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THE INTEGRAL HOMOLOGY OF SMALE DIFFEOMORPHISMS

M. MALLER† and M. SHUB‡

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THE dynamics and topology of diffeomorphisms are closely related. In this paper we show how to deduce information about the topology of a large class of diffeomorphisms from local information about the dynamics.

Let M be a connected closed manifold of dimension n, and $f: M \to M$ a diffeomorphism. A closed f-invariant subset $\Lambda \subset M$ is said to be *hyperbolic* if there exists an invariant splitting of the tangent bundle over Λ, $T_\Lambda M = E^s \oplus E^u$, and constants $C > 0$, $0 < \lambda < 1$ such that $\|Tf^n|E^s\| \leq C\lambda^n$ for all $n \geq 0$, and $\|Tf^n|E^u\| \leq C\lambda^n$ for all $n \leq 0$. f is said to satisfy *Axiom A* if the non-wandering set, $\Omega(f) = \{x \in M$: for all neighborhoods U of x, $f^k(U) \cap U \neq \emptyset$, for some $k > 0\}$, is hyperbolic and the periodic points of f are dense in $\Omega(f)$. For these diffeomorphisms Smale's spectral decomposition theorem says $\Omega(f)$ is a finite disjoint union of closed f-invariant subsets called *basic sets*, $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_s$ [14]. The *index* of Ω_i is the fiber dimension of E^u/Ω_i.

For each Ω_i let $W^s(\Omega_i) = \{x \in M: d(f^n(x), \Omega_i) \to 0$ as $n \to +\infty\}$ and $W^u(\Omega_i) = \{x \in M: d(f^n(x), \Omega_i) \to 0$ as $n \to -\infty\}$ One writes $\Omega_i \geq \Omega_j$ if $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \emptyset$. f is said to have *no cycles* if $\geq$ can be extended to a total ordering of the basic sets. If in fact $W^u(\Omega_i) \cap W^s(\Omega_j) = \emptyset$ whenever index $(\Omega_i) <$ index (Ω_j) we will say f has an *index-compatible* ordering. Finally, if $W^u(x)$ and $W^s(y)$ have transverse intersection for all $x, y \in \Omega(f)$, then f is said to satisfy the strong transversality condition.

A diffeomorphism is said to be *Smale* if it satisfies Axiom A and strong transversality and has zero dimensional Ω. Our results hold under somewhat weaker hypotheses. We will write $f \in \mathscr{F}$ if f satisfies Axiom A and no cycles, has zero dimensional Ω and an index compatible ordering on the basic sets.

Basic sets of Axiom A diffeomorphisms admit Markov partitions. When Ω_i is zero-dimensional the partition can be constructed so that Ω_i is topologically conjugate to the subshift of finite type Σ_{A_i}, where A_i is the 0–1 intersection matrix of f on the partition. Furthermore E^u/Ω_i is orientable, and if the partition is sufficiently fine the orientation numbers of $Tf|E^u$ are constant on rectangles, so the intersection numbers of f can be recorded with signs. The resulting integral matrix B_i is called a *signed representative* of Ω_i [3].

Let C_* be a free finitely generated Z-complex $0 \to C_m \to \ldots \to C_0 \to 0$. We will say C_* is a *complex of* M if $H_*(C_*) \cong H_*(M; Z)$. It follows that $H_*(C_*; R) \cong H_*(M; R)$ for all coefficient rings R. Given a chain map $E: C_* \to C_*$ we will say the pair (C_*, E) is an R-*endomorphism of* f if there exists an isomorphism which conjugates E_* and f_*

$$\begin{array}{ccc} H_*(C_*; R) & \xrightarrow{\cong} & H_*(M; R) \\ \downarrow E_* & \circlearrowleft & \downarrow f_* \\ H_*(C_*; R) & \xrightarrow{\cong} & H_*(M; R) \end{array}$$

