_i, \quad \bar{v}_i \neq 0,$$

$$\text{trdeg}_Q T(\bar{u}_1, \ldots, \bar{u}_s, \bar{v}_1, \ldots, \bar{v}_s) \leq \{|i \leq s: \beta_i \text{ instruction for an addition or subtraction}|\}.$$

Let us sketch the induction step "$s > 0$": If $\beta_s = (+; i, j)$ with $i, j < s$ then (w.l.o.g. $b_i \neq 0$)

$$b_s = b_i + b_j = \bar{\alpha}_i (\bar{u}_i \bar{v}_j + \bar{\alpha}_j (\bar{v}_i \bar{u}_j) / (\bar{u}_i \bar{v}_j));$$

if $\beta_s = (*; i, j)$ with $i, j < s$ then

$$b_s = b_i \cdot b_j = (\bar{\alpha}_i \bar{\alpha}_j) (\bar{u}_i \bar{u}_j) / (\bar{v}_i \bar{v}_j).$$

In the case of the other operations one has analogous representations for $b_s$. From these the definition of $\bar{\alpha}_s, \bar{u}_s$, and $\bar{v}_s$ is evident.

Analogously, Let $\beta = (\beta_1, \ldots, \beta_s)$ be an $\Omega_{(a)}$-straight line program. Assume that $\beta$ is executable on some input of the form $(\xi_1, \ldots, \xi_n; x_1, \ldots, x_m)$ with $\xi \in \mathbb{C}^n$ and result sequence

$$(b_{-n-m+1}, \ldots, b_0, b_1, \ldots, b_s).$$

Then there are

$$u_i, v_i \in \mathbb{C}[x_1, \ldots, x_m], \alpha_i \in \mathbb{C} \quad (i = 1, \ldots, s)$$

satisfying the two conditions

$$b_i = \alpha_i + u_i/v_i, \quad v_i \neq 0,$$
The theorem is an immediate consequence of these statements. ■

3. Proofs

In the following let \( \text{ord}_p : C(x_1, \ldots, x_m)^x \to \mathbb{Z} \) denote the discrete valuation associated with an irreducible polynomial \( p \in C[x_1, \ldots, x_m] \), i.e., \( f = p^{\text{ord}_p(f)}A/B \), where \( p \) does not divide \( A \) and \( B \).

Proof of Theorem 1. Let \( X \subseteq C^m \) be a \( Q \)-generic hypersurface of degree format \((d_1, \ldots, d_t) \in \mathbb{N}^t \). Then we may assume for the decomposition \( X = X_1 \cup \cdots \cup X_t \) of \( X \) into irreducible components that

\[
X_i = \text{Zeroset}(f_i), \quad f_i \in C[x_1, \ldots, x_m] \text{ irreducible},
\]

\[
\deg f_i = d_i \quad (i = 1, \ldots, t),
\]

and the coefficients of \( f_1, \ldots, f_t \) are algebraically independent over \( Q \).

For the upper bound we compute all the \( f_i - f_i(0) \) with

\[
\sum_{i=1}^t \left( \binom{d_i + m}{m} - 2 \right) \text{ additions}
\]

and then compare them with \(-f_i(0)\).

To prove the lower bounds assume \( T \) to be a tree deciding membership in \( X \). We denote by \( \pi_i \) the typical path of \( X_i \) \((i = 1, \ldots, t) \) (i.e., the path taken by Zariski-almost all elements of \( X_i \)) and by \( \pi_0 \) the typical path of \( C^m \). The node where \( \pi_i \) and \( \pi_0 \) separate will be called \( \nu_i \). After permuting \( X_1, \ldots, X_t \) we can assume that the sequence of nodes \((\nu_1, \ldots, \nu_t)\) is ordered according to the partial order \( < \) defined by the tree \( T \) (predecessor relation), say

