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C^r STABILITY OF PERIODIC SOLUTIONS AND SOLUTION SCHEMES

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The structural stability of a periodic solution γ of an ordinary differential equation (O.D.E.) whose Poincare map has no eigenvalue of modulus one is well known. Less well known is the fact that γ persists as an invariant embedded circle for small discrete perturbations of the time- τ map of the flow, ϕ_τ, $\tau \neq 0$. And yet less well known is the manner in which γ depends smoothly on the perturbation. Issues like these arise naturally in the study of solution schemes for O.D.E.'s. See Braun and Hershenov (1977), Shub (1984), or Eirola (1987). To be precise, let

$$\frac{dx}{dt} = X(x)$$

be an autonomous O.D.E. defined on an open subset U_0 of R^m, or for that manner on a manifold. Assume that X is of class $C^r, 1 \leq r \leq \infty$, and let ϕ be the X-flow. That is, $t \rightarrow \phi_t(x)$ is the solution to the O.D.E. with the initial condition $\phi_0(x) = x$. As a function of (t, x), ϕ is of class $C^{r+1,r}$ in the sense that its partial derivatives respecting t and x exist and are continuous (jointly) up to order $r+1$ and r respectively. The domain of definition ϕ is an open subset of $R \times U_0$.

We want to speak of the curve of time-t maps $t \rightarrow \phi_t$. To do so all the maps ϕ_t should be defined on the same domain. We can assure this by assuming that U is an open subset of U_0. U has compact closure in U_0, and t is small enough that $\phi_t|_U$ is an embedding $U \rightarrow R^m$. Alternatively, we could assume that X has compact support contained in some open $U \subset U_0$. In either case,

$$t \rightarrow \phi_t|_U$$

is a curve in $Emb^r(U, R^m)$, the space of C^r embeddings of U into R^m. It is a special kind of curve because of the group property of a flow, $\phi_{t+s} = \phi_t \circ \phi_s$. The space of embeddings is a subspace of the bounded C^r mappings $U \rightarrow R^m$ and it carries the C^r topology induced by the C^r norm

$$\|f\|_r = \sup|f(x)| + \sup\|(Df)_x\| + \cdots + \sup\|(D^r f)_x\|,$$

where x ranges over U. More generally, a curve in $Emb^r(U, R^m), t \rightarrow E_t$, is called a C^r compositional solution scheme for X if E_0 is the inclusion of U into R^m and

$$\|\phi_t - (E_{t/k})^k\|_r \rightarrow 0 \text{ as } k \rightarrow \infty.$$

(Here and below we take $r < \infty$. If X is C^∞ then we say E_t is C^∞ when it is C^r for all r.) A curve of embeddings $t \rightarrow E_t$ is said to be a C^r tangential solution scheme provided that E_0 is the inclusion and

$$\|\phi_t - E_t\|_r \leq ct^2 \text{ as } t \rightarrow 0,$$

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c being some constant. Examples of tangential solution schemes are the k^{th} order Taylor methods for C^{2k} vector fields, $0 < k < \infty$. This is fairly easily seen because the partials commute. See Shub (1984). The k^{th} order Taylor methods are also compositional solution schemes for C^{k+1} vector fields.

The relation between the two types of solution schemes is this. A C^r tangential solution scheme for a C^{r+1} O.D.E. is a compositional solution scheme. The extra differentiability assumption (C^{r+1} O.D.E.) is annoying, but we believe it is necessary. We view either type of solution scheme $t \to E_t$ as a discrete perturbation of ϕ_t and that will be our application of the following theorem.

THEOREM 1.. *Let γ be a periodic orbit of the C^r O.D.E. $\dot{x} = X(x)$ and suppose that γ is hyperbolic – no eigenvalue of the Poincare map around γ has modulus one. Fix a time $\tau \neq 0$ of the X-flow ϕ. Then there exists a neighborhood ν^r of ϕ_τ in $Emb^r(U, R^m)$ and a C^r map*

$$\Gamma : \nu^r \times \gamma \to R^m$$

such that $\Gamma(\psi, \cdot) : \gamma \to R^m$ is a C^r diffeomorphism of γ onto an embedded circle β. It is ψ- invariant in the sense that $\psi(\beta) = \beta$. Also, $\Gamma(\psi, x) \equiv x$ if $\psi = \phi_\tau$. Finally, as a function of $\tau \neq 0$, the neighborhood ν^r contains a ball of radius $\geq c|\tau|$ at ϕ_τ in the C^r metric, c being some positive constant.