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If $F_1, \ldots, F_s$ are square integral matrices, we will say an integral matrix E is a *nilpotent extension* of $F_1, \ldots, F_s$ if E is similar over the integers to a matrix of the form

$$\begin{pmatrix} N_1 & & & & & & & \\ & F_1 & & & & * & & \\ & & N_2 & & & & & \\ & & & F_2 & & & & \\ & & & & N_3 & & & \\ & 0 & & & & \ddots & & \\ & & & & & & F_s & \\ & & & & & & & N_{s+1} \end{pmatrix}$$

where the N_i are square nilpotent matrices and the above diagonal entries * are arbitrary.

THEOREM 1. *Suppose M is orientable and $f \in \mathscr{F}$. For $0 \le k \le n$ let $B_1^k, \ldots, B_{s_k}^k$ be signed representatives of the basic sets of index k. Then there exists a Z-endomorphism of f (C_*, E) such that for each k E_k is a nilpotent extension of the $B_1^k, \ldots, B_{s_k}^k$.* □

This theorem extends previous results of Bowen and Franks [3] who studied a single basic set on an orientable manifold. If f is actually a Smale diffeomorphism, Theorem 1 can also be proved using Pixton's theory of fitted rectangular decompositions, without requiring that M be orientable [10].

Fitted diffeomorphisms are Smale diffeomorphisms which preserve a handle decomposition of the manifold [13]. They are dense in the C^0-topology on $\mathrm{Diff}^r(M)$ and exhibit a particularly simple structure and a close connection between dynamics and homology theory. Examples of Bowen [2], Newhouse [9], and Pixton [11] show there exist Smale diffeomorphisms which are not fitted. In Pixton's terminology these diffeomorphisms are dynamically wild. From Theorem 1 and the techniques of [13] we obtain

THEOREM 2. *Let M be a 2-connected manifold with torsion free homology, and $\dim M \ge 6$. If $f \in \mathscr{F}$ has at least one fixed source and one fixed sink, then f is isotopic to an omega-conjugate fitted diffeomorphism.* □

In principle this theorem reduces to algebra the problem of determining the omega-conjugacy types of Smale diffeomorphisms in the component of f, using the known machinery for fitted diffeomorphisms.

Let A and B be square integral matrices; they are said to be *shift equivalent* ($A \underset{\text{shift}}{\sim} B$) if there exist integral matrices R and S and an integer $k > 0$ such that $A^k = RS$, $B^k = SR$, and $SA = BS$, $AR = RB$.

Bowen and Franks [3] proved that for a single basic set Ω_i of index k, the signed B_i^k is shift equivalent to the map induced by f in the relative k-homology of a filtration pair for Ω_i (see section 2). A main tool in the proof of Theorem 1 is a characterization of shift equivalence of integral matrices.

For square integral matrices A and B, we will say that A is *nil-equivalent* to B ($A \underset{\text{nil}}{\sim} B$) if there are nilpotent extensions

$$\begin{pmatrix} N_1 & & * \\ & A & \\ 0 & & N_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} N_3 & & * \\ & B & \\ 0 & & N_4 \end{pmatrix}$$

of A and of B which are similar over Z.

PROPOSITION. *If A and B are square integral matrices then $A \underset{\text{shift}}{\sim} B$ if and only if $A \underset{\text{nil}}{\sim} B$.* □

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§1. NILPOTENT EXTENSIONS

Suppose L is a finitely generated Z-module and $\alpha: L \to L$ a linear map. Let $\mathrm{Nil}(\alpha) = \{v \in L:$ for some $k \geq 0\ \alpha^k(v) = 0\}$. Then $\mathrm{Nil}(\alpha)$ is invariant under α. Let $\bar{L}$ be the quotient module $L/\mathrm{Nil}(\alpha)$ and $\bar{\alpha}: \bar{L} \to \bar{L}$ the injective quotient map. If L is free then $\bar{L}$ is free as well, so the sequence $0 \to \mathrm{Nil}(\alpha) \to L \to \bar{L} \to 0$ splits and α can be represented by a matrix

$$\begin{pmatrix} \alpha/\mathrm{Nil}(\alpha) & * \\ 0 & \bar{\alpha} \end{pmatrix}.$$