\[
\nu_1 = \cdots = \nu_{t_0} < \nu_{t_0+1} = \cdots = \nu_{t_1} < \cdots < \nu_{t_{i-1}+1} = \cdots = \nu_t,
\]

where \( t_0 := 0 < t_1 < t_2 < \cdots < t_s = t \). By following the path \( \pi_0 \) up to the node \( \nu_{t_i} \) \((i = 1, \ldots, s)\) we obtain rational functions \( g_i^{(1)}, g_i^{(2)} \in C(x) \) such that

\[
g_i^{(1)} = g_i^{(2)}?
\]

is tested at node \( \nu_{t_i} \) and
The $g_i := g_1^{(1)} - g_2^{(2)}$ are obviously not zero. Furthermore, for every $i \in \{1, \ldots, s\}$ the rational function $g_i$ is defined almost everywhere on $X_{t_i-1+1}, \ldots, X_t$ and, among these sets, vanishes exactly on $X_{t_i-1+1}, \ldots, X_{t_i}$. By the Nullstellensatz we have for all $i \in \{1, \ldots, s\}, j \in \{1, \ldots, t\}$,

\begin{align*}
\text{ord}_f(g_i) > 0 & \quad \text{if } t_{i-1} < j \leq t_i, \\
\text{ord}_f(g_i) & \quad \text{if } t_i < j.
\end{align*}

The matrix $[\text{ord}_f(g_i)]_{i,j} \in \mathbb{Z}^{s \times t}$ is therefore a blockwise lower triangular matrix where the block diagonal entries are elementwise positive. Hence it is possible to choose $m_1, \ldots, m_s \in \mathbb{N}$ such that

\[ \sum_{i=1}^{s} m_i \text{ord}_f(g_i) > 0 \quad \text{for all } j \in \{1, \ldots, t\}. \]

So the nonzero rational function $h := g_1^{m_1} g_2^{m_2} \cdots g_s^{m_s}$ satisfies the condition $\text{ord}_f(h) > 0$ for all $j$. Since $h$ may be obtained from $g_1^{(1)}, \ldots, g_s^{(2)}$ with $s$ subtractions and $s - 1$ operations $(t_1, t_2) \mapsto t_1^{t_1} t_2^{t_2}$ of arity 2 we have

\[ L_+(h) \leq C_+(-T), \quad L_{(2)}(h) \leq C_+(-T) - 1. \]

Theorem 3 implies now that there exist

\[ \alpha, \hat{\alpha} \in \mathbb{C}, \quad u, \hat{u} \in \mathbb{C}[x_1, \ldots, x_m], \quad v, \hat{v} \in \mathbb{C}[x_1, \ldots, x_m] \setminus \{0\} \]

satisfying

\[ h = \alpha \frac{\hat{u}}{\hat{v}}, \quad \text{trdeg}_Q T(\hat{u}, \hat{v}) \leq L_+(h), \]

\[ h = \alpha + \frac{u}{v}, \quad \text{trdeg}_Q T(u, v) \leq 2 L_{(2)}(h). \]

The polynomials $f_1, \ldots, f_t$ are prime factors of $\hat{u}$ and of $\alpha v + u$. Lemma 1 implies therefore that

\[ \text{trdeg}_Q T(f_1, \ldots, f_t) - t \leq \text{trdeg}_Q T(\hat{u}), \]

\[ \text{trdeg}_Q T(f_1, \ldots, f_t) - t \leq \text{trdeg}_Q T(\alpha v + u). \]
Since we assume the algebraic independence over $\mathbb{Q}$ of the coefficients of the polynomials $f_i$ we have

$$\text{trdeg}_Q T(f_1, \ldots, f_i) = \sum_{i=1}^r \left( \frac{d_i + m}{m} \right).$$

