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EVALUATING RATIONAL FUNCTIONS: INFINITE PRECISION IS FINITE COST AND TRACTABLE ON AVERAGE*

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Abstract. If one is interested in the computational complexity of problems whose natural domain of discourse is the reals, then one is led to ask: what is the "cost" of obtaining solutions to within a prescribed absolute *accuracy* $\varepsilon = 1/2^s$ (or *precision* $s = -\log_2 \varepsilon$)? The *loss of precision* intrinsic to solving a problem, independent of method of solution, gives a lower bound on the cost. It also indicates how realistic it is to assume that basic (arithmetic) operations are exact and/or take one step for complexity analyses. For the relative case, the analogous notion is the *loss of significance.*

Let $P(X)/Q(X)$ be a rational function of degree d, dimension n and real coefficients of absolute value bounded by ρ. The loss of precision in evaluating P/Q will depend on the input x, and, in general, can be arbitrarily large. We show that, w.r.t. normalized Lebesgue measure on B_r, the ball of radius r about the origin in R^n, the average loss is small: loglinear in d, n, ρ, r, and K_Q, a simple constant.

To get this, we use techniques of integral geometry and geometric measure theory to estimate the volume of the set of points causing the denominator values to be small. Suppose $\varepsilon > 0$ and $d \geqq 1$. Then:

THEOREM. *Normalized volume* $\{x \in B_r \mid |Q(x)| < \varepsilon\} < \varepsilon^{1/d} K_Q^{1/d} nd(d+1)/2r$.

An immediate application is a loglinear upper bound on the average loss of significance for solving systems of linear equations.

Key words. loss of precision/significance, condition of a problem, rational functions, average case, integral geometry, models of real computation

1. Introduction. One approach to the analysis of the computational complexity of algebraic problems, such as the cost of evaluating rational functions over the reals, presupposes a model of computation with all arithmetic operations exact and of unit cost, for example, the real number model (Borodin-Munro [1], Knuth [6]). *How do results concerning such an idealized infinite continuous model apply to the finite discrete process of computing on actual machines*? For example, in the real number model there is no problem to decide given real x if $x \neq 0$, and if so then to compute $1/x$. However, on any real computer, when x is ultimately presented digit by digit, the "marking time" to observe the first nonzero digit of x, as well as the "input precision" for x needed to compute $1/x$ to within absolute accuracy $1/2^s$, can be arbitrarily large. If x is presented in floating point notation, the computation of $1/(1-x)$ in the unit interval to relative accuracy ε exhibits analogous problems.

How do these factors influence the actual cost and reliability of real computation, particularly as we move away from these simple examples to more interesting ones? For instance, given an invertible $n \times n$ real matrix A, what is the "input precision" needed to specify the entries of A in order to compute $1/\det A$ to within accuracy $1/2^s$ (Moler [9])? Such questions, dealing with tolerable round-off error and achievable accuracy are often avoided in approaches to algebraic complexity theory that assume infinite precision. As Knuth says in [6, p. 486], they are "beyond the scope of this book".

In this paper we address some of these questions with regard to the problem of evaluating rational functions. One approach might be to analyze achievable accuracy

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and related costs assuming given input size or machine precision. In an attempt to perhaps more fully interpret and apply results concerning the continuous model, we take a somewhat opposite approach. That is, we ask, given a rational function $P(X)/Q(X)$ of n variables over the reals, and desired accuracy $1/2^s$, what "input precision" is needed to achieve this desired accuracy? Clearly, the answer will depend on s, on P and Q (i.e., the number of variables, the degrees and coefficients), and on the input $x=(x_1, \cdots, x_n)$. It is unbounded in general. Surprisingly, however, we show (Theorem 4 in § 5) that the average input precision sufficient for x in B_r, the ball of radius r about the origin in $\mathbf{R}^n$, is finite and exceedingly "tractable". Here average means with respect to normalized Lebesgue measure on B_r. This tractability result, as well as others in this paper, is explicit rather than asymptotic.

Our main tool is a formula which enables us to estimate the volume of the set of inputs causing the denominator values to be small. In its normalized version, we get the following elegant estimate (in § 4).

MAIN THEOREM.

$$\frac{\operatorname{Vol}\{x \in B_r \,|\, |Q(x)| < \varepsilon\}}{\operatorname{Vol}\{B_r\}} < C_Q \frac{\varepsilon^{1/d}}{r}$$

where $Q: \mathbf{R}^n \to \mathbf{R}$ *is a polynomial of degree* $d \geqq 1$, r, ε *are positive real numbers,* $C_Q = K_Q^{1/d} nd(d+1)/2$ *and* K_Q *is a simple constant depending on the coefficients of* Q (*see* § 4).

This result allows us to estimate, for example, the average "marking time" to determine that $Q(x)$ is not zero for $x \in B_r$ (Theorem 3 in § 5).

We prove the Main Theorem using techniques from geometric measure theory and integral geometry (Federer [2] and Santalo [11]), thus using methods that are somewhat new to computational complexity theory.

We address questions both of absolute and relative accuracy. However, unless otherwise stated, "accuracy" will mean "absolute accuracy". The results on "relative accuracy" are included in § 6.

Remark 1. *Computational complexity and the condition/loss of precision of a problem.* If one is interested in the computational complexity of problems whose natural domain of discourse is the reals (or complex numbers), then one is led, both naturally and necessarily, to ask: What is the "cost" of obtaining solutions to within a *prescribed accuracy* $\varepsilon = 1/2^s \leqq 1$ (or equivalently, to within a *prescribed precision* $s = -\log_2 \varepsilon$)?

Clearly, a satisfactory answer must resolve a number of issues relating to tolerable error and the number of bit and/or basic operations required. In particular, we at least must answer the following two questions:

(1) What is the necessary and sufficient *input accuracy* δ (or equivalently, *input precision* $-\log \delta$) required for the data in order to obtain a solution to within *output accuracy* ε? Here δ will in general depend on both ε and the data, and we assume that both ε and δ are positive real numbers less than or equal to 1. (If $\delta > 1$, then we will define the *associated precision* to be 0.) The ratio ε/δ is a measure of the *condition* of the problem (Henrici [4], Wilkinson [17]) and will be "very large" for *ill-conditioned problems*, i.e., for problems where the necessary *input precision* $|\log \delta|$ is "much larger" than the prescribed *output precision* $|\log \varepsilon|$ or where the *loss of precision*

$$\log^+ \frac{\varepsilon}{\delta} = \begin{cases} |\log \delta| - |\log \varepsilon| & \text{if } \varepsilon/\delta > 1, \\ 0 & \text{otherwise} \end{cases}$$

is "great". *Thus, ill-conditioned problems are generally not computationally tractable* (nor are they computationally stable—small perturbations of the input can result in large changes in the output).[1]

(2) Given input data of precision $-\log \delta$, what is the cost, (e.g. in terms of the number of basic arithmetic operations needed) of evaluating the solution of within output precision $-\log \varepsilon$?