Let $\mathbf{M}$ be the category whose objects are pairs (L, α) and whose morphisms i: $(L, \alpha) \to (M, \beta)$ are linear maps $i: L \to M$ such that $\beta i = i\alpha$. Then $\bar{\cdot}$ is a functor from $\mathbf{M}$ to itself. $\bar{\cdot}$ fails to preserve exactness. Given a short exact sequence in $\mathbf{M}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{j} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{j} & N & \longrightarrow & 0 \end{array}$$

in the quotient sequence $\bar{i}$ is 1-1 and $\bar{j}$ is onto but in general image $(\bar{i}) \neq$ kernel $(\bar{j})$. If $\bar{w} \in$ kernal $(\bar{j})$ then for some $k \geq 0$, $0 = \gamma^k(jw) = j(\beta^k w)$ so there exists $v \in L$, $i(v) = \beta^k(w)$. Therefore the map induced by $\bar{\beta}$ on kernel $(\bar{j})$/image $(\bar{i})$ is always nilpotent. If $\bar{\alpha}$ is onto then exactness is preserved: for if $\bar{\alpha}^k(\bar{z}) = \bar{v}$ then $\bar{\beta}^k(\overline{iz}) = \bar{i}(\bar{v}) = \bar{\beta}^k(\bar{w})$ so $\bar{i}(\bar{z}) = \bar{w}$. In particular, exactness is preserved in the category of finite dimensional vector spaces. Similarly, if α is nilpotent then $\bar{j}$ is 1-1, and again exactness is preserved. However consider the example

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{i} & Z \oplus Z & \xrightarrow{j} & Z & \longrightarrow & 0 \\ & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ & & (2) & & \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} & & (0) & & \end{array}$$

where $i(v) = (v, 0)$ and $j(v, w) = w$. The quotient sequence is $0 \to Z \xrightarrow{\bar{i}} Z \xrightarrow{\bar{j}} 0$ which is not exact since $\overline{(0, 1)} \notin$ image $(\bar{i})$. We will need the following fact.

Lemma 1. *Given two exact sequence in* $\mathbf{M}$ $0 \to (A, \alpha) \xrightarrow{i} (B, \beta) \xrightarrow{j} (C, \gamma) \to 0$ *and* $0 \to (B, \beta) \xrightarrow{k} (D, \delta) \xrightarrow{l} (E, \varepsilon) \to 0$, *suppose that* A *and* E *are finite*, α *and* ε *are nilpotent, and* $\gamma: C \to C$ *is an isomorphism. Then* $Nil(\delta)$ *is finite and* $(\bar{D}, \bar{\delta}) \cong (C, \gamma)$.

Proof: Observe that $\mathrm{Nil}(\beta) = i(A)$; therefore $(C, \gamma) \cong (\bar{B}, \bar{\beta})$, and exactness is preserved by $\bar{\cdot}$ in the second sequence. Since ε is nilpotent $(\bar{B}, \bar{\beta}) \cong (\bar{D}, \bar{\delta})$. If $\mathrm{Nil}(\delta)$ were infinite, so would kernel $(l) \cap \mathrm{Nil}(\delta) = \mathrm{im}(k) \cap \mathrm{Nil}(\delta) \cong \mathrm{Nil}(\beta) = i(A)$ but A is finite. □

