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THE NEWTONIAN GRAPH OF A COMPLEX POLYNOMIAL*

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Abstract. In a recent paper [4] Smale posed as an important problem in complexity theory, characterization of the graph G_f of the Newtonian vector field N_f for a complex polynomial f. Such graphs are known to be connected and acyclic and Smale conjectured that these two properties completely characterize them. The purpose of this paper is to prove this conjecture, after adding an additional hypothesis (part 3 of the definition of "dynamic graph," § 2). In additon we give an example and prove a proposition to show this is necessary (see "Counterexamples" in § 2).

We present the proof as Theorem C in § 5 using the topological characterization of analytic maps given by Stöilow [5] in 1929. Bill Thurston pointed us in this direction, though considering the fact that G. T. Whyburn was the major professor of one of us, we should not have needed this help. In addition we present direct proofs of three special cases as Theorem A, Theorem B and Example 7. While this was being written an independent proof was given in the generic case (Theorem A) in [2].

Sections 1 and 2 are devoted to basic properties and to a list of examples designed to acquaint the reader (and the writers) with various aspects of Newtonian Graphs.

Key words. polynomial, root, Newtonian vector field, Newtonian Graph

AMS(MOS) subject classifications. 58F, 65H, 68Q

1. The Newtonian vector field. Given a smooth map $f: \mathbf{R}^n \to \mathbf{R}^n$ the Newtonian vector field for f, N_f, is defined by

$$N_f(x) = -(Df_x)^{-1}(f(x)).$$

Let $\phi_t: \mathbf{R}^n \to \mathbf{R}^n$ be the corresponding flow. Then a computation carried out below using the chain rule shows that $f(\phi_t(x)) = e^{-t}f(x)$. Thus f maps orbits of ϕ_t into rays pointing toward 0. This is essentially the geometric content to Newton's method for seeking zeros of f.

Alternatively, one defines

$$V_f(x) = -\tfrac{1}{2}\operatorname{grad} \|f(x)\|^2 = -(Df_x)^{\mathrm{tr}} f(x).$$

For conformal maps, and in particular analytic maps, of one complex variable (by the Cauchy-Riemann equations) the inverse of a matrix and its transpose differ only by a scalar multiple. Therefore, the vector fields V_f and N_f also differ only by a scalar multiple, except where $(Df)^{-1}$ is undefined.

We next collect some facts for f a polynomial of one complex variable:

1. N_f or V_f have the same solution curves as

(a) $-f(z)\bar{f}'(z)$ or

(b) $-\sum (z-a_j)/|z-a_j|^2$, $\{a_j\}$ the zeros of f.

Thus the field is the sum of forces toward a_j, each inversely proportional to the distance from a_j.

To see (a),

$$N_f(z) = -\frac{f(z)}{f'(z)} \cdot \frac{\bar{f}'(z)}{\bar{f}'(z)} = -(1/|f'(z)|^2)f(z)\bar{f}'(z)$$

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so that N_f differs from $-f(z)\bar{f}'(z)$ only by the scalar function $|f'(z)|^{-2}$. Note however that this scalarization converts poles of N_f to zeros of V_f.

(b)
$$-f/f' = -[(\log f)']^{-1} = -\left[\sum \frac{1}{z-a_i}\right]^{-1}$$
$$= -\left[\sum \frac{\overline{z-a_i}}{|z-a_i|^2}\right]^{-1} = -\frac{\bar{A}}{\bar{A}}\left[\sum \frac{\overline{z-a_i}}{|z-a_i|^2}\right]^{-1}$$
$$= \left(-\sum \frac{z-a_i}{|z-a_i|^2}\right)(1/|A|^2),$$

where $A = \sum \overline{z-a_i}/|z-a_i|^2$.

2. Properties of N_f and V_f:

(a) $f(\phi_t(x)) = e^{-t} f(x)$

(b) N_f and V_f have attractors (sinks) at the zeros a_j of f.

(c) The only other rest points of V_f are the zeros θ_j of f':

(i) generic zeros of f' are hyperbolic saddles of V_f;

(ii) at multiple zeros of f' V_f has multipronged saddles ("monkey saddles" and worse);

(iii) For θ a zero of f', the orbits leaving θ, called together the unstable manifold $W^u(\theta)$, consist of n algebraic curves emanating from θ at equal angles because $f(z) = c_1 + c_2(z-\theta)^n +$ higher order terms, $c_2 \neq 0$.