The condition of a problem (1) is often investigated in numerical analysis; the basic number of steps required (2) is generally investigated in complexity theory. Clearly, a deeper understanding of the complexity issues arising in the real (or complex) case requires an understanding of *both* issues.

The condition and loss of precision are measures intrinsic to a problem and independent of method of solution. However, we remark that from a computational point of view, the *loss of precision* appears more natural than the *condition.* For one, the loss of precision gives a lower bound on the complexity of a problem. But also, the mathematics helps affirm its naturalness even more. For example, a simple computation shows that, while the average (with respect to the uniform distribution) condition of the problem of evaluating $1/x$ in the unit interval is infinite, the average loss of precision (i.e., average $\log^+$ of the condition) is very small, consistent with our intuition. (See § 2 and also § 6 for an example in the relative case.) Thus, we propose that one focus on the loss of precision, rather than the condition of a problem, for purposes of computational complexity.

In this paper we investigate the condition, more precsiely, the loss of precision in evaluating rational functions with respect to an average case analysis. This point of view is new. Our results show that, although the problem of evaluating rational functions can be arbitrarily badly ill-conditioned, the average loss of precision (and standard deviation) is small.

An anlysis of average cost now just incorporates known results about the number of arithmetic operations to evaluate rational functions, as well as the cost of multiplying m-bit numbers (Borodin-Munro [1], Knuth [6]). Thus even on the level of bit operations required, our results imply that the problem of evaluating, to accuracy $1/2^s$, rational functions of n variables, of degree d, whose coefficients lie in a ball of radius ρ, on a point x in a ball of radius r about the origin in R^n, is on the average tractable in s, n, d, and T, the maximum number of nonzero terms.

Remark 2. *Interpretation and application.* We can interpret our results as saying that computation of rational functions of infinite precision real numbers is tractable on the average, as long as one only uses as much precision as one needs. It suggests a methodology for the design of algorithms to compute such functions: start with the input precision sufficient on the average to achieve the desired accuracy and increase as necessary, e.g. by the standard deviation.

We give a general formula for input precision sufficient on the average that holds for all rational functions. Hence the result implies the average tractability of many problems such as the computation of $1/\det A$ for an invertible $n \times n$ matrix A, the inversion of the matrix by Cramer's rule, the estimation of the average logarithm of the condition number of such a matrix, and other problems expressable by a polynomial number of rational functions. Of course, in any specific problem one expects to do better. For example, Norman Weiss [16] has shown us that the exponent of ε in the

[1] For a problem whose underlying function P is differentiable, an infinitesimal version of the *condition* of the problem at input x is the size $|P'(x)|$ of the linear operator $P'(x)$ [Henrici]. The analogous measure of *loss of precision* would be $\log^+ |P'(x)|$. Results similar to those in this paper can be achieved using these notions.

computation of the volume estimate for the determinant in any dimension can be taken equal to 1 instead of $1/d$ and for the discriminant of polynomials of one variable of degree d, the exponent is trivially seen to be no worse than $\frac{1}{2}$.

In general, sharper volume estimates for polynomials should be possible depending on such criteria as irreducibility, the lower coefficients, etc. and these would be interesting. However, the example $P(x) = x^d$ demonstrates the sharpness of our results for the general case with regard to the exponent of ε.

Remark 3. *Abstract models of computation: recursive analysis.* Our results have implications for recursive analysis, the abstract theory of computation of functions on the reals (or complex numbers). A model for such a theory of computation could be based on the notion of function-oracle Turing machines. (See Ko–Friedman [7] for the development of a formal theory). *Informally,* we imagine a Turing machine M_f for computing a real function f being fed, by oracle, a real input x digit by digit as necessary. Computations are performed by M_f in the usual oracle Turing machine manner. Outputs are produced digit by digit converging to a sequence representing the real value $f(x)$. Computable real functions are then those functions that can be realized in this model, and are necessarily continuous (on their domains). The complexity of functions is measured by the cost (in terms of time and space) of producing outputs that are accurate to within ε. Hence, in this model, polynomial-time computable functions must have polynomially bounded moduli of continuity (i.e., input precision $-\log \delta$ which is polynomial in the desired output precision $-\log \varepsilon$). Thus, rational functions are not in general polynomial-time computable.

Our results show that if in this model we extend the notion of "tractable" to mean "polynomial time computable on the average", then we significantly extend the class of tractable functions in a way that is both natural and useful. In so doing, we also show how methods from integral geometry and probability theory can be used to obtain results in recursive analysis.

Remark 4. *Relation to other work.* Much of our work and techniques are motivated by Smale [13] and Shub–Smale [12]. Myong Hi Kim [5] is doing an analysis of the finite precision analogue of the real number model results on average tractability of Newton–Euler iteration schemes for finding approximate zeros of polynomials. Smale [14] is also investigating general notions for the condition (or loss of precision) of a problem.

The first draft of this paper, including the main results on sufficient input precision, was finished in the fall of 1983. In Steve Smale's seminar in the spring of 1984, Blum suggested that the log of the condition was the appropriate concept for study. Smale then introduced the expression "loss of precision" and asked what the average loss was for linear systems. This motivated the facile application of our main results in § 7. Using special techniques Eric Kostlan [8] and Adrian Ocneanu [10] get sharper results for linear systems.

Remark 5. *Further directions and results.* The results can be easily extended to the complex case, the exponent of ε changes to $2/d$, and generalized to semi-analytic functions. R. Hardt [3] has proven a volume estimate for these functions which should be sufficient to conclude the average tractability of the loss of precision function for a single f. Also, averages could be computed with respect to other probability measures, reflecting nonuniform distributions inherent in classes of naturally arising problems.

Remark 6. *Organization of paper.* We begin with three simple examples (in § 2) to illustrate key points. In § 3 we outline our procedure for the analysis of input precision sufficient to evaluate a rational function to within accuracy ε. In § 4 we estimate the volume of the set of points on which a polynomial has values near zero.

