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On the Existence of Generally Convergent Algorithms

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To motivate our result, consider Newton's method N for solving the equation $f(z) = 0$, where f is a complex polynomial, $f(z) = \Sigma_{i=0}^{d} a_i z^i$. We write N: $\mathcal{F}_d \times S \to S$, where $\mathcal{F}_d$ is the space of polynomials of degree $\leq d$ and S is the Riemann sphere $\mathbb{C} \cup \infty$. Then $N(f, z) = N_f(z) = z - f(z)/f'(z)$ is rational over $\mathbb{C}$ in f and z; that is, N can be formed from the complex rational operations $(+, -, \times, \div)$ from the coefficients of f and z.

If z is sufficiently close to a zero ζ of f, then the iterates $z_k = N_f^k(z)$ converge to ζ as k tends to ∞. However, as is well known there is an open set U in $\mathcal{F}_d \times \mathbb{C}$ (if $d > 2$) such that this convergence will not happen for (f, z) in U. See, e.g., Smale (1985). In this paper it was conjectured that no such algorithm could be generally convergent. Curt McMullen settled the question by proving the following result.

THEOREM (McMullen). *Let $d > 3$ and T: $\mathcal{F}_d \times S \to S$ be any map rational over $\mathbb{C}$ in f and z. Then there is no open set $U \subset \mathcal{F}_d \times S$ of full measure with this property: If $(f, z) \in U$, then $T_f^k(z) = z_k$ converges to a root of f as $k \to \infty$.*

Here a "set of full measure" means one whose complement has Lebesque measure zero.

McMullen's result can be paraphrased as saying there is no generally convergent purely iterative algorithm, rational over $\mathbb{C}$, for finding roots of polynomials of degree ≥ 4. Here "purely iterative" means that the algorithm can be expressed as a discrete dynamical system on S parameterized by the polynomial. Equivalently, the algorithm is one point stationary.

The goal of this paper is to show that if one adds the operation of complex

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conjugation, then there do exist generally convergent purely iterative algorithms for finding zeros of polynomials. This gives a complement of McMullen's theorem. Moreover our theorem works for n variables while McMullen's result, which depends on a recent one-variable theorem of Mañe, Sad, and Sullivan (1983), remains unproved for two or more variables.

THEOREM. *For any d, there is a map T: $\mathcal{F}_d \times S \to S$ formed from the complex rational operations and complex conjugation from the coefficients of $f \in \mathcal{F}_d$ and $z \in S$ with the following property: there is an open set of full measure $U \subset \mathcal{F}_d \times S$ such that if $(f, z) \in U$, then the iterates $z_k = T_f^k(z)$ converge to a zero of f.*

Theorem 2 (complemented by Theorem 1) in Section 2 below is a slightly sharper version of this result and moreover contains the n-variable case.

In Section 3 we give another example of a generally convergent purely iterative algorithm which is presumably more efficient. This second example is a modification of Newton's method so that it has quadratic convergence near a zero of a polynomial of multiplicity one. However, this algorithm uses square roots of positive numbers as well as complex conjugation. Also, we have not been able to extend the general convergence proof to more than one variable, thus leaving open a problem which seems to us important and challenging.

Of course there is a long history of results related to our work, a few of which are mentioned in Smale (1985). Also, there are works of Kim (1985), Hirsch and Smale (1979), Murota (1982), Wasilkowski (1983), and Wisniewski (1984). In Kim (1985) an algorithm similar to the one-variable case of Section 3 is proposed and studied with respect to general convergence.

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Let $\mathcal{F}_d$ be the linear space of all polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$ of degree $\leq d$, $d > 1$ (more abstractly one could say: let $\mathbb{E}$, $\mathbb{F}$ be complex Hilbert spaces of dimension n and $\mathcal{F}_d$ the space of all maps $\mathbb{E} \to \mathbb{F}$ whose $(d + 1)$st derivative is identically zero).

Let U_d be the subset of $\mathcal{F}_d$ of those f: $\mathbb{C}^n \to \mathbb{C}^n$ which satisfy these three conditions:

(a) The dth homogeneous parts of the coordinate functions f_i, $i = 1, \ldots, n$, of f have no common zeros except the origin. This implies that f is proper (see Hirsch and Smale, 1979, for example).

(b) If $f(z) = 0$, then the derivative $Df(z)$: $\mathbb{C}^n \to \mathbb{C}^n$ is nonsingular (our calculus notation follows Lang, 1983).

(c) The map g: $\mathbb{C}^n \to \mathbb{R}$ defined by $g(z) = \|f(z)\|^2$ is a Morse function. Here we use the Hermitian inner product and norm on $\mathbb{C}^n$. A Morse function is one with nondegenerate critical points. (Milnor, 1963, is a good reference for Morse theory.)