(d) Multiple zeros a of f have no geometric effect on the corresponding sinks. Only the *velocity* of the flow toward $\underline{a}$ is increased.

(e) (Gauss-Lucas Theorem). The convex hull of the sinks $\{a_j\}$ contains the saddles $\{\theta_j\}$. The flow of N_f or V_f is inwardly transverse to any convex curve containing the $\{a_j\}$.

Proof of 2. To prove (a) we compute

$$\frac{d}{dt} f(\phi_t(x)) = Df(\phi_t(x)) \cdot \frac{d}{dt}\phi_t(x)$$
$$= Df(\phi_t(x)) \cdot (-Df(\phi_t(x)))^{-1} \cdot f(\phi_t(x))$$
$$= -f(\phi_t(x)).$$

Thus $f(\phi_t(x)) = \rho_t$ where

$$\frac{d\rho}{dt} = -\rho \quad \text{at } t=0, \quad \rho = f(x)$$

which has $e^{-t} f(x)$ as solutions.

Parts (b) and (d) follow from the "attracting force" version of N_f, $-\sum(z-a_j)/|z-a_j|^2$. Similarly, if all the a_j are on the side of a line, the vector field is transverse to this line, which proves the second part of (e). The first part follows from the second part.

The form $-f(z)\bar{f}'(z)$ shows that the zeros of f and f' are the only rest points. Part (a) implies (c)(ii)-(c)(iii); the inverse image of a directed line under the map $z \to c_1 + c_2(z-\theta)^n +$ h.o.t. consists of $2n$ directed curves pointing alternatively toward and away from θ, with tangents evenly spaced at θ.

In fact, property 2(a) is proven for general $C^1 f$ and 2(b) is true for simple zeros of $C^1 f$ since at such points the derivative of N_f is $-I$.

When $f'(\theta)=0$, $DV_f(\theta)(w)=-\overline{f''(\theta)w}f(\theta)$. If $f(\theta)\neq 0$ and $f''(\theta)\neq 0$ then the 2×2 real matrix representing this linear transformation has strictly negative determinant and trace 0. Thus the eigenvalues of $DV_f(\theta)$ are real and have opposite sign.

We have borrowed here from [3] and [1] where a desingularization of Newton's method is discussed in more variables as well.

2. Newtonian graphs and special terminology. Let f be a complex polynomial with $\{a_j\}$ its zeros and $\{\theta_k\}$ the zeros of f' which are not zeros of f. Let $V_f=-f(z)\overline{f'(z)}$ be the gradient vector field of f as in § 1. Let $W^u(\theta_k)$ be the "unstable manifold" of θ_j, i.e., the union of all solutions which limit on θ_k as $t\to-\infty$. Note that they in turn limit on some a_j or some other θ_k, as $t\to+\infty$. Define

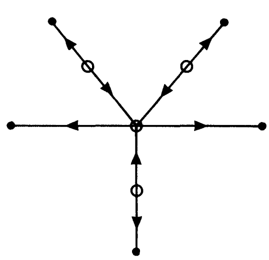
$$G_f=\{a_j\}\cup\{\theta_j\}\cup W^u(\theta_j).$$

Then G_f is a finite graph with distinguished vertices a_j, θ_k and directed "edges" $W^u(\theta_k)$.

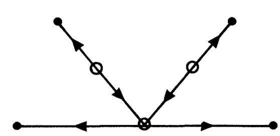
For nonrepeated zeros θ_k of f', $W^u(\theta_k)\cup\{\theta_k\}$ is a smooth curve but in degenerated cases $W^u(\theta_k)$ consists of 3 or more prongs.

The sinks a_j of G_f have weights α_j where α_j is the multiplicity of a_j.

Counterexamples. CE1 below is not homeomorphic to any Newtonian graph G_f, f a polynomial. CE2 is not isotopic to any such G_f.



CE1



CE2

These facts follow from the general proposition.

PROPOSITION. *For f any complex polynomial and v a vertex of G_f at most one incoming edge can lie between any two outgoing edges.*

Proof. Suppose the contrary. Then choose a small circle J around v and note that it passes through points A, B, C, D where

(i) $A<B<C<D$ on J;

(ii) there is no other point of $J\cap$ (outgoing edge) between A and D on J;

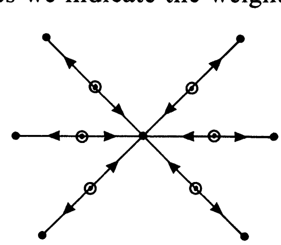
(iii) A and D lie on rays beginning at v;

(iv) B and C lie on rays terminating at v.