This proves the asserted lower bounds. $\blacksquare$

**Proof of Theorem 2.** Let $T$ be a tree deciding membership in a proper algebraic subset $X \subset \mathbb{R}^m$. Without loss of generality we can assume that for every path $\pi$ from the root to a leaf in $T$ the set

$$D_{\pi} := \{ \xi \in X : \text{the input } \xi \text{ defines the path } \pi \text{ in } T \}$$

is nonempty. Obviously $X = \bigcup\{D_{\pi} : \pi \text{ path leading to a yes-leaf}\}$. Consider a path $\pi$ that leads to a yes-leaf as being fixed for a moment. Let us denote by $V_\leq$ the set of $\leq$-branching nodes of $\pi$ and let $g_{\nu}^{(1)}, g_{\nu}^{(2)}$ be rational functions for $\nu \in V_\leq$ such that

$$g_{\nu}^{(1)} \leq g_{\nu}^{(2)}.$$

is the test performed at node $\nu$. The set $V_\geq$ and the rational functions $g_{\nu}^{(1)}, g_{\nu}^{(2)}$ for $\nu \in V_\geq$ are defined similarly. We have $g_{\nu}^{(1)} \neq g_{\nu}^{(2)}$ for every $\nu \in V_\leq \cup V_\geq$ since the $D_{\pi}$ are nonempty. Furthermore

$$L_+ (g_{\nu}^{(1)}, g_{\nu}^{(2)}) \leq C_{+, \leq}(T) - 1, \quad L_+ (g_{\nu}^{(1)}, g_{\nu}^{(2)}) \leq C_{+, \leq}(T) - 1$$

for all $\nu \in V_\leq \cup V_\geq$. There are partitions

$$V_\leq = V_{\leq, \text{true}} \cup V_{\leq, \text{false}}, \quad V_\geq = V_{\geq, \text{true}} \cup V_{\geq, \text{false}}$$

with the property that an element $\xi$ lies in $D_{\pi}$ if and only if the following conditions are satisfied:

$$\forall \nu \in V_{\leq, \text{true}} g_{\nu}^{(1)}(\xi) \leq g_{\nu}^{(2)}(\xi), \quad \forall \nu \in V_{\leq, \text{false}} g_{\nu}^{(1)}(\xi) > g_{\nu}^{(2)}(\xi),$$

$$\forall \nu \in V_{\geq, \text{true}} g_{\nu}^{(1)}(\xi) = g_{\nu}^{(2)}(\xi), \quad \forall \nu \in V_{\geq, \text{false}} g_{\nu}^{(1)}(\xi) \neq g_{\nu}^{(2)}(\xi).$$

We show now that

$$D_{\pi} \subset \bigcup \{ \{ \xi \in \mathbb{R}^m : g_{\nu}^{(1)}(\xi) = g_{\nu}^{(2)}(\xi) \} : \nu \in V_{\leq, \text{true}} \cup V_{\geq, \text{true}} \}.$$  \hfill (9)

If $V_{\text{true}}$ is nonempty the statement (9) is obvious. So let us assume that $V_{\text{true}}$ is empty. If (9) were violated then $D_{\pi}$ would contain a nonempty
open subset of $\mathbb{R}^m$ which would contradict the assumption that $X$ is a proper algebraic subset of $\mathbb{R}^m$.

Now we use that $X$ is an irreducible hypersurface. Therefore there exists a path $\pi$ such that $D_\pi$ is Zariski-dense in $X$. Moreover, also by the irreducibility of $X$, there is a $\nu \in V_{\mathbb{R}, \text{true}} \cup V_{\mathbb{C}, \text{true}}$ such that

$$D_\pi \subset \{ \xi \in \mathbb{R}^m : g^{(1)}(\xi) = g^{(2)}(\xi) \}.$$ 

Because the vanishing ideal of $X$ equals $\langle f \rangle$ (cf. Bochnak, Coste, and Roy, 1987, Théorème 4.5.1, p. 85) we get $\text{ord}_f(g^{(1)} - g^{(2)}) > 0$. The rest of the proof is identical to the one of Theorem 1.