Using this, we estimate, in § 5, the average marking time to determine that a polynomial is not zero. This enables us to estimate the input precision sufficient on average. Section 6 deals with floating point notation and the relative case. In § 7 we give an immediate application of the main results of the paper to the problem of solving linear systems.

2. Three simple examples. We start with three elementary examples in order to illustrate key points.

(1) Let $x \in [0, 1]$ and let $l(x) = |\log_2(x)|$. Then $m(x) = \lfloor l(x) \rfloor + 1$ is sufficient "marking time" to observe the first nonzero entry in a binary expansion of x. Although $m(x)$ can be arbitrarily large, its average or expected value Av m is less than 2.5 since

$$\text{Av}\, m < \int_0^1 (l(x)+1)\, dx = \log_2 e + 1.$$

(2) Suppose that an integer $s \geqq 1$ is given and that $x \in [0, 1]$. We wish to compute $1/x$ to within absolute accuracy $1/2^s$. If $|x - x^*| < 1/2^{s+1+2l(x)+\varepsilon}$ where $\varepsilon = 1/2^{s+1}$, it is not hard to see that $|1/x - 1/x^*| < 1/2^{s+1}$ (and if $|1/x^* - y^*| < 1/2^{s+1}$ then $|1/x - y^*| < 1/2^s$). Thus, an "input precision" $\Pi_s(x)$ for x sufficient to compute $1/x$ to within absolute accuracy $1/2^s$ (allowing truncation in the answer) is $s + 1 + \lceil 2l(x) + \varepsilon \rceil$, which again can be arbitrarily large depending on x. In this case, the average is

$$\begin{aligned}\text{Av}\, \Pi_s &= \int_0^1 \Pi_s(x)\, dx \leqq \int_0^1 (s+1+\lceil 2l(x)+\varepsilon \rceil)\, dx < \int_0^1 (s+1+2l(x)+1)\, dx \\ &= s + 2\log_2 e + 2\end{aligned}$$

which is approximately $s + 4.885$, and anyway, less than $s + 5$, again exceedingly tractable.

Thus, the average "loss of precision" in evaluating $1/x$ in $[0, 1]$ is finite and does not exceed $2\log_2 e + 2$, consistent with our intuition. On the other hand, using the fact that given $\varepsilon > 0$, $\delta(\varepsilon, x) = \varepsilon x^2/(1 + \varepsilon x)$, a simple calculation shows that the average "condition" ε/δ for the problem is infinite.

In these examples, we can also easily estimate the variance. Here it is useful to note that for square integrable functions f and f^* on $[0, 1]$ if $f^* \leqq f \leqq f^* + 1$ then $|f - \int f| < |f^* - \int f^*| + 1$. So, a crude estimate by Cauchy-Schwarz gives $\text{Var}(f) \leqq \text{Var}(f^*) + 2\sigma(f^*) + 1$ where

$$\text{Var}(f) = \int_0^1 \left(f(x) - \int_0^1 f(x)\, dx\right)^2 dx$$

is the variance of f, and $\sigma(f) = \sqrt{\text{Var}(f)}$ is the standard deviation.

Thus, in example 2, letting $\Pi_s^*(x) = s + 2l(x) + 1$, a straightforward calculation shows that

$$\text{Var}(\Pi_s) = \int_0^1 (\Pi_s(x) - \text{Av}\, \Pi_s)^2\, dx \leqq 4(\log_2 e)^2 + 4\log_2 e + 1.$$

Hence, $\sigma(\Pi_s) < 4$.

(3) Suppose that a positive integer s is given, and real numbers P and Q are chosen at random in the unit interval $[0, 1]$. We wish to compute P/Q to within relative accuracy $1/2^s$. What precision $\Pi_s(P, Q)$ is sufficient for P and Q? If $|P| > 1/2^l$ and

$|Q| > 1/2^l$ then

$$\left|\frac{P^*/Q^*}{P/Q} - 1\right| < \frac{1}{2^{s+1}}$$

as long as $|P - P^*| < 1/2^A$ and $|Q - Q^*| < 1/2^A$, where $A > s + l + 3$.

Thus, $\Pi_s(P, Q) \leqq s + \max(m(P), m(Q)) + 3$. The average value of Π_s in the unit square is

$$\begin{aligned}\text{Av}\,\Pi_s &\leqq s + \left[\Sigma m\left(\left(1 - \frac{1}{2^m}\right)^2 - \left(1 - \frac{1}{2^{m-1}}\right)\right)\right] + 3 \\ &\leqq \Sigma m\left(-\frac{1}{2^{m-1}} + \frac{1}{2^{m-2}}\right) + s + 3 \\ &= \Sigma m\left(\frac{1}{2^{m-1}}\right) + s + 3 = \left(\frac{1}{1-1/2}\right)^2 + s + 3 = s + 7.\end{aligned}$$

In the above examples we have basically used the following, which we also use later on.

LEMMA 1. *Let s, m, v, θ and A be positive integers and P, Q, P^*, Q^* be real numbers none of which is zero. Then*:

1) *If* $|Q| \geqq 1/2^m$ *and* $|Q - Q^*| \leqq 1/2^\theta$ *then* $|1/Q - 1/Q^*| < 1/2^s$ *as long as* $\theta > s + 2m$.

2) *If* $|Q| \geqq 1/2^m$, $|P| \leqq 2^V$, $|Q - Q^*| \leqq 1/2^\theta$ *and* $|P - P^*| \leqq 1/2^A$ *then* $|P/Q - P^*/Q^*| < 1/2^s$ *as long as* $\theta > s + 2m + v + 1$ *and* $A > s + m + 1$.

3) *If* $|P - P^*| < (1/2^{s+2})|P|$ *and* $|Q - Q^*| < (1/2^{s+2})|Q|$, *then*

$$\left|\frac{P^*/Q^*}{P/Q} - 1\right| < \frac{1}{2^s}.$$

Proof.