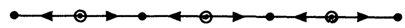
Then f has a singularity at v (of index $\geqq 2$) and f maps the open arc (B,C) of J completely around $f(v)$ since it intersects the ray $\{mf(v)\mid m \text{ real } m>1\}$ in the two points $f(B), f(C)$. $f\mid(B,C)$ does not intersect the ray $\{mf(v)\mid m \text{ real}, 0<m<1\}$ which is a contradiction.

Examples. In our sketches we indicate the weights only if different from 1.

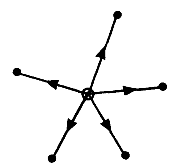
1. z^n-z $\qquad n=7$



2. Real roots

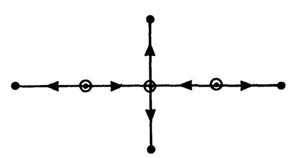


3. $z^d - 1$ $\qquad$ $d = 5$

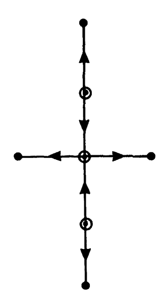


multiple saddle or
unstable star with 5 prongs.

4. $(z^2 + a^2)(z^2 - b^2)$ $\qquad$ saddle connections



$a < b$

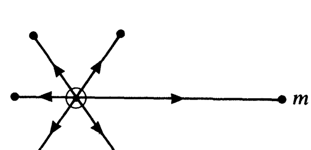


$a > b$

5. $z^n(z-c)^m$

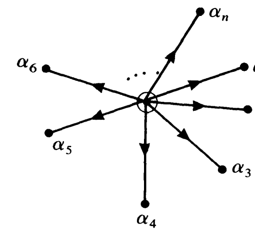
sinks of weights n and m.

6. There is a polynomial P, monic and of degree n such that $((z-c)^m P)' = (n+m)(z-c)^{m-1}z^n$ m, n positive integers. (This is proved in Lemma 4.1; we sketch the graph for such a P, where $n = 5$ next.)



$n = 5$

7. Given positive integers $\alpha_1, \cdots, \alpha_n$ there is a polynomial of degree $\alpha_1 + \cdots + \alpha_n$ with graph

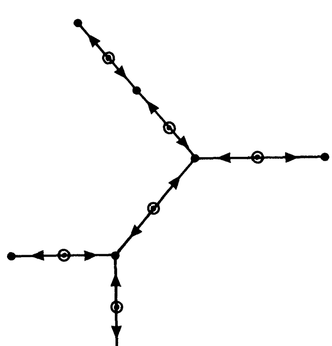


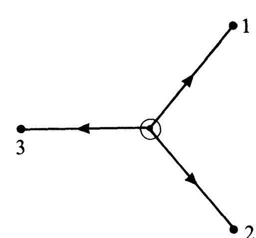
unstable star with weights

Remark. For any complex polynomial f, the graph G_f is connected and acyclic.

Proof. A cycle in the graph, G_f, would bound a finite region in the plane, but this contradicts 2(a). The flow is a gradient flow with ∞ as the only source. Thus the plane is the union of G_f and $W^u(\infty)$, the unstable manifold of ∞. Thus G_f has the homotopy type of the plane (or a point) and so G_f is connected.

Remark. Any connected acyclic finite graph can be embedded in the plane, and sometimes in distinct-*nonisotopic* ways. Embeddings $f: G \to \mathbf{R}^2$ and $g: G \to \mathbf{R}^2$ are *isotopic* provided there is a continuous 1-parameter family f_t, $0 \leqq t \leqq 1$ such that $f_0 = f$, $f_1 = g$ and f_t is an embedding of G for each t, $0 \leqq t \leqq 1$. This last property, that each f_t be an embedding, distinguishes isotopy from homotopy; were it to be dropped, one could reverse the orientation of the next two examples, by pushing one of the legs through another one. The following two examples have two isotopy classes of embedding determined by the cyclic order at the circled vertices.