Note that (c) implies grad g has finitely many zeros.

THEOREM 1. *U_d is an open set of $\mathcal{F}_d$ containing the complement of a real algebraic subvariety; thus U_d is an open set of full measure.*

Proof. First work over the complex numbers.

Let $A \subset \mathcal{F}_d$ be the set of f such that the dth homogeneous parts of the f_i have a nontrivial common zero. Let $B \subset \mathcal{F}_d$ be the set of f such that the f_i and Det $Df(z)$ have a common zero. By elimination theory of algebraic geometry (see Van der Waerden, 1950, p. 15), A and B are each contained in algebraic subvarieties of $\mathcal{F}_d$ of complex codimension 1. See Renegar (1984) (also Smale, 1981) for this (in particular Renegar's Proposition 5.1). Thus it remains to deal with (c).

For this, the same procedure works, using the real numbers instead of the complex numbers. The equations (polynomial, real) this time are given by

(i) $Dg(z)$: $\mathbb{R}^{2n} \to \mathbb{R}$, $Dg(z) = 0$ (real derivative),
(ii) Det $D^2 g(z) = 0$. Q.E.D

For $f \in \mathcal{F}_d$, define an endomorphism T_f: $\mathbb{C}^n \to \mathbb{C}^n$ by

$$T_f(z) = T(z) = z - h_z \operatorname{grad} g(z), \qquad g(z) = \|f(z)\|^2,$$

where

$$\frac{1}{h_z} = \sum_{i=2}^{d} \left(1 + \frac{\|D^i g(z)\|_0^2}{i!}\right)(1 + \|\operatorname{grad} g(z)\|^2)^{i-2}.$$

Here $\|\ \|_0^2$ denotes the sum of the squares of the corresponding components, which is greater than or equal to the operator norm squared $\|\ \|^2$. The following argument shows this.

Let V, W be inner product spaces. Express $L(V, W)$ as matrices with respect to an orthonormal basis of V and W. Let $\|\ \|_E$ denote the Euclidean norm and $\|\ \|_{op}$ the operator norm. If A: $V \to L(V, W)$ is linear, and $L(V, W)$ has the operator norm, then the multilinear norm of A is the operator norm of A,

$$\|A\|_{op} = \sup_{v_1} \frac{\|A(v_1)\|_{op}}{\|v_1\|} = \sup_{v_2, v_1} \frac{\|A(v_1)(v_2)\|}{\|v_1\|\,\|v_2\|}.$$

Now since $\|\ \|_E \geq \|\ \|_{op}$ on $L(V, W)$ the operator norm of A: $V \to L_E(V, W)$ is $\geq$ operator norm of A and $\|A\|_E \geq \|A\|_{op}$. Now induction finishes the argument.

THEOREM 2. *For each $f \in U_d$, there is an open set $V_f \subset \mathbb{C}^n$ of full measure such that for $z \in V_f$, $T^k(z) = z_k$ converges to a zero of f as $k \to \infty$.*

The proof of Theorem 2 uses two propositions.

PROPOSITION 1. *Let $z \in \mathbb{C}^n$ with* grad $g(z) \neq 0$ *and let $z' = T(z)$. Then $g(z') < g(z)$.*

Proof. Expand g by a Taylor series about z, and evaluate it at $z' = T(z)$,

$$g(z') = g(z) - h_z |\operatorname{grad} g(z)|^2 + \sum_{i=2}^{d} (-h_z)^i \frac{D^i g(z)}{i!} \operatorname{grad} g(z)^i.$$

Then Proposition 1 is a consequence of this lemma:

LEMMA 1. *If* grad $g(z) \neq 0$, *then*

$$h_z |\operatorname{grad} g(z)|^2 > \left\| \sum_{i=2} -1^i h_z^i \frac{D^i g(z)}{i!} (\operatorname{grad} g(z))^i \right\|.$$

Proof of Lemma 1. Dividing by the left-hand side, it is sufficient to show

$$h_z \sum_{i=2}^{d} h_z^{i-2} \frac{\|D^i g(z)\|}{i!} \|\operatorname{grad} g(z)\|^{i-2} < 1.$$

(Here we use the operator norm on $D^i g(z)$; cf. Lang, 1983.) Since $h_z < 1$, this amounts to

$$h_z \sum_{i=2}^{d} \frac{\|D^i g(z)\|}{i!} \|\operatorname{grad} g(z)\|^{i-2} < 1.$$

The last follows from the definition of h_z, the fact that $1 + x^2 > x$ for any $x > 0$, and the fact that $\|A\| \leq \|A\|_0$.