1)

$$\left|\frac{1}{Q} - \frac{1}{Q^*}\right| = \left|\frac{Q^* - Q}{Q \cdot Q^*}\right| \leqq \frac{1/2^\theta}{(1/2^m)(1/2^m - 1/2^\theta)} = \frac{2^m}{2^{\theta-m} - 1}$$

$$< \frac{1}{2^s} \quad \text{as long as } \theta > s + 2m.$$

2)

$$\left|\frac{P}{Q} - \frac{P^*}{Q^*}\right| \leqq |P|\left|\frac{1}{Q} - \frac{1}{Q^*}\right| + \left|\frac{1}{Q^*}\right| \cdot \frac{1}{2^A}.$$

Now by 1)

$$\left|\frac{1}{Q} - \frac{1}{Q^*}\right| < \frac{1}{2^{s+v+1}}$$

and thus

$$\left|\frac{1}{Q^*}\right| < 2^m + \frac{1}{2^{s+v+1}}.$$

Hence

$$\left|\frac{P}{Q} - \frac{P^*}{Q^*}\right| < \frac{1}{2^{s+1}} + \left(2^m + \frac{1}{2^{s+v+1}}\right)\frac{1}{2^A} < \frac{1}{2^s}.$$

3) $|1-P^*/P|<1/2^{s+2}$ and $|1-Q^*/Q|<1/2^{s+2}$. So,

$$1-\frac{1}{2^{s+2}}<\frac{P^*}{P}<1+\frac{1}{2^{s+2}}$$

and

$$1-\frac{1}{2^{s+2}}<\frac{Q^*}{Q}<1+\frac{1}{2^{s+2}}.$$

So

$$1-\frac{1}{2^{s+2}+1}<\frac{Q}{Q^*}<1+\frac{1}{2^{s+2}}.$$

Now multiply and subtract 1 to see that $|P^*Q/Q^*P-1|<1/2^s$.

3. Procedure for the analysis. We define the *precision associated with accuracy* $\delta>0$ to be $|\log_2\delta|$ if $\delta\leqq 1$ and to be 0 otherwise.

Our procedure for the analysis of sufficient "input precision" (and hence a lower bound on the "cost" involved) on average, is as follows.

First, using Lemma 1.2 in § 2, we estimate the (output) precision $O_s(P, Q, x)$ for $P(x)$ and $Q(x)$ (depending on the values of $P(x)$ and $Q(x)$) necessary and sufficient for the value $f(x)=P(x)/Q(x)$ to be within accuracy $1/2^s\leqq 1$:

$$O_s(P, Q, x)\leqq s+2m_Q(x)+\log_2|P(x)|+1$$

where

$$m_Q(x)=\begin{cases}|\log_2|Q(x)|| & \text{if } |Q(x)|\leqq 1,\\ 0 & \text{otherwise}\end{cases}$$

is the *marking time* to determine that the denominator of f is not zero.

Next, we let $I_t(P, Q, x)$ be the (input) precision for x *and the coefficients of* P *and* Q, necessary and sufficient for the values $P(x)$ and $Q(x)$ to be within accuracy $1/2^t\leqq 1$. Even the most naive method of evaluating P and Q by multiplying out the monomials and adding them up will show that, for every positive integer t, $I_t(P, Q, x)\leqq t+H(d, T_f, \rho, r)$ where

$$H(d, T_f, \rho, r)\leqq\log_2 d+\log_2 T_f+d\log_2^+ r+\log_2^+\rho+2.$$

Here $\log^+=\max\{\log, 0\}$, $d=\max\{\deg P, \deg Q\}$, T_f is the maximum of the number of nonzero terms of P and Q and is bounded by $\binom{n+1}{d}$, where n is the maximum number of variables of P and Q, ρ is an upper bound on the absolute value of the coefficients of P and Q, r is a positive real and $|x|<r$. Here $|x|$ is the Euclidean norm.

So, letting $\Pi_s(P, Q, x)$ be the *input precision* for x *and the coefficients of* P *and* Q necessary and sufficient to evaluate $f(x)$ to within accuracy $1/2^s$, we have $\Pi_s(P, Q, x)\leqq I_t(P, Q, x)$ where $t=O_s(P, Q, x)$. And so,

$$\begin{aligned}\Pi_s(P, Q, x)&\leqq O_s(P, Q, x)+H(d, T_f, \rho, r)\\ &\leqq s+2m_Q(x)+\log_2|P(x)|+1+H(d, T_f, \rho, r)\\ &\leqq s+2m_Q(x)+[\log_2 T_f+d\log_2^+ r+\log_2\rho]+1+H(d, T_f, \rho, r)\\ &\leqq s+2m_Q(x)+2[\log_2 T_f+d\log_2^+ r+\log^+\rho]+\log_2 d+3.\end{aligned}$$

Now, we also note that computing $P(x)/Q(x)$ to within accuracy $1/2^s$ using the naive method suggested above and fast multiplication requires no more than $2(d+1)T_f$ arithmetic operations times $O(\Pi_s(P, Q, x) \log \Pi_s(P, Q, x) \log\log \Pi_s(P, Q, x))$ bit operations.

Thus, in order to show that tractability of the average input precision necessary and sufficient to evaluate f to within accuracy $1/2^s$ and hence the tractability of evaluating f on average, our main problem, and the focus of this paper, is to estimate and show the tractability of the average marking time to decide that a polynomial is not zero. To do this we first show that the volume of the set of points on which a polynomial has values near zero is "small".

4. The volume estimate. Given a polynomial $Q: \mathbf{R}^n \to \mathbf{R}$ of degree $d \geqq 1$, express Q as

$$Q(X) = \sum_I a_I X_1^{i_1} \cdots X_n^{i_n} + a_0$$

where

$$I = \{i_1, \cdots, i_n\}, \qquad 0 \leqq i_1, \cdots, i_n \leqq d$$

and

$$|I| = \sum_{j=1}^{n} i_j \leqq d.$$

Let $b_I = i_1! \cdots i_n! a_I$ (where $0! = 1$) and $K_Q = 1/\max_{|I|=d} |b_I|$.

Let r be a positive real number and let $B_r = \{x \in R^n \,|\, |x| \leqq r\}$ be the ball of radius r about the origin in R^n. Let A_{n-1} be the $(n-1)$-dimensional surface area of ∂B_1, the boundary of B_1. Let V_n be the volume of B_1. So, the surface area of ∂B_r is $A_{n-1} r^{n-1}$ and the volume of B_r is $V_n r^n$.