PROPOSITION. *Generically V_f is structurally stable, having*

(a) *hyperbolic saddles,*

(b) *no saddle connections, and*

(c) *all weights* $= 1$.

Proof. Generically f and f' have no repeated zeros which proves (a) and (c). Any one saddle connection can be removed by an arbitrarily small perturbation of one of the vertices involved, noting for example that a saddle connection implies that two critical values lie on the same ray. Thus proceeding one at a time, one can remove all saddle connections by a perturbation so small as not to effect those already broken. Structural stability now follows from Peixoto's theorem (see Palis-de Melo [8]).

In order for an abstract finite acyclic graph to be the Newtonian graph of a complex polynomial, it must have certain properties best described in dynamical terms.

By a *dynamic graph* we mean a finite directed graph with two types of vertices, which we call *saddles* and *sinks* subject to the following conditions:

1) At a sink, all edges are directed inward (i.e., toward the sink).

2) Saddles have at least two outwardly directed edges.

3) At a saddle, any two adjacent outwardly directed edges have at most one inwardly directed edge between them. (Each such edge must then connect two saddles, and is thus called a *saddle connection.*)

4) Each sink has a weight which is a positive integer, often 1. The weights have no effect on the geometry of a dynamic graph.

It follows that a dynamic graph falls into natural units each consisting of a saddle together with all edges directed away from it. These will be called *unstable stars* or *k-prongs* where k is the number of issuing edges, $k = 2, 3, \cdots$. A 2-prong is also called a *hyperbolic saddle.*

3. The generic case. Our first theorem is a special case of each of the two others, but its proof is easy. In addition, while this was being written, other authors [2] have found a proof independently of this part of our results.

THEOREM A. *Given an acyclic dynamic graph with all saddles hyperbolic, no saddle connections and all weights* $= 1$, *there is a complex polynomial* f *whose graph* G_f *is isotopic to* G.

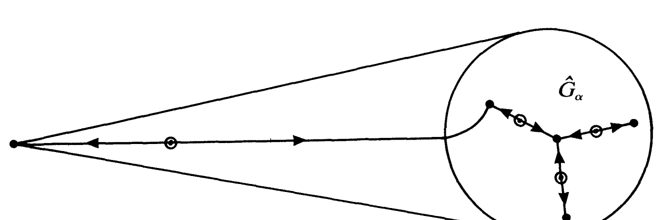
Proof. First, there exist sinks $v_0, v_1, \cdots, v_m \in G$, and subgraphs $G_\alpha \subset G$ such that for each α, $v_\alpha \in G_\alpha$ and $G_{\alpha+1}$ consists of G_α together with one additional hyperbolic saddle having v_α as one of its 2 sinks.

We proceed by induction on α. The graph $\{v_0\}$ is realized by the polynomial z. Thus assume we have a generic polynomial P_α such that the graph $\hat{G}_\alpha$ of P_α is isotopic to G_α. Let D be a (round) disk containing the zeros of P_α. Then $D \supset \hat{G}_\alpha$ and the field V_{P_α} is inwardly transverse to the boundary ∂D of D.

Let $f(z) = P_\alpha(\lambda(z-c))$ where $|\lambda| = 1$ and c is real. Note that the graph of f is conjugate to $\hat{G}_\alpha$ by a linear conjugacy: $f'(z) = \lambda P'_\alpha(\lambda(z-c))$ so that $V_f(z) = -\bar{\lambda} P_\alpha(\lambda(z-c))\bar{P}'_\alpha(\lambda(z-c)) = \bar{\lambda} V_{P_\alpha}(\lambda(z-c))$ which gives $V_f(z) = \lambda^{-1} V_{P_\alpha}(\lambda(z-c))$.

Let $g(z) = zf(z)$. We claim that for c sufficiently large and certain λ, $|\lambda| = 1$, the graph of g is isotopic to $G_{\alpha+1}$. In fact the field V_g on D differs from V_f only by the summand $-1/z$ and this is essentially constant $= -1/c$. Thus for $c \gg 1$, the part of the graph of g related to the zeros and saddles of f is isotopic to $\hat{G}_\alpha$ by structural stability.

Note that the "ice-cream cone" region sketched below contains the zeros of g and hence the dynamics of G_g. It follows that the graph G_g of g consists of $\hat{G}_\alpha$ together with a single edge added and that for c large, the saddle of this new edge is outside D.