If A is an $(n \times n)$ integral matrix then $\bar{A}$ is defined up to similarity over Z by allowing A to act on Z^n. Recall that in the category of integral matrices shift equivalence coincides with the *a priori* stronger relation of strong shift equivalence. Let $A \approx_1 B$ if there exist integral matrices R and S such that $A = RS$ and $B = SR$, i.e. $A \underset{\text{shift}}{\sim} B$ with the lag $k = 1$. *Strong shift equivalence* ($A \approx B$) is the transitive closure of $\approx_1$. These relations were introduced by Williams in [16]. The so called "Williams problem" is whether the two relations also coincide in the category of non-negative matrices [17]. In the category of integral matrices similarity over Z implies strong shift equivalence; if $PAP^{-1} = B$ let $S = PA$ and $R = P^{-1}$. Therefore shift equivalence is also a relation on linear maps.

LEMMA 2. *Let A, B be square integral matrices. Then* $\bar{A} \underset{Z}{\sim} \bar{B} \Rightarrow A \underset{nil}{\sim} B \Rightarrow A \underset{shift}{\sim} B \Rightarrow \bar{A} \underset{Q}{\sim} \bar{B}$.

Proof: (i) $A \underset{Z}{\sim} \begin{pmatrix} A/\mathrm{Nil}(A) & * \\ 0 & \bar{A} \end{pmatrix}$ so $A \underset{\mathrm{nil}}{\sim} \bar{A} \underset{Z}{\sim} \bar{B} \underset{\mathrm{nil}}{\sim} B$.

(ii) Let $*$ represent an arbitrary additional final column. Then

$$(A^*)\begin{pmatrix} Id \\ 0 \end{pmatrix} = A \quad \text{and} \quad \begin{pmatrix} Id \\ 0 \end{pmatrix}(A^*) = \begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix}, \quad \text{so } A \approx_1 \begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix}. \quad (\text{"bordering"})$$

Similarly

$$A \approx_1 \begin{pmatrix} 0 & * \\ 0 & A \end{pmatrix}.$$

Any nilpotent block N is similar over Z to a strictly upper triangular matrix so by iterating bordering $A \underset{\mathrm{shift}}{\sim} \begin{pmatrix} N_{1_A} & * \\ 0 & N_2 \end{pmatrix}$ which proves the second implication. (iii) Bowen and Franks [3] proved that if $A \underset{\mathrm{shift}}{\sim} B$, then for any abelian group G, regarding A and B as maps $G^n \to G^n$, $\underset{G}{\varprojlim} A \cong \varprojlim B$. Let $G = Q$; taking inverse limits over Q we have $\bar{A} \cong \varprojlim A \otimes l_Q \cong \varprojlim B \otimes l_Q \cong \bar{B}$. □

LEMMA 3. *Let B be an $(n \times n)$ integral matrix and suppose $L \cong Z^k$ is a B-invariant sublattice of Z^n such that $B(Z^n) \subset L \subset Z^n$. Then $B \approx_1 B/L$.*

Proof: Let A be a matrix representing $B/L: L \to L$ in the basis inherited from Z^k, and let R be the matrix of B regarded as a map $Z^n \to L \cong Z^k$, and let S be the matrix of the inclusion $Z^k \cong L \subset Z^n$. Then $RS = A$ and $SR = B$. □

It follows that the $(k+n) \times (k+n)$ matrices

$$\begin{pmatrix} A & R \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \mathrm{R} \\ 0 & \mathrm{B} \end{pmatrix}$$

and similar over Z. For:

$$\begin{pmatrix} Id & 0 \\ S & Id \end{pmatrix}\begin{pmatrix} A & R \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & R \\ 0 & B \end{pmatrix}\begin{pmatrix} Id & 0 \\ S & Id \end{pmatrix}$$

Example: It seems natural to ask whether the non-singular quitient map $\bar{\cdot}$ can be changed by a nilpotent extension, i.e. if N is nilpotent is

$$\overline{\begin{pmatrix} A & * \\ 0 & N \end{pmatrix}} \underset{Z}{\sim} \bar{A}?$$