MAIN THEOREM 1. *Let $Q: R^n \to R$ be a polynomial of degree $d \geqq 1$. For any real numbers $r > 0$ and $\varepsilon > 0$,*

$$\operatorname{Vol}\{x \in B_r \,|\, |Q(x)| < \varepsilon\} < 2\varepsilon^{1/d} K_Q^{1/d} \left[V_{n-1} + \left(\frac{d(d+1)}{2} - 1 \right) \frac{A_{n-1}}{2} \right] r^{n-1}.$$

Proof outline. The proof is by induction on the degree d. The linear case $(d = 1)$ is straightforward. The inductive step uses the co-area formula (Federer [2]) to show that the $\operatorname{Vol}\{x \in B_r \,|\, |Q(x)| < \varepsilon\}$ can be computed by first computing the volumes of the fibers $Q^{-1}(w) \cap B_r$ for $(|w| < \varepsilon)$, and then integrating over the fibers. Then the methods of integral geometry (Santalo [11]) are used to compute the volumes of the fibers. The "capsule" $\{x \in B_r \,|\, |Q(x)| < \varepsilon\}$ is partitioned in such a way as to make use of the inductive hypothesis.

Proof. Since $\{x \in B_r \,|\, |Q(x)| < \varepsilon\} = \{x \in B_r \,|\, |K_Q Q(x)| < \varepsilon K_Q\}$, and noting that $K_{K_Q Q} = 1$, it suffices to prove the theorem when $K_Q = 1$. We prove this theorem by induction on the degree d. If the degree of Q is 1, $Q(X) = \sum_{i=1}^n a_i X_i + a_0$ and $1 = 1/\max_{i>0} |a_i|$. The gradient $\nabla Q = (a_1, \cdots, a_n)$ and $1/|\nabla Q| \leqq 1$. $\operatorname{Vol}\{x \in B_r \,|\, |Q(x)| < \varepsilon\}$ is therefore less than the volume of the ball of radius r in the $n-1$ plane orthogonal to $(a_1, \cdots, a_n)$ with $(-a_0/|\nabla Q|^2)$ $(a_1, \cdots, a_n)$ as center, times $2\varepsilon/|\nabla Q|$. So, $\operatorname{Vol}\{x \in B_r \,|\, |Q(x)| < \varepsilon\} < 2\varepsilon V_{n-1} r^{n-1}$. Now we proceed by induction. Assuming the inequality proven for $d \geqq 1$ we attempt to prove it for Q of degree $d+1$. Choose x_i such that $K_{\partial Q/\partial x_i} = K_Q = 1$.

Let $S_1 = \{x \in B_r \mid |Q(x)| < \varepsilon\}$ and $S_2 = \{x \in B_r \mid |\partial Q/\partial x_i| < \varepsilon^{d/(d+1)}\}$. $\mathrm{Vol}(S_1) \leqq \mathrm{Vol}(S_1 - S_2) + \mathrm{Vol}(S_2)$. By the inductive hypothesis

$$\mathrm{Vol}(S_2) \leqq 2(\varepsilon^{d/(d+1)})^{1/d}\left[V_{n-1} + \left(\frac{d(d+1)}{2} - 1\right)\frac{A_{n-1}}{2}\right] r^{n-1}.$$

On $S_1 - S_2$, $1/|\nabla Q| \leqq 1/|\partial Q/\partial x_i| < 1/\varepsilon^{d/(d+1)}$. So, by the co-area formula (see Federer [2, Thm. 3.2.12, p. 249]),

$$\begin{aligned}\mathrm{Vol}(S_1 - S_2) &= \int_{-\varepsilon}^{\varepsilon}\left[\int_{Q^{-1}(w)\cap(S_1-S_2)} \frac{1}{|\nabla Q|}\right] dw \\ &\leqq 2\varepsilon \frac{1}{\varepsilon^{d/(d+1)}} \max\left(\mathrm{vol}\left(Q^{-1}(w) \cap (S_1 - S_2)\right)\right) \\ &= 2\varepsilon^{1/(d+1)}\left(\max \mathrm{vol}\left(Q^{-1}(w) \cap (S_1 - S_2)\right)\right)\end{aligned}$$

where $\mathrm{vol}\, Q^{-1}(w) \cap (S_1 - S_2)$ and the first integral are with respect to the induced $(n-1)$-dimensional volume on the hypersurface $Q^{-1}(w)$. Thus we will be done as soon as we show that $\mathrm{Vol}(Q^{-1}(w)) \cap (S_1 - S_2) \leqq (d+1)(A_{n-1}/2)r^{n-1}$. This follows from the following proposition which is essentially contained in Smale [13]; the general case was shown to us in a personal communication from Smale.

PROPOSITION 1 (Smale). *Let $Q: \mathbf{R}^n \to \mathbf{R}$ be a polynomial of degree $d > 1$. For any real number $r > 0$*

$$\mathrm{Vol}(Q^{-1}(w) \cap B_r) \leqq d\frac{A_{n-1}}{2} r^{n-1}.$$

For the sake of completeness, we include a sketch of this proposition. The typical line L_1 in R^n can meet $Q^{-1}(w) \cap B_r$ in at most d points. Of the ones that do, the typical one intersects $\partial B_r = S_r^{n-1}$ in two points. Using results from Santalo [11, 14.70, p. 245] we have

$$\begin{aligned}\frac{A_n}{A_1}\mathrm{Vol}(Q^{-1}(w) \cap B_r) &= \int_{Q^{-1}(w)\cap B_r \cap L_1 \neq \varnothing} \#(Q^{-1}(w) \cap B_r \cap L_1)\, dL_1 \\ &\leqq \frac{d}{2}\int_{Q^{-1}(w)\cap B_r\cap L_1 \neq \varnothing} 2\, dL \\ &\leqq \frac{d}{2}\int_{S_r^{n-1}\cap L_1 \neq \varnothing} 2\, dL_1 \\ &= \frac{d}{2}\frac{A_n}{A_1}\mathrm{Vol}(S_r^{n-1}) \\ &= \frac{d}{2}\frac{A_n}{A_1} A_{n-1} r^{n-1}.\end{aligned}$$

Q.E.D

MAIN THEOREM 2. *Let $Q: \mathbf{R}^n \to \mathbf{R}$ be a polynomial of degree $d \geqq 0$. For any real numbers $r > 0$ and $\varepsilon > 0$*

$$\frac{\mathrm{Vol}\{x \in B_r \mid |Q(x)| < \varepsilon\}}{\mathrm{Vol}\{B_r\}} < \frac{C_Q}{r}\varepsilon^{1/d}$$

where $C_Q = K_Q^{1/d} n((d(d+1))/2)$.