Now as λ varies in the unit circle, the disk D and the dynamics of V_{P_α} rotates through a full circle as well, by structural stability. Thus our new edge arrives at any one of the entering orbits, for the appropriate choice of λ. This is important below, but here we have more leeway, because there is an open set of orbits limiting on the sink v_α, and have the correct deployment with respect to $\hat{G}_\alpha$. Thus for some choice of λ the graph G_g of g is isotopic to $G_{\alpha+1}$.

This completes the inductive step and the proof of Theorem A.

4. Nongeneric saddles and Theorem B. The purpose of this section is to prove Thoerem B. This uses the fact that our example 6 is the Newtonian graph of a complex polynomial which we prove as Lemma 4.1. Example 6 is used in the proof of Theorem B (implicitly) and in example 7, below.

LEMMA 4.1. *Given integers m, n and $c \in \mathbf{C}$, there exists a monic polynomial P of degree $n+m$ such that $P(0) \neq 0$ for $c \neq 0$ and $((z-c)^m P(z))' = (n+m)z^n(z-c)^{m-1}$.*

Proof. This linear ODE reduces to $(z-c)P' + mP = (n+m)z^n$. We try a solution of the form $P = z^n + \sum_{i=0}^{n-1} b_i z^i$ and note that the coefficient of z^n is $n+m$ on both sides. Proceeding downward, for $j = n-1, n-2, \cdots, 0$ one has the formulas $jb_j - c(j+1)b_{j+1} + mb_j = 0$ for the coefficient of z^j. Thus

$$b_j = \frac{c(j+1)}{m+j} b_{j+1}$$

gives our solution, which terminates with b_0 as $b_{-1} = 0$.

THEOREM B. *Given an oriented acyclic embedded dynamic graph G with no saddle connections and all weights* 1, *there exists a polynomial f with graph G_f isotopic to G.*

Proof. We proceed as before with v_α, $G_\alpha \subset G$ by induction on α. Here, however, $G_{\alpha+1}$ is G_α with a *multiple saddle* attached at $v_\alpha \in G_\alpha$. Suppose then that P_α is a polynomial yielding the graph G_α and that D is a disk around 0 containing all G_α (as above) and w, the field of P_α transverse to ∂D. Let $f(z) = P_\alpha(\lambda(z-c))$. For simplicity we assume 0 is a zero of P_α and set $f(z) = P_\alpha(\lambda(z-c))$.

Now $G_{\alpha+1}$ is G_α with a saddle edge added at the vertex v_α. Say the new unstable star has $(n+1)$ exiting edges, for some $n \geqq 1$. Then we want our next polynomial g to have derivative $z^n f'(z)$, or $g(z) = \int_c^z w^n f'(w)\,dw$, some $c \in \mathbf{C}$ which we choose to be real. Here we use the same c as that in $f(z) = P_\alpha(\lambda(z-c))$. Then integrating by parts,

$$g(z) = z^n f(z) - n\int_c^z w^{n-1} f(w)\,dw,$$

as $f(c) = 0$.

Then the gradient field for g is given by

$$V_g = -|z|^{2n} f(z)\bar{f}'(z) + n\bar{z}^n \bar{f}'(z)\int_c^z w^{n-1} f(w)\,dw.$$

Now near c we scalarize to $V_g/|z|^{2n}$ which gives

$$-f(z)\bar{f}'(z) + \frac{n\bar{f}'(z)}{z^n}\int_c^z w^{n-1} f(w)\,dw,$$

so that we have our given field with an error term,

$$E = \frac{n\bar{f}'(z)}{z^n}\int_c^z w^{n-1} f(w)\,dw.$$

Then one can estimate E on D_c by

$$|E| \leqq \frac{n(c+\delta)^{n-1}|f'(z)f(\zeta)|\delta}{(c-\delta)^n} \quad \text{some } z, \zeta \in D_c$$

which goes to 0 for large c, where $\delta =$ diameter of D.

Next to check the C^1 part of the error, we note that E is differentiable as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$, as conjugation is real analytic. Hence

$$E' = \frac{nz^{n-1}\bar{f}'(z)f(z)}{z^n} + \frac{n\bar{f}''(w)\int_c^z w^{n-1}f(w)\,dw}{z^n} - \frac{n^2\bar{f}'(z)\int_c^z w^{n-1}f(w)\,dw}{z^{n+1}}.$$

These clearly go to zero on D_c as $c \to \infty$. Thus the field of g on the disk is near that $f(\lambda(c-z))$ and has exactly the same saddles. It follows that for c large enough, the graph G_g is isotopic to G_α with an unstable star of $n+1$ prongs added.