In fact β is the unique embedded ψ-variant circle that lies near γ in the C^1 sense. Smoothness of Γ with respect to ψ is what is novel. See also Shub (1987), p. 70, exercise 4, for a similar idea in the case of stable manifolds. Note that we do not assert that $\psi \to \Gamma(\psi)$ is a C^r map $\nu^r \to C^r(\gamma, R^m)$. This would be too much. The topology of $C^r(\gamma, R^m)$ is too strong. Rather, it is the evaluated joint map $(\psi, x) \to \Gamma(\psi, x)$ that is C^r. However, it does follow from standard Banach calculus methods that if we drop the target differentiability from r to $r - j$ then

$$\psi \to \Gamma(\psi, \cdot) \qquad \nu^r \to Emb^{r-j}(\gamma, R^m)$$

is $C^j, 0 \leq j \leq r$.

Corollary 1. If E_t is a C^r tangential solution scheme for X then for each small $\tau \neq 0$ there is a unique E_τ-invariant embedded circle β_τ near γ, and there is a C^r map $(\tau, x) \to b(\tau, x)$ sending γC^r diffeomorphically onto β_τ

PROOF. *Define $b(\tau, x) = \Gamma(E_\tau, x)$. Since $\|\phi_\tau - E_\tau\|_r \leq c\tau^2$ for small τ, we see that E_τ lies in the neighborhood where Γ is defined.* *QED*

Corollary 2. Let E_t be a C^r compositional solution scheme for X and let $\tau \neq 0$ be given. If k is large then $(E_{\tau/k})^k$ has a unique invariant embedded circle $\beta_{\tau,k}$ near γ and there is a C^r map $(\tau, k, x) \to b(\tau, k, x)$ sending γC^r diffeomorphically onto $\beta_{\tau,k}$ as $k \to \infty, b(\tau, k, \cdot)$ tends to the inclusion of γ into R^m in the C^r sense. Lowering the target differentiability form r to $r - 1$ we conclude that the C^1 distance from β to γ is no more than a constant times $\|\phi_\tau - (E_{\tau/k})^k\|_r$

PROOF. DEFINE. $b(\tau, k, x) = \Gamma\left((E_{\tau/k})^k, x\right)$. *QED*

Theorem 1 is a special case of a more general result where the periodic orbit γ is replaced by a boundaryless compact sub-manifold V of a manifold M. (For instance, V could be the 2-torus in R^3.) Let U be a compact neighborhood of V and suppose that $f : U \to M$ is a C^r embedding which leaves V invariant in the sense that $f(V) = V$. Additionally, we assume that if r-normally hyperbolic, which means the following. The tangent bundle of M along V splits as the sum of three bundles

$$T_V M = E^u \oplus TV \oplus E^s.$$

They are invariant under the derivative of f, $Tf(E^u) = E^u$, $Tf(TV) = TV$, and $Tf(E^s) = E^s$. There are constants $\lambda, \lambda^*, \mu, \mu^*$ such that $0 < \mu < \mu^* \leq 1 \leq \lambda^* < \lambda$ and

$$\lambda \|w\| \leq \|Tf(w)\| \qquad \text{if } w \; \varepsilon E^u$$

$$\mu^* \|w\| \leq \|Tf(w)\| \leq \lambda^* \|W\| \qquad \text{if } w \; \varepsilon TV$$

$$\|Tf(w)\| \leq \mu \|w\| \qquad \text{if } w \; \varepsilon E^s$$

$$\mu < (\mu^*)^r \qquad (\lambda^*)^r < \lambda.$$

The behavior of f is the normal direction to V r-dominates the behavior of f along V. The norm is some continuous norm on the tangent bundle of M, "adapted" to f. See Hirsch, Pugh, Shub (1977), p. 3. Using an unadapted norm would only necessitate replacing f with a power f^k. We note that a hyperbolic periodic orbit γ is r-normally hyperbolic for $f = \phi_\pi$, π being the period of γ. In fact, f restricted to γ is the identity map, so the constants μ^*, λ^* can be taken to be 1 in this case.