Proof. Recall (e.g. Santalo [11, p. 976] $V_0 = 1$, $V_2 = 2$, $V_2 = \pi$ and generally $V_n = 2\pi^{n/2}/n\Gamma(n/2)$ where Γ is the gamma function. Also, $A_0 = 2$, $A_1 = 2\pi$, $A_2 = 4\pi$ and generally $A_n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$. Thus for $n = 1$,

$$\operatorname{Vol}(x \in B_r \,|\, |Q(x)| < 2\,\varepsilon^{1/d} K_Q^{1/d} \frac{d(d+1)}{2}.$$

So, if we normalize, we have for $n = 1$,

$$\frac{\operatorname{Vol}\{x \in B_r \,|\, |Q(x)| < \varepsilon\}}{\operatorname{Vol}\{B_r\}} < \frac{\varepsilon^{1/d}}{r} K_Q^{1/d} \frac{d(d+1)}{2},$$

and for $n > 1$,

$$\frac{\operatorname{Vol}\{x \in B_r \,|\, |Q(x)| < \varepsilon\}}{\operatorname{Vol}\{B_r\}} < \frac{2\varepsilon^{1/d}}{r} K_Q^{1/d} \left[\frac{V_{n-1}}{V_n} + \left(\frac{d(d+1)}{2} - 1\right)\frac{A_{n-1}}{2V_n}\right].$$

Now $A_{n-1}/V_n = n$ and

$$\frac{V_{n-1}}{V_n} = \left(\frac{n}{n-1}\right)\frac{\Gamma(n/2)}{\Gamma((n-1)/2)\pi^{1/2}} < \frac{n}{2}. \qquad \text{Q.E.D.}$$

5. Average marking time and average input precision. We continue with the notation and setting introduced in §§ 3 and 4. We now wish to estimate the average value of $m_Q(x)$, the "marking time" to determine that $Q(x)$ is not zero for $x \in B_r$.

Recall,

$$m_Q(x) = \begin{cases} |\log_2 |Q(x)|| & \text{if } |Q(x)| \leqq 1, \\ 0 & \text{otherwise.} \end{cases}$$

From the normalized volume estimate, and assuming normalized Lebesgue measure on B_r, we see that for $0 \leqq \varepsilon \leqq 1$, $m_Q(x) \leqq |\log_2 \varepsilon|$ with probability at least $1 - C_Q\varepsilon^{1/d}/r$. This enables us to estimate the average value of $m_Q(x)$.

THEOREM 3. *Let $Q: \mathbf{R}^n \to \mathbf{R}$ be a polynomial of degree $d \geqq 1$ and let r be a positive real number. Then* Av m_Q, *the average of $m_Q(x)$ in B_r with respect to normalized Lebesgue measure, satisfies*

$$\operatorname*{Av}_{|x|<r} m_Q \leqq d\left(\log_2^+\left(\frac{C_Q}{r}\right) + \log_2 e\right)$$

where $C_Q = K_Q^{1/d} nd(d+1)/2$ and $\log^+ = \max(\log, 0)$.

Also, $\sigma(m_Q)$ the standard deviation of $m_Q(x)$ in B_r, satisfies

$$\sigma(m_Q) \leqq d \log_2 e\left(\left(\ln^+ \frac{C_Q}{r}\right)^2 + 2\ln^+ \frac{C_Q}{r} + 2\right)^{1/2}.$$

Proof. We first recall the following (from Shub-Smale [12, Part II]):

DEFINITION. Let (X, μ) be a probability space with no atoms. Let $m: X \to \mathbf{R}_+$ be a real valued nonnegative measurable function and let $f: (0, 1) \to \mathbf{R}$ be decreasing and Riemann integrable. We say that $m(x) \leqq f(\mu)$ with probability $1 - \mu$ if $\mu\{x \,|\, m(x) \leqq f(y)\} \geqq 1 - y$ for all $0 < y < 1$.

PROPOSITION 2. *Suppose as above that $m(x) < f(\mu)$ with probability $1 - \mu$. Then*
(1)

$$E(m) = \int_X m(x)\mu(dx) \leqq \int_0^1 f(\mu)\, d\mu;$$

(2)

$$\mathrm{Var}\,(m)=\int_X (m(x)-E(m))^2\mu(dx)\leqq \int_0^1 f^2(\mu)\,d\mu-(E(m))^2.$$

Returning to our proof, we let $e(\mu)=(\min(1, r\mu/C_Q))^d$ for $0\leqq\mu\leqq 1$. Thus, letting $f(\mu)=|\log_2 e(\mu)|$, we get $m_Q(x)\leqq f(\mu)$ with probability $1-\mu$. And so, by above $\int_{B_r} m_Q(x)\,dx\leqq\int_0^1 f(\mu)\,d\mu$.

Now, $\int_0^1 f(\mu)\,d\mu=-d\int_0^\gamma \log_2(\gamma\mu/C_Q)\,d\mu=\log_2 e(d\gamma+d\gamma\ln(C_Q/\gamma r))$, where $\gamma=\min(C_Q/r, 1)$. Thus, if $C_Q/r\geqq 1$, then $\gamma=1$ and we have

$$\int f(\mu)\,d\mu=d\log_2 e\left(\ln\frac{C_Q}{r}+1\right).$$

And, if $C_Q/r<1$ then $\gamma=C_Q/r$ and $\int f(\mu)\,d\mu<d\log_2 e$. So, $\int m_Q(x)\,dx\leqq\int f(\mu)\,d\mu\leqq d\log_2 e(\ln^+(C_Q/r)+1)=d(\log_2^+(C_Q/r)+\log_2 e)$.

The second estimate follows in a similar fashion, now using part 2 of Proposition 2 and the fact that $\sigma(m_Q)=\sqrt{\mathrm{Var}\,(m_Q)}$. Q.E.D.

And thus we have also shown:

COROLLARY. *Let Q: $\mathbf{R}^n\to\mathbf{R}$ be a polynomial of degree d. Then (with respect to normalized Lebesgue measure)*

$$\int_{B_1\cap Q^{-1}[-1,1]} |\ln|Q(x)\||\leqq\begin{cases}2d\ln(d)+d\ln(n)+d+\ln(K_Q) & \text{if } C_Q\geqq 1,\\ C_Q d & \text{if } C_Q<1.\end{cases}$$

By § 3 and Theorem 3 we get:

THEOREM 4. *Let $f(X)=P(X)/Q(X)$ be a rational function $f:\mathbf{R}^n\to\mathbf{R}$ of degree $d\geqq 1$. Suppose that the maximum absolute value of the coefficients of P and Q is ρ and that a real number $r\geqq 0$ and an integer $s>0$ are given. Then $\mathrm{Av}_{|x|<r}\Pi_S(P, Q, x)$, the average input precision for x and the coefficients of P and Q necessary and sufficient to evaluate $f(x)$ to within accuracy $1/2^s$ on B_r with normalized Lebesgue measure, is finite and satisfies*

$$\begin{aligned}\mathop{\mathrm{Av}}_{|x|<r}\Pi_s(P, Q, x)&\leqq s+2\mathop{\mathrm{Av}}_{|x|<r} m_Q+\log_2\max_{|x|<r}|P(x)|+1+H(d, T_f, \rho, r)\\ &\leqq s+2d\left(\log_2^+\frac{C_Q}{r}+\log_2 e\right)+2[\log_2 T_f+d\log^+ r+\log_2^+\rho]+\log_2 d+3\end{aligned}$$

where $C_Q=K_Q^{1/d}nd(d+1)/2$ and T_f, the maximum number of nonzero terms of P and of Q, is bounded by $\binom{n+1}{d}$.