But just as in the proof of Theorem A, we can choose λ so that this last edge is added in the correct "angle."

This completes the proof of Theorem B.

PROPOSITION (*Example* 7). *For any (integral) weights $\alpha_1, \cdots, \alpha_k$ there is an unstable star with these weights. Equivalently, there is a solution $a_1, a_2, \cdots, a_k$, $a_i \neq 0$ of the equation*

$$\left(\prod_{n=1}^{k}(z-a_n)^{\alpha_n}\right)' = (\alpha_1 + \cdots + \alpha_k)z^{k-1}\prod_{n=1}^{k}(z-a_n)^{\alpha_n - 1}.$$

Furthermore, any cyclic ordering of the weights (see the above remark) can be realized.

Proof. This ODE leads to an unstable star with weight α_i at a_i, provided it has a solution with the a_i distinct. To solve it is equivalent to solving a set of $k-1$ equations in k unknowns, which we augment by one equation.

$$(j)\ \sum_{n=1}^{k}\alpha_n \sum_{\substack{i_1<i_2<\cdots<i_j \\ i_\alpha \neq i_n}} a_{i_1}a_{i_2}\cdots a_{i_j} = 0, \qquad j = 1, 2, \cdots, k-1;$$

$$a_k - 1 = 0.$$

Now the function $F_\alpha : \mathbf{C}^k \to \mathbf{C}^k$ defined by the left-hand sides, where $\alpha = \alpha_1, \cdots, \alpha_k$ is a proper map and has a positive Jacobian (see [1, p. 294]). Thus F_α can be extended to the $2k$-sphere $\mathbf{C}^k \cup \{\infty\}$ so that it has a solution $a_1, \cdots, a_k$.

In fact, by counting degrees we see that there are $(k-1)!$ solutions. To see that these contain all of the isotopy classes we need a homotopy argument. First, we may as well suppose that the α_n's are numbered in their desired cyclic order. Next let $\alpha(t)$ be a path, $0 \leqq t \leqq 1$, where

$$\alpha(0) = (1, 1, \cdots, 1) \quad \text{and}$$

$$\alpha(1) = (\alpha_1, \alpha_2, \cdots, \alpha_k) \quad \text{and}$$

$$\alpha_i(t) \neq \alpha_j(t) \quad \text{for } i \neq j \text{ and } 0 < t < 1.$$

Now by Lemma 4.1, above, there is a singular graph of weights $1, 1, \cdots, 1$, that is, the regular unstable star of k petals, and thus there is a solution $a_n(0)$ $n = 1, \cdots, k$ of the equation $F_{\alpha(0)} = 0$. Then $F_{\alpha(t)}$ is an analytic isotopy of this algebraic function, so there is a unique arc of solutions $\{a_n(t)\}$ to the equations $F_{\alpha(t)} = 0$. This gives a 1-parameter path of functions $f_t(z) = \prod_1^k (z-a_n)^{a_n}$, and Newtonian graphs G_t, whose end G_1 has its weights in the correct order as G_t is an isotopy of graphs. The weights $\{\alpha_n(t)\}$ do vary with t, of course, but G_t is an isotopy of the graphs if we disregard weights. One could also interpret G_t as an isotopy of weighted graphs. The $f_t(z)$ are proper and their Newtonian graphs are acyclic, connected, etc.

5. The general case via Stöilow's theorem.

THEOREM C. *Given an acyclic dynamic graph $G \subset \mathbf{R}^2$ there is a polynomial f such that the Newton graph G_f is isotopic to f.*

DEFINITIONS (see Whyburn [7]). A map $f:X\to Y$ is *light* if $f^{-1}(y)$ is finite (or totally disconnected; for surfaces X, Y it comes to the same thing) for each $y\in Y$ and *open* provided $f(u)$ is open for each open set $U\subset X$. In these terms there is the classical (1929) result as follows.