THEOREM 2.. *Let V be an r-normally hyperbolic invariant manifold for the embedding $f \varepsilon Emb^r(U, M)$. Then there is a C^r neighborhood ν^r of f and a C^r map*

$$\nu^r \times V \to M$$

$$(g, x) \to \sigma_g(x)$$

such that $\sigma_g(V)$ is a g-invariant r-normally hyperbolic submainfold and σ_g sends VC^r diffeomorphically onto $\sigma_g(V)$. If $g = f$ then σ_g is the inclusion of V into M. Moreover, if f lies on a one-parameter subgroup ϕ_t which is tangent to V (and therefore $\phi_t(V) \equiv V$), then each ϕ_t, $t \neq 0$, is r-normally hyperbolic at V and the neighborhood ν_t^r may be chosen to contain a ball of radius $c|t|$ at ϕ_t, c being a positive constant.

sketch of the proof. Let N be a C^r tubular neighborhood of V. The map σ_g will be a section of $N, \sigma_g : V \to N$. It is produced by the usual invariant manifold theory and is the unique section of N that is g-invariant. See Hirsch, Pugh, Shub (1977), p. 39. (In fact it is not necessary that f be injective on U so long as it is a diffeomorphism when restricted to V). To prove that $(g, x) \to \sigma_g(x)$ is jointly C^r we consider a new map. Let $E = Emb^r(U, M)$ and define

$$F : E \times U \to E \times M$$

$$(g, x) \to (g, g(x)).$$

F is C^r and leaves invariant the compact submainfold $W = f \times V = \{(f, x) : x \in V\}$. We are going to apply Center Manifold Theory to F where W plays the role of the fixed point. We think of the bundles E^u, TV, and E^s as defined over W by

$$E^u_{f,x} = 0 \oplus E^u_x \qquad T_{f,x}V = 0 \oplus T_xV \qquad E^s_{f,x} = 0 \oplus E^s_x.$$

They are TF-invariant. The "0" indicates the component in the E direction. We claim that there exists a bundle E^c over W such that

$$T_W(E \times U) = E^u \oplus E^c \oplus E^s,$$

E^c contains TV, and TF restricted to $E^u \oplus E^s$ r-dominates TF restricted to E^c. To find E^c we use § 2 of Hirsch, Pugh, Shub (1977). A bundle map acts naturally on the space of bounded sections of the bundle, so $TF : T_W(E \times U) \to T_W(E \times U)$ induces $(TF)_\# : \sec(T_W(E\times U)) \to \sec(T_W(E\times U))$. Corresponding to the splitting $T_W(E\times U) = T_W(E) \oplus E^u \oplus TV \oplus E^s$ we express $(TF)_\#$ in block matrix form as

$$(TF)_\# = \begin{bmatrix} I & 0 & 0 & 0 \\ * & (T^uf)_\# & 0 & 0 \\ * & 0 & (T_Vf)_\# & 0 \\ * & 0 & 0 & (T^sf)_\# \end{bmatrix}.$$

By T^uf, T_Vf, T^sf we mean Tf restricted to E^u, TV, E^s, and I is the identity map on the space of sections of $T_W(E)$. The norms of the blocks are governed by $\lambda, \mu, \lambda^*, \mu^*$, except for the entries labelled "*". They are bounded. Recall from Hirsch, Pugh (1970), p. 154, that a similarity of block matrices

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \sim \begin{bmatrix} A' & 0 \\ 0 & D \end{bmatrix}$$

results from the assumption that either $\|D\|\|A^{-1}\| < 1$ or $\|D^{-1}\|\|A\| < 1$. Grouping together the nine upper left blocks as A and the block $(T^sf)_\#$ as D let us eliminate the lowest entry "*". This gives a TF- invariant bundle E^{cu} such that $T_W(E\times U) = E^{cu} \oplus E^s$. Arguing similarly with E^u, we get an invariant bundle E^{cs} with $E^u \oplus E^{cs} = T_W(E \times U)$. The intersection is E^c, $E^c = E^{cu} \cap E^{cs}$. Again, T^uF and T^sF r-dominate T^cF. Now we apply a version of the Center Mainfold Theorem.

We have a C^r compact sub-manifold W of a Banach manifold $M = E \times$ M, it is invariant by a C^r mapping F, the tangent to F leaves invariant a splitting $T_W(M) = E^u \oplus E^c \oplus E^s$ with $E^c \supset TW$, and T^uF, T^sF r-dominate T^cF. From this we construct via the graph transform method an F-invariant manifold W^c containning W and nearly tangent to E^c at W. (In the classical case W is a point.) In general, center manifolds are not unique, even in finite dimensions. However, they always contain the orbits which are both forward and backward bounded. This is true here too, and we see that W^c contains the sets $g \times \sigma_g(V)$. In fact, this means that near W, W^c consists exactly of C^r. (In the finite dimensional case, the W^c we construct via graph transform is always, C^r, in the Banach space case, we need to use this double Lyapunov stability.)

But C^r-ness of $W^c = \cup g \times \sigma_g(V)$ means that the map $(x, g) \to \sigma_g(x)$ is C^r. QED

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