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On the Distance to the Zero Set of a Homogeneous Polynomial*

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In this paper we prove simple estimates relating the value of a complex homogeneous polynomial at a point to the distance of the point to the zero set of the polynomial, and also the distance to the zero set along a projective line to the distance in projective space. Our motivation for doing this was to give a quantitative aspect to the algorithm of W. Zulehner and to relate it to the algorithms of J. Renegar, J. Canny, and M. Kim. © 1989 Academic Press, Inc.

Let F: $\mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial of degree m. Let $H \subset S^{2n-1} = F^{-1}(0) \cap S^{2n-1}$, where S^{2n-1} is the unit sphere in $\mathbb{C}^n$. Given a complex 2 plane $\mathbb{C}^2$ through the origin it intersects S^{2n-1} in a 3 sphere, S^3. For any point $x \in S^3$ we have the distance $d_{S^3}(x, H)$ and $d_{S^{2n-1}}(x, H)$ the distances from x to $H \cap S^3$ in S^3 and the distance from x to H in S^{2n-1}. Obviously, $d_{S^3}(x, H) \geq d_{S^{2n-1}}(x, H)$.

THEOREM 1. *Let R denote the furthest distance of a point in S^3 from H, as measured in S^{2n-1}. Then*

$$\pi \csc R \, d_{S^{2n-1}}(x, H)^{1/m} m^{1/m} \geq d_{S^3}(x, H) \geq d_{S^{2n-1}}(x, H).$$

To prove Theorem 1 we use:

PROPOSITION 2. *Let F: $\mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial of degree m. Let $\|F\| = \max_{x \in S^{2n-1}} |F(x)|$. Suppose for x_0 in the unit sphere S^{2n-1}*

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that the distance from x_0 to $F^{-1}(0) \cap S^{2n-1}$ in S^{2n-1} is $R > 0$. Then

$$\|F\| \leq |F(x_0)|(\csc R)^m \text{ or } \sin R \leq \left(\frac{|F(x_0)|}{\|F\|}\right)^{1/m}. \qquad (*)$$

Proof: We may rotate the sphere S^{2n-1} so that the point $x_0 = (1, 0, 0, \ldots, 0) \in \mathbb{C}^n$ and we may assume $|F(x_0)| = 1$ and that $F(x_1, \ldots, x_n) = x_1^m + \Sigma_{i=0}^m a_i(x_2, \ldots, x_m)x_1^i$, where $a_i(x_2, \ldots, x_m)$ is a homogeneous polynomial of degree $m - i$ in the variables $x_2, \ldots, x_{n+1}$. F has no zeros inside the double cone, K, on the sphere of radius $\tan R$ in the plane $x_1 = 1$ centered at $(1, 0, \ldots, 0) \in \mathbb{C}^n$. In fact, since F is homogeneous, any point $(\alpha, 0, \ldots, 0)$ is at distance R from H for $\alpha \in \mathbb{C}$ of norm 1. Thus we may assert that F has no zero in the orbit of the cone K, under the action of the unit complex members, $S^1 \subset \mathbb{C}$, on the first coordinate. The orbit is the cone on $S^1 \times D^{2n-2}$. Here S^i denotes the i-sphere and D^j the j disk. The complementary cone on $D^2 \times S^{2n-3}$ in the $2n - 1$ sphere has D^2 fibers over $\mathbb{C}^{n-1}$, and the radius of the two disk at $\hat{x} \in \mathbb{C}^{n-1}$ is $\|\hat{x}\|/\tan R$.

We write $\mathbb{C}^n$ as $\mathbb{C} \times \mathbb{C}^{n-1}$. With $\hat{x} = (x_2, \ldots, x_n) \in \mathbb{C}^{n-1}$ fixed, the polynomial $F(x_1, \hat{x})$ in x_1 has all its zeros in the disk of radius $\|\hat{x}\|/\tan R$ and since the ith coefficient is a symmetric function of the roots

$$|a_i(\hat{x})| \leq C_i^m\left(\frac{\|\hat{x}\|}{\tan R}\right)^i,$$

where $\|\ \|$ is the Euclidean norm and C_i^m is the ith bionomial coefficient. Thus for $x = (x_1, \hat{x})$ in the unit sphere.

$$|F(x)| \leq \sum_{i=0}^{m} C_i^m \left(\frac{\|\hat{x}\|}{\tan R}\right)^i |x_1|^{m-i} = \left(|x_1| + \frac{\|\hat{x}\|}{\tan R}\right)^m$$

and

$$|F(x)| \leq \left(\frac{(\tan^2 R + 1)^{1/2}}{\tan R}\right)^m$$

by calculus.

Now we return to the proof of Theorem 1. We may assume that the point $(1, 0, \ldots, 0) \in \mathbb{C}^n$ is at distance R from H and that $F((1, 0, \ldots, 0)) = 1$.

By Kellog (1928), (*) implies

$$|DF(x)| \leq m\,(\csc R)^m \tag{**}$$

for x in the unit ball.

Fix $x_0 \in S^3$ and suppose that $x_0 = (z, \hat{x}) \in \mathbb{C} \times \mathbb{C}^{n-1}$ with $\hat{x} \neq 0$.
Let $d_{S^{2n-1}}(x_0, H) = c$; then by (**)

$$|F(x_0)| < cm\,(\csc R)^m.$$

So with $\hat{x}$ fixed there is a z' with

$$|z - z'| \leq c^{1/m} m^{1/m}\,(\csc R)$$

and $F(z', \hat{x}) = 0$.

The distance from $x_0 = (z, \hat{x})$ to $(z', \hat{x})$ is likewise at most $c^{1/m}m^{1/m}$ (csc R). As x_0 is on the unit sphere the projection of $(z', \hat{x})$ to the unit sphere has distance from x_0 magnified at most by π. This finishes the proof for those points $(z, \hat{x})$ with $\hat{x} \neq 0$. The rest follows by continuity.

Theorem 1 has a natural statement in projective space, which follows immediately by employing the S^1 action on S^{2n-1} to define $\mathbb{C}P(n-1)$. The metrics are the unitarily invariant metrics.

THEOREM 2. *Let $H \subset \mathbb{C}P(n)$ be a hypersurface of degree m and L be a projective line. Let R be the furthest distance of a point in L from H. Then for $x \in L$*

$$m^{1/m}\pi(\csc R)d_{\mathbb{C}P(n)}(x, H)^{1/m} \geq d_L(x, H \cap L) \geq d_{\mathbb{C}P(n)}(x, H).$$

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