PROPOSITION 2. *Let $f \in U_d$, $\theta \in \mathbb{C}^n$ satisfy $f(\theta) \neq 0$, and* grad $g(\theta) =$ 0. *Then the set $W^s(\theta)$ of all z such that $T^k(z) \to \theta$ as $k \to \infty$ has measure zero.*

For the proof we use some lemmas.

LEMMA 2. *θ is not a local minimum of g.*

Proof. This is a consequence of the maximum principle.

LEMMA 3. *$DT(\theta)$ has an eigenvalue greater than 1.*

Proof. $DT(\theta) = I - h_\theta D^2 g(\theta)$, so Lemma 3 follows from Lemma 2.

From the center manifold theory (see, e.g., Hirsch, Pugh, and Shub, 1977), it follows that there are arbitrarily small neighborhoods U of θ such that $W^s(\theta) \cap U$ is contained in a closed set of measure 0, in fact a differentiable disc of codimension one $W^c_f(\theta, U) = W^c_f(U) = W^c(U)$ with the property that $T^{-1}(W^c(U)) \cap U \subseteq W^c(U)$.

Next note that T is real algebraic and nondegenerate in that its image contains an open set by checking near the roots as below; in particular the Jacobian determinant $\det(DT)$ vanishes on a real subvariety of codimension one.

Proposition 2 now follows. $W^s(\theta) \subseteq W^c(\theta) \equiv \bigcup_0^\infty T^{-k}(W^c(U))$, which has measure zero since it is the countable union of measure zero sets.

Now for the proof of Theorem 2. Let $f \in U_d$ and $g = \|f\|^2$. If z_0 is a critical point of g let $W^s(z_0)$ be the set of $z \in \mathbb{C}^n$ such that $T^k(z) \to z_0$ as $k \to \infty$. Then define

$$W_f = \bigcup_{\substack{\operatorname{grad} g(z)=0 \\ f(z)\neq 0}} W^s(z) \qquad \text{and} \qquad V_f = \mathbb{C}^n - W_f.$$

Let $z \in V_f$. We claim that $z_k = T^k(z)$ converges to a zero of f as $k \to \infty$. By property (a) of f (since $f \in U_d$) the set of z_k is bounded; therefore by Proposition 1, z_k must converge to a zero of grad g. Since $z \notin W_f$, this zero of grad g is also a zero of f. By property (c) of f any zero z_0 of f is a sink of $-\operatorname{grad} g$; that is, all the eigenvalues of $-D(\operatorname{grad} g)(z_0)$ are negative. By the definition of h_z all the eigenvalues of $-h_{z_0} D \operatorname{grad} g(z_0)$ are negative but greater than -1. Thus all the eigenvalues of $DT(z_0) = I - h_{z_0} D \operatorname{grad} g(z_0)$ are greater than zero but less than one. Thus z_0 is a sink for T.

This shows that $W^s(z_0)$ is open, and V_f is open. Moreover, there is a disc D_0 around z_0 mapped into its interior by a contraction for any g in U_d close enough to f. It follows by continuity that if $z \in W^s(z_0)$ and $f \in U_d$ then (f, z) is in the interior of $U = \{(f, z) \mid f \in U_d \text{ and } z \in V_f\}$. Thus U is open and of full measure in $\mathcal{F}_d \times \mathbb{C}^n$. Q.E.D

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Let f be a polynomial of one variable, $f(z) = \Sigma_0^d a_i z^i$, $z \in \mathbb{C} \cup \infty = S$. Define

$$k_z = \frac{\phi(|z|)|f'(z)|^2}{2\phi'(|z|)^2|f(z)|\,\|f\|_{\max}},$$

where

$$\phi(r) = \sum_{i=0}^{d} r^i \qquad \text{and} \qquad \|f\|_{\max} = \max_i |a_i|.$$

Let $\rho(z) = \min(1, k_z)$ and define $T_f\colon S \to S$ by $T_f(z) = z - \rho(z) \times$

$(f(z)/f'(z))$. Here note that the max and min of positive numbers may be expressed in terms of square roots, e.g.,

$$\max(a, b) = \frac{|a-b|}{2} + \frac{|a+b|}{2}, \qquad \sqrt{(a-b)^2} = |a-b|.$$

Next let G_d be the space of polynomials of one variable, degree $\leq d$, with zeros and critical points all distinct.

THEOREM 3. *Let $f \in G_d$. Then there is a closed set W_f of measure zero such that if $z \notin W_f$, then $T_f^k(z)$ converges to a zero of f as k tends to ∞. Moreover T_f is Newton's method in a neighborhood of each zero of f.*

Proof. The last statement follows from the definitions. We now prove the rest.