So, the average loss of precision in evaluating $P(x)/Q(x)$ in B_r is loglinear and crudely bounded by

$$2d(2\log_2 d+2\log_2(n+1)+\log_2^+ r+\log_2 e)+\log_2 d+2(\log_2^+\rho+\log_2 K_Q^+)+3.$$

We may average as well over the polynomials themselves. Let $P(n, d)$ denote the vector space of real polynomials $Q:\mathbf{R}^n\to\mathbf{R}$ of degree d, and F a vector subspace determined by allowing a fixed subset of the coefficients to be nonzero, with at least one of these the coefficient of a term of degree d. Let k be the number of nonzero coefficients of degree d and m the dimension of f. Let $C(\rho)$ be the cube of side 2ρ in F i.e., $C(\rho)=\{Q\in F|$ the maximum absolute value of a coefficient of Q is less than or equal to $\rho\}$. Thus the volume of $C(\rho)=2^m\rho^m$. We normalize this volume to one, fix $r>0$ and average $\log_2^+(C_Q/r)$ over $C(\rho)$.

LEMMA 2. $\log_2^+(C_Q/r) < (1/d)\log_2^+(1/r\rho\mu^{1/k}) + \log_2 n + \log_2(d(d+1)/2)$ *with probability* $1-\mu$.

Proof. $K_Q \leqq 1/\max_{I=d}|a_I| \leqq 1/\rho\mu^{1/k}$ with probability at least $1-\mu$ since the normalized volume of

$$\{Q \in C(\rho) | \max_{|I|=d} |a_I| < \rho\mu^{1/k}\}$$

is equal to

$$\frac{2^m \rho^{m-k}(\rho\mu^{1/k})^k}{2^m \rho^m} = \mu.$$

So,

$$\log_2^+\left(\frac{C_Q}{r}\right) = \log_2^+\left[\frac{K_Q^{1/d} n d(d+1)/2}{r}\right] \leqq \frac{1}{d}\log_2^+\left(\frac{K_Q}{r}\right) + \log_2 n + \log_2 \frac{d(d+1)}{2}$$

$$\leqq \frac{1}{d}\log_2^+ \frac{1}{r\rho\mu^{1/k}} + \log_2 n + \log_2 \frac{d(d+1)}{2}$$

with probability $1-\mu$. Q.E.D.

Now let

$$\psi_k(\rho, r) = \begin{cases} \log_2(e)\left(\dfrac{1}{k} - \ln \rho r\right) & \text{if } \rho r \leqq 1, \\ \log_2(e)\dfrac{1}{k}\left(\dfrac{1}{\rho r}\right)^k & \text{if } \rho r \geqq 1. \end{cases}$$

Now Proposition 2 gives

LEMMA 3.

$$\int_{C(\rho)} \log_2^+ \frac{C_Q}{r} \leqq \log_2 n + \log_2 \frac{d(d+1)}{2} + \frac{1}{d}\psi_k(\rho, r).$$

Proof.

$$\int \log_2^+\left(\frac{1}{r\rho\mu^{1/k}}\right) = \log_2(e)\int_0^\gamma \ln\left(\frac{1}{r\rho\mu^{1/k}}\right) d\mu$$

where $\gamma = \min(1, (1/\rho r)^k)$ and the two cases give the two integrals.

THEOREM 5. *Let* F *be as above and* r, ρ *real numbers bigger than zero. Then the average marking time for a point* (Q, x) *in* $C(\rho)xB_r$ *is less than or equal to* $d\log_2 n + d\log_2(d(d+1)/2) + \psi_k(\rho, r) + d\log_2 e$.

Proof. The double integral $\iint_{C(\rho)xB_r} M$ is just the iterated single intervals by Fubini's theorem and thus Theorem 3 and Lemma 3 finish the argument.

6. The relative case. We now consider the relative case. The *relative condition* of evaluating a function f at x is the ratio ε/δ where $|\Delta x|/|x| < \delta$ implies $|\Delta f|/|f(x)| < \varepsilon$. Its logarithm$^+$ can be considered a measure of the *loss of significance* for x in evaluating $f(x)$ since it represents the loss of significant digits when input x and output $f(x)$ are given in scientific or floating point notation. Rewriting the ratio $|\Delta f|/|f|/|\Delta x|/|x|$ we get $|\Delta f|/|\Delta x| \cdot |x|/|f(x)|$ which is approximated by $|Df| \cdot |x|/|f(x)|$, the *infinitesimal condition* of the problem. (Note that by the mean value theorem, we have $|\Delta f|/|x| \leqq \sup|Df(x^*)|$ over all x^* on the line segment between x and $x+\Delta x$.) We call the logarithm$^+$ of the infinitesimal condition, the *infinitesimal loss of significance.*

As an example, we consider the problem of evaluating $f(x)=1/(1-x)$ in the unit interval. Here the infinitesimal condition is $x/(1-x)$ and the infinitesimal loss of significance is $\log^+(x/(1-x))$. So, analogous to the absolute case, we have a problem were the average loss of significance is small ($\int_0^1 \log_2^+(x/(1-x))\,dx=1$), although the average condition is infinite.

We now wish to estimate the average loss of significance for x in evaluating a general rational function $f=P/Q$. Suppose we know that $|\Delta x|<\delta'$ implies $|\Delta f|/|f|<\varepsilon$. Then we have, $|\Delta x|/|x|<\delta$ implies $|\Delta f|/|f|<\varepsilon$ where $\delta=\delta'/|x|$. So,

$$\log^+\left(\frac{\varepsilon}{\delta}\right)=\log^+\left(\frac{\varepsilon}{\delta'/|x|}\right)\leqq \log^+\left(\frac{\varepsilon}{\delta'}\right)+\log^+|x|.$$

By Lemma 1 and § 3 we have,

$$\log^+\left(\frac{\varepsilon}{\delta'}\right)\leqq \max(m_P(x), m_Q(x))+2+H(d, T_f, \rho, r).$$

So, Theorem 6 follows.