STÖILOW'S THEOREM [7, p. 103]. *If $f:M^2\to \mathbf{C}$ is a light open map from the surface $M(\partial M=\varnothing)$ to the complex plane then there is an analytic function $\phi:R\to\mathbf{C}$, R a Riemann surface and a homeomorphism $h:R\to M^2$ such that $\phi=f\circ h$.*

We next outline the proof of Theorem C. We construct a light open map $f:\mathbf{R}^2\to\mathbf{C}$ such that

(a) f is zero only at the sinks $v\in G$ and has degree m at v, m the weight of the sink v.

(b) The degree of f at a saddle θ is $k=k(\theta)$ where θ is a k-prong in G.

(c) f has no points of degree >1 except as in (a) and (b).

(d) The f-image of each directed edge is a (straight) ray pointed toward the origin in $\mathbf{C}$.

(e) f is proper.

Now applying Stöilow's theorem we obtain an analytic map $\phi:R\to\mathbf{C}$, and a homeomorphism $h:R\to\mathbf{R}^2$. Now R must be $\mathbf{C}$ or the interior of a disk in $\mathbf{C}$. But the latter case is ruled out as ϕ is proper. Then ϕ is a polynomial since it is entire and has poles only at ∞. But G_ϕ is just $h^{-1}(G)$ by our construction. That is, the zeros occur only at the sinks of $h^{-1}(G)$ and the other singular points are $\phi(h^{-1}(\theta))$, θ a saddle of G. Finally the solution curves of V_ϕ being those curves which map onto rays of $\mathbf{C}$ pointing toward the origin, include the directed edges of $h^{-1}(G)$.

This completes the proof of Theorem C; it remains only to construct the light open map f satisfying (a)-(e).

Construction of f. The construction will use induction on the number of saddle points in G.

Suppose there are no saddle points in the graph G. Then G consists of one root of weight m. In this case $G_f=G$ for $f(z)=z^m$.

We make the induction hypothesis that such f exist is true for dynamic graphs with less than or equal to n saddle points.

An equivalent form of the induction hypothesis which will be convenient is the following. Let G be a dynamic graph with less than or equal to n saddle points. For any disc D containing G there exists a light open map f such that (1) $f(D)=D_1=\{z\mid |z|\leqq 1\}$, (2) f is a covering map from the boundary of D onto the boundary of D_1 and (3) f satisfies (a), (b), (c), (d) relative to G.

In order to see that the second form of the induction hypothesis follows from the first, observe that a large enough disc $\{z\mid |z|\leqq R\}$ in the range of f has for preimage a disc containing D. Adjusting f by an isotropy in the domain and by a radial isotopy in the range gives $f:D\to D_1$ with the desired G_f.

Suppose we have a dynamic graph G which is connected and acyclic, with $n+1$ saddle points. The points of G are partially ordered by the directed edges. Choose a saddle point θ at which no edge terminates. Let $\{\gamma_i\}$, $1\leqq i\leqq k$ be the edges emanating from θ. For each γ_i, $G-\{\text{interior of }\gamma_i\}$ is the union of two dynamic graphs. Let G_i be the component of $G-\{\text{interior of }\gamma_i\}$ not containing θ. Let $\{D_i\}$, $1\leqq i\leqq k$, be a family of pairwise disjoint discs such that D_i contains G_i, $\partial D_i\cap\gamma_i$ is a single point p_i in the interior of γ_i, and $D_i\cap\gamma_j=\phi$ for $i\neq j$. Such discs D_i can be found by taking small neighborhoods of the G_i. Since G_i is connected and acyclic, by the induction hypothesis there is a light open map $f_i:D_i\to D_1$ such that $G_i=G_{f_i}$.

Denote by q_i the terminal end of γ_i.

LEMMA. *We can alter γ_i by an isotopy so that $f(q_i) = \lambda_i \cdot f(p_i)$, for some real number $\lambda_i \geqq 0$, and so that the part of γ_i from p_i to q_i is mapped by f_i onto the radial segment from $f_i(p_i)$ to $f_i(q_i)$. Furthermore, we can assume that $f_i(p_i) = 1$ for $1 \leqq i \leqq k$.*