Remark. Assume $k = \mathbb{R}$ or $k = \mathbb{C}$. Eve's algorithm (Eve, 1964) shows for a univariate polynomial $f \in k[x]$ of degree $d$ the upper bound

$$L_{*, k[x]}(f) \leq d/2 + 2. \quad (10)$$

From this can be easily concluded that for fixed $m \in \mathbb{N}$ there is a sequence $(\rho_d) = o(1) (d \to \infty)$ such that for any polynomial $f \in k[x_1, \ldots, x_m]$ of degree $d$

$$L_{*, k[x]}(f) \leq \frac{d^m}{2(m!)} (1 + \rho_d). \quad (11)$$

Assume now that $m, t \in \mathbb{N}$ are fixed. The estimate (11) implies immediately that there is a sequence $(\sigma_{(d_1, \ldots, d_t)}) = o(1) (\min_{1 \leq i \leq t} d_i \to \infty)$ such that for all hypersurfaces $X \subset \mathbb{C}^m$ of degree format $(d_1, \ldots, d_t)$

$$C_{*, t}(X) \leq \frac{1}{2(m!)} \sum_{i=1}^{t} d_i^m (1 + \sigma_{(d_1, \ldots, d_t)}).$$

The lower bounds (4), (6) in Theorem 1 and 2 are therefore asymptotically sharp.

Proof of Proposition 1. The upper bounds in Proposition 1 follow from the obvious bisection algorithm. The lower bound in (8) is a consequence of Ben-Or's result (Ben-Or, 1983). Our proof of statement (7) is based on the following theorem due to Grigoriev (1982) and Risler (1985) (see Benedetti and Risler, 1990):

For all $f \in \mathbb{R}(x)^*$ we have for the additive complexity $k = L_+(f)$ of $f$

$$|\{ \xi \in \mathbb{R} : f(\xi) - 0 \}| \leq (k + 2)^{2k + 1} 2^{2k^2 + 2k + 1}. \quad (12)$$
We now show the lower bound in (7). Let $T$ be a tree deciding membership in $X$. By the first part of the proof of Theorem 2, applied to the finite subset $X \subset \mathbb{R}$, we have from relation (9) for each path $\pi$ of $T$ leading to a yes-leaf

$$D_\pi \subset \bigcup \{ \{ \xi \in \mathbb{R} : g^{(1)}_\nu(\xi) = g^{(2)}_\nu(\xi) \} : \nu \in V_{\text{true}} \cup V_{\text{false}} \}$$

(13)

(with the notation adopted from there). We put $\rho := C_{+,\infty}(T)$. As the right-hand side of (13) is the zeroset of

$$h_\pi := \prod_{\nu \in V_{\text{true}} \cup V_{\text{false}}} (g^{(1)}_\nu - g^{(2)}_\nu)$$

and $L_+(h_\pi) \leq \rho$ we conclude from statement (12) that

$$|D_\pi| \leq (\rho + 2)^{2^p + 1} 2^{2^{p^2} + 2^p + 1}.$$ 

Since there are at most $2^p$ paths $\pi$ we see that

$$|X| \leq 2^p(\rho + 2)^{2^p + 1} 2^{2^{p^2} + 2^p + 1},$$

and a routine calculation shows that

$$\frac{1}{2} \sqrt{\log |X|} \leq \rho = C_{+,\infty}(X)$$

(if $\rho = 1$ one sees directly that $|X| \leq 3$).

Assume now that the elements of $X$ are algebraically independent over $\mathbb{Q}$. Theorem 3 and Lemma 1 imply

$$|D_\pi| - 1 \leq L_+(h_\pi)$$

as $D_\pi \subset \text{Zeroset}(h_\pi)$. If there is a path $\pi$ leading to a yes-leaf such that $|D_\pi| > \log t$ we are done. Otherwise the tree $T$ has at least $t/\log t$ yes-leaves, and there is a path $\pi$ with at least $\log(t/\log t)$ comparisons.

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