THEOREM 6. *Let $f=P/Q\colon \mathbf{R}^n\to\mathbf{R}$ be a rational function of degree $d\geqq 1$ and let $r>0$ be a real number. Then, the average loss of significance for x in evaluating $f(x)$ in B_r is bounded by*

$$\underset{|x|<r}{\mathrm{Av}}\log^+|x|+\underset{|x|<r}{\mathrm{Av}}\max(m_P(x), m_Q(x))+H(d, T_f, \rho, r)+2$$

$$\leqq \log_2 r-\frac{1}{n}+d\log_2^+\left[\frac{C_P+C_Q}{r}\right]+d\log_2 e+H(d, T_f, \rho, r)+2.$$

Proof. To integrate $\int_{B_r}\log^+|x|/\mathrm{vol}\,B_r$ use polar coordinates and integration by parts.

Also, by Theorem 2 we have,

$$\frac{\mathrm{Vol}\,\{x\in B_r \mid \min(|P(x)|, |Q(x)|)<\varepsilon\}}{\mathrm{Vol}\,B_r}\leqq\frac{(C_P+C_Q)\varepsilon^{1/d}}{r}.$$

Now proceed as in Theorem 3.

7. Loss of significance in solving linear equations. An immediate consequence of Theorem 3 is a tractable upper bound for the average loss of significance in solving systems of linear equations. This was a question of great interest to von Neumann [15]. Here the problem is: *Given $A\in\mathbf{R}^{n^2}$*, a real $n\times n$ invertible matrix and a vector $b\in\mathbf{R}^n$, *solve $Ax=b$* for x. What is the loss of significance?

Suppose an error Δb in input causes an error Δx in solution, i.e., $A(x+\Delta x)=b+\Delta b$. By linearity, we have $A(\Delta x)=\Delta b$ and by invertibility, we have $A^{-1}(\Delta b)=\Delta x$. So, the *relative condition* of the problem is

$$\frac{\dfrac{|\Delta x|}{|x|}}{\dfrac{|\Delta b|}{|b|}}=\frac{|\Delta x||b|}{|\Delta b||x|}=\frac{|A^{-1}(\Delta b)|}{|(\Delta b)|}\frac{|Ax|}{|x|}$$

where $||$ is some standard norm. We note that the right-hand side is bounded above by $K_A=\|A^{-1}\|\,\|A\|$, where $\|A\|=\max|Ax|/|x|=\max_{|x|=1}|Ax|$ is the *operator norm* of A. Numerical analysts, see e.g. Moler [9], call K_A the *condition number* of the matrix A, and it can be considered the worst case measure of the relative condition for the problem of solving $Ax=b$ (with fixed A, varying b). Analogously, we consider $\log K_A$ which is a measure of the worst case *loss of significance* of the system.

THEOREM 7. *With respect to the Euclidean norm* $||$ *in* $\mathbf{R}^n$ *and* $\mathbf{R}^{n^2}$ *and normalized Lebesgue measure,*

$$\underset{A\in B_1\subseteq \mathbf{R}^{n^2}}{\operatorname{Av}} \ln K_A \leqq \ln n - \left(\frac{n-1}{2}\right)\ln(n-1)+n+4n\ln n.$$

Proof. Let $A=(a_{ij})$ and $A^{-1}=(b_{ij}/\operatorname{Det} A)$ where b_{ij} is the ijth cofactor of A. Suppose $A\in B_1$. Then, $\|A\|\leqq 1$: For suppose $|x|=1$. Let a_i be the vector $(a_{i1},\cdots,a_{in})$. Then,

$$|Ax|=\left(\sum_{i=1}^n (a_i\cdot x)^2\right)^{1/2}\leqq\left(\sum_{i=1}^n |a_i|^2|x|^2\right)^{1/2}=\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}=\left(\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2\right)^{1/2}\leqq 1$$

since $A\in B_1\subset \mathbf{R}^{n^2}$.

To estimate $\|A^{-1}\|$ we estimate $|b_{ij}|$: Suppose $b_{ij}=\operatorname{Det}(c_{lk})$ where c_{lk} are entries from A. Let $C_k=(\sum_{l=1}^{n-1} c_{lk}^2)^{1/2}$ be the length of the kth column vector. Then $|b_{ij}|\leqq \prod_{k=1}^{n-1} C_k$. Since $A\in B_1$, $\sum_{k=1}^{n-1} C_k^2\leqq 1$. And so, $\prod_{k=1}^{n-1} C_k\leqq (1/\sqrt{n-1})^{n-1}$. Thus, $|b_{ij}|\leqq (1/\sqrt{n-1})^{n-1}$. So we have,

$$K_A=\|A^{-1}\|\,\|A\|\leqq\|A^{-1}\|\leqq\frac{n}{\operatorname{Det} A}\max|b_{ij}|$$

$$\leqq\frac{n}{\operatorname{Det} A}\frac{1}{(n-1)^{(n-1)/2}}.$$

Therefore,

$$\ln K_A\leqq \ln n-\left(\frac{n-1}{2}\right)\ln(n-1)+|\ln|\operatorname{Det} A||.$$

Now, $\operatorname{Det} A$ is a polynomial in n^2 variables of degree n and $K_{\operatorname{Det} A}=1$. By the corollary to Theorem 3,

$$\underset{A\in B_1}{\operatorname{Av}}\ln K_A\leqq \ln n-\left(\frac{n-1}{2}\right)\ln(n-1)+2n\ln n+n\ln n^2+n$$

$$=\ln n-\left(\frac{n-1}{2}\right)\ln(n-1)+n+4n\ln n.$$

This estimate follows from very general considerations, and so one would expect better results for this particular problem. Indeed, using methods different from ours, Eric Kostlan [8] has shown that although the average condition number (with respect to the standard Gaussian measure or the space of $n\times n$ complex matrices) is infinite, the average loss of significance is less than $\frac{5}{2}\ln n$. Adrian Ocneanu [10] has calculated sharp upper and lower bounds, $(3+\varepsilon)\ln n$ and $(\frac{2}{3}-\varepsilon)\ln n$ respectively, where ε can be made as small as desired by setting a lower bound on n.

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