Define for each $z \in \mathbb{C}$, f a polynomial with $f'(z) \neq 0$,

$$\alpha(z, f) = \sup_{k \geq 2} \left| \frac{f^{(k)}(z)}{k!\, f'(z)} \right|^{1/(k-1)} \frac{|f(z)|}{|f'(z)|}.$$

PROPOSITION 3. *Let f be a polynomial and $z \in \mathbb{C}$, with $f(z) \neq 0$, $f'(z) \neq 0$. Let h satisfy*

$$0 < h < 1/2\alpha, \qquad \alpha = \alpha(z, f).$$

Then for

$$z' = z - h\frac{f(z)}{f'(z)}, \qquad |f(z')| < |f(z)|.$$

Proof. Expand f by Taylor's series about z so

$$f(z') = (1-h)f(z) + \sum_{i=2}^{d} \frac{1}{i\,!} f^{(i)}(z) \left(\frac{hf(z)}{f'(z)} \right)^i$$

and

$$\begin{aligned} \frac{|f(z')|}{|f(z)|} &\leq 1 - h + h \sum_{i=2}^{d} \frac{h^{i-1}}{i!} \frac{|f^{(i)}(z)|\,|f(z)|^{i-1}}{|f'(z)|^i} \\ &\leq 1 - h + h \sum_{i=2}^{d} (h\alpha)^{i-1} \\ &\leq 1 - h + h(h\alpha)\left(\frac{1}{1-h\alpha} \right) < 1. \end{aligned}$$

Q.E.D

This generalizes easily to polynomial maps from one Banach space to another.

PROPOSITION 4. (a) *Let f be a polynomial, $z \in S$, and $\phi(r) = \Sigma_{i=0}^{d} r^i$. Then*

$$\alpha(z, f) \leq \frac{|f(z)|\, \|f\|_{\max} \phi'(|z|)^2}{|f'(z)|^2 \phi(|z|)}.$$

(b) Let $r > 0$. Then

$$\frac{\phi(r)\phi''(r)}{2\phi'(r)^2} \leq 1.$$

This is proved in Smale (1986), where in fact (a) is proved for Banach spaces.

PROPOSITION 5. *Let $f \in G_d$, $f'(\theta) = 0$, $W^s(\theta) = \{z \mid T^k(z) \to \theta$, as $k \to \infty\}$. Then $W^s(\theta)$ is a closed set of measure* 0.

Proof. By the argument of Proposition 2 of Section 2, it is sufficient to prove Proposition 5 locally, in a neighborhood of θ.

To that end we calculate the derivative $DT(\theta)$: $\mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbb{R}^2$ is just $\mathbb{C}$. For z in a neighborhood of θ, $\rho(z) = k_z$ and we may write $T(z)$ as

$$T(z) = z - \frac{\phi(|z|)f(z)}{2\phi'(|z|)^2\|f\|_m|f(z)|}\overline{f'(z)}$$

so

$$DT(\theta)(v) = v - \frac{\phi(|\theta|)f(\theta)}{2\phi'(|\theta|)^2\|f\|_m|f(\theta)|}(D\overline{f'(z)})_{z=0}(v)$$

for $v \in \mathbb{R}^2$. Now $D(\overline{f'(z)})_{z=0}(v) = \overline{f''(\theta)v}$.

Thus the linear map $DT(\theta)$ has the form $DT(\theta)(v) = v - \beta\bar{v}$, where $\beta = f(\theta)\overline{f''(\theta)}\phi(|\theta|)/2\phi'(|\theta|)^2\|f\|_m|f(\theta)|$.

LEMMA 4. *The linear map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $v \to \beta\bar{v}$ has trace* 0 *and determinant $-|\beta|^2 \leq 0$. Thus its eigenvalues are $\pm|\beta|$.*

The proof is simple and direct.

From the lemma it follows that the eigenvalues of $DT(\theta)$ are $1 \pm |\beta|$. But

$$|\beta| = \frac{|f''(\theta)|\phi(|\theta|)}{2\|f\|_m\phi'(|\theta|)^2} \leq \frac{|f''(\theta)|}{\|f\|_m}\frac{1}{\phi''(|\theta|)},$$

by Proposition 4(b), which can be seen to be less than 1. Therefore θ is a saddle point for T_f, proving Proposition 5.

For the proof of Theorem 3, note that $k_z < 1/2\alpha(f, z)$ by using Proposition 4(a). Thus Proposition 3 applies to T_f. Now the same argument used in the end of the proof of Theorem 2, using Proposition 5, yields Theorem 3.