Proof of Lemma. Let $\{\theta_j^i\}$ be the saddle points of f_i. Then $f_i(G_i)$ is the star formed by the union of the radial segments from $f(\theta_j^i)$ to 0. We consider separately two cases: (1) q_i is a saddle of G_i and (2) q_i is a sink of G_i. In case 1, there is a unique radial segment J from a point of the circle $\{z \mid |z| = 1\}$ to $f(q_i)$. Let I be the union of all curves in $f_i^{-1}(J)$ that terminate at the saddle point q_i. There is one and only one such curve between each pair of successive edges emanating from q_i. If J contains some $f(\theta_j^i)$ then one of the curves of I will contain an edge of G_i which terminates at q_i. Suppose γ_i arrives at q_i between the two successive outward edges e_0, e_1. By property 3 of the definition of a saddle connection for the dynamic graph G, no edge of G_i can arrive at q_i between e_0 and e_1. Let $I_{0,1}$ be the curve in I that arrives between e_0 and e_1. Then $I_{0,1}$ does not contain an edge of G_i. Since G_i is connected and acyclic γ_i can be isotoped (that is, G can be isotoped, fixing $G - \gamma_i$) so that γ_i agrees with $I_{0,1}$ in D_i.

In case 2, γ_i arrives at the sink q_i and $f_i(\gamma_i \cap D_i)$ is a topological line segment proceeding from 0 to some point $p_i' \in \partial D$. Of course the map f_i, which we know exists, is only weakly related to the arc γ_i since γ_i is *not* a part of the graph G_i. We construct a more appropriate arc as follows. Choose a point p_i' near q_i in the correct "angle" or cone at q_i. Then $f(p_i') \neq 0$; let J be the line interval from 0 to $f(p_i')$. Then there is a unique lifting of J to an arc I_γ joining p_i' to q_i. This arc lifting property is essentially trivial to understand since we know f_i is a branched covering.

Let R_i be the rotation of $\mathbf{C}$ centered at $z = 0$ such that $R_i(f(p_i)) = 1$. Define $\bar{f}_i = R_i \circ G_i$. Note that $G_{\bar{f}_i} = G_f$ because R_i preserves radial lines. This completes the proof of the lemma.

We now have the situation pictured in Fig. 5.1, which we can think of as a map f defined on the union of the disks $D_1, \cdots, D_i$. Thus it is a simple matter to define a star-shaped disk D_δ^* bounded by the dotted lines in Fig. 5.1 and small arcs on the disks $D_1, \cdots, D_i$. This disk D^* is in turn mapped into the disk D' bounded by the dotted line $A = F(A_i)$ and a small arc of D, by a covering map, branched at $F(\theta)$ (see Fig. 5.2). The resulting map F is a light open map having the appropriate properties except that it is defined only on a compact disk $D_0 = D^* \cup D_1 \cup \cdots \cup D_i$. But since $F(D_0) = D \cup D'$ is a covering on the boundary, it is a simple matter to extend it to the whole plane by a covering map. This completes the induction and thus the proof of Theorem C.

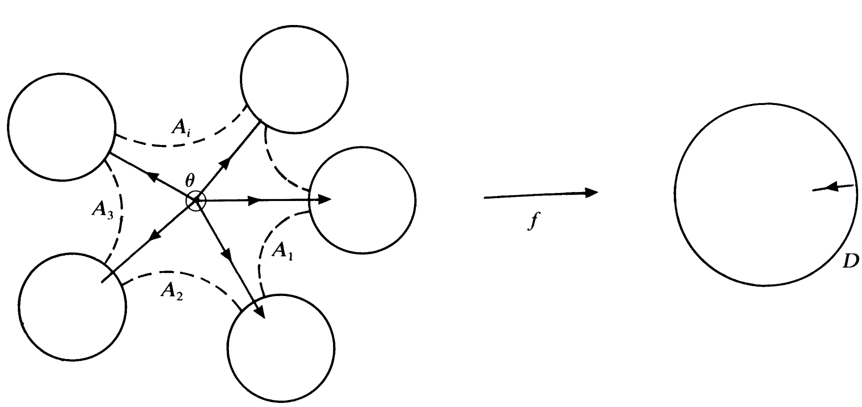


FIG. 5.1

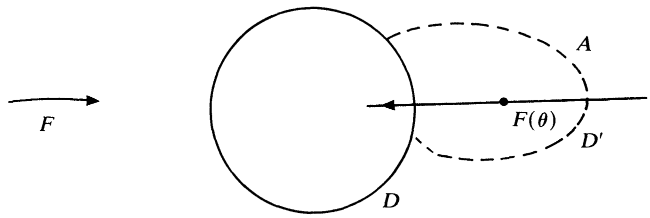


FIG. 5.2

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