One needs to remark that the operations involved in the definition of $T_f(z)$, besides the complex rational operations only require complex conjugation and the square root of a positive real number.

The preceding arguments for Theorem 3 extend to the n-variable case except for the local argument of Proposition 5. We give a short discussion.

The Newton vector field $N(z) = -Df(z)^{-1}f(z)$ is not generally globally defined on $\mathbb{C}^n$ because $Df(z)$ may not be invertible. We desingularize N as follows: Given the $n \times n$ complex matrix A, let A be the $n \times n$ matrix whose (i, j)th entry is $(-1)^{i+j} \det A_{ji}$, where A_{ji} is the (j, j)th cofactor of A. The standard proof of Cramer's rule for inverting a matrix gives

$$\hat{A}A = A\hat{A} = (\det A)I. \tag{a}$$

Now define $\mathring{N}(z) = -\widehat{Df(z)}f(z)$. $\mathring{N}(z)$ is globally defined, and

$$\frac{1}{\det Df(z)}\mathring{N}(z) = N(z).$$

Note that $\mathring{N}(z)$ is zero in the following cases.

(i) $Df(z)$ has corank 2 or more; then $\widehat{Df(z)}$ is identically zero.

(ii) $Df(z)$ has corank 1 and $f(z) \in \text{Image } Df(z) = \text{kernel } \widehat{Df(z)}$.
(iii) $f(z) = 0$.

DEFINITION. *Let*

$$K_f(z) = \frac{\phi(\|z\|)}{2\|f(z)\| \, \|f\|_{\max} \phi'(\|z\|)^2 \|Df^{-1}(z)\|^2},$$

where $\|f\|_{\max} = \max_i(\|D^i f(o)\|/i!)$ and let $\rho(z) = \rho_f(z) = \min(1, K_f(z))$.

For a polynomial f such that $Df(z)$ is invertible at the roots of f, $\rho(z)$ extends continuously to be identically one in a neighborhood of the roots of f and $\rho(z)\|Df^{-1}(z)\|$ extends to be zero on the variety Σ of z such that $\operatorname{Det} Df(z) = 0$. Now let

$$T(z) = T_f(z) = z - \rho(z)Df(z)^{-1}f(z).$$

For f with nondegenerate roots, T is Newton's method near the roots of f and the identity on Σ. Near Σ,

$$T(z) = z - K_f(z)Df^{-1}(z)f(z) = z + h(z)\,\overline{\operatorname{Det} Df(z)}\,\mathring{N}(z),$$

where

$$h(z) = \frac{\phi(\|z\|)}{2\|f(z)\|\,\|f\|_{\max}\phi'(\|z\|)^2\|\widehat{Df(z)}\|^2}.$$

$K_f(z) \le 1/2\alpha(z,f)$, so Proposition 3 still applies. Add the additional hypothesis that f is proper. Then the crucial question for the global behavior of T is the nature of the set of points which tend to Σ under the iteration of f.

PROBLEM 1. For all f in the complement of an algebraic subvariety of $\mathcal{F}_d$ of codimension ≥ 1, is it true that $W^s(\Sigma) = \{z \mid T^k(z) \to \Sigma \text{ as } k \to +\infty\}$ is in a closet set of measure zero?

If we assume that $\theta \in \Sigma$, that Df has corank one at θ and is transversal to the corank one matrices there, and moreover that $f(\theta) \notin \underline{\text{Image } Df(\theta)}$ and Ker $Df(\theta)$ is not tangent to Σ then $T'(\theta)v = v + h(\theta)\overline{L(\theta)v}\,\mathring{N}(\theta)$, where $L(\theta)$ is a linear map $L(\theta)$: $\mathbb{C}^n \to \mathbb{C}$, $h(\theta) \neq 0$, and $\mathring{N}(\theta) \neq 0$ and thus $T'(\theta)$ has an eigenvalue larger than one. Now the theory of partially hyperbolic fixed points (Hirsch, Pugh, and Shub, 1977) shows that locally near θ, $W^s(\Sigma)$ has measure zero; i.e., there is a neighborhood U of θ such that $\{z \in U \mid f^k(z) \in U \text{ for all } k > 0 \text{ and } f^k(z) \to \Sigma\}$ has measure 0. This takes care of most of the points in Σ, but generically there are points not satisfying these hypotheses even for two variables.

PROBLEM 2. If $\theta \in \mathbb{C}^n$ and $Df(\theta)$ is singular, under generic conditions on f, is the set of z such that $T^k(z) \to \theta$ as $k \to \infty$ contained in a closed set of measure 0?

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