We show this fails as follows. Let C and D be non-singular $(n \times n)$ integral matrices which are similar over Q but not over Z. Let $PC = DP$ where P is integral; then $C \underset{Z}{\sim} D/P(Z^n)$ and there exists an integer k such that $kD(Z^n) \subset P(Z^n)$. Letting $B = kD$ and $L = P(Z^n)$ we have

$$\begin{pmatrix} kC & * \\ 0 & 0 \end{pmatrix} \underset{Z}{\sim} \begin{pmatrix} 0 & * \\ 0 & kD \end{pmatrix} \quad \text{but } kC \underset{Z}{\not\sim} kD.$$

Thus

$$\overline{\begin{pmatrix} kC & * \\ 0 & 0 \end{pmatrix}} \underset{Z}{\not\sim} \overline{kC}.$$

For example, let

$$C = \begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{D} = \begin{pmatrix} -1 & 3 \\ -2 & 1 \end{pmatrix}$$

which represent distinct ideal classes of the ring $Z[\sqrt{(5)}\,i]$. Then

$$\overline{\begin{pmatrix} 0 & -15 & -5 & 0 \\ 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \underset{Z}{\simeq} \overline{\begin{pmatrix} 0 & 0 & -5 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & -6 & 3 \end{pmatrix}} \underset{Z}{\simeq} \begin{pmatrix} -3 & 9 \\ -6 & 3 \end{pmatrix}$$

PROPOSITION 1. *In the category of integral matrices* $A \underset{nil}{\simeq} B$ *if and only if* $A \underset{shift}{\simeq} B$.

Proof: $\underset{\text{nil}}{\simeq} \Rightarrow \underset{\text{shift}}{\simeq}$ was proved in Lemma 2. To prove the converse, suppose that $A \approx_1 B$ and $A = RS$, $B = SR$. Then, as above:

$$\begin{pmatrix} A & R \\ 0 & 0 \end{pmatrix} \underset{Z}{\simeq} \begin{pmatrix} 0 & R \\ 0 & B \end{pmatrix}$$

Therefore

$$A \underset{\text{nil}}{\simeq} \begin{pmatrix} A & R \\ 0 & 0 \end{pmatrix} \underset{Z}{\simeq} \begin{pmatrix} 0 & R \\ 0 & B \end{pmatrix} \underset{\text{nil}}{\simeq} B \qquad \square$$

§2. Z-ENDOMORPHISMS

Let $f \in \mathscr{F}$. Number the basic sets Ω_i^k, $0 \le k \le n$, $1 \le i \le s_k$, where $k = \text{index}\ \Omega_i^k$ and if $i < j$ then $W^u(\Omega_i^k) \cap W^s(\Omega_j^k) = \emptyset$. There exist filtrations of M with one basic set added at each stage, that is, a sequence of submanifolds with boundary M_i^k such that

(i) $M_{i-1}^k \subset M_i^k$ (if $i = 1$ let $M_0^k = M_{s_{k-1}}^{k-1}$).

(ii) $f(M_i^k) \subset \text{int}\, M_i^k$

(iii) $\bigcap_{n \in Z} f^n(\overline{M_i^k - M_{i-1}^k}) = \Omega_i^k$

If we require in addition that the boundaries of M_i^k and M_{i-1}^k meet transversely, then in the language of [3] M_i^k, M_{i-1}^k are a filtration pair for Ω_i^k.

We suppose that signed representatives B_i^k are given for Ω_i^k. Let $M_k = \bigcup_{i=1}^{s_k} M_i^k = M_{s_k}^k$. We will show that $f_{*_k}: H_k(M_k, M_{k-1}, Z)\circlearrowleft$ is shift equivalent to a matrix B_k which is itself a nilpotent extension of the signed representatives $B_1^k, \ldots, B_{s_k}^k$.

We recall the following facts: Bowen proved that $f_{*j}: H_j(M_i^k, M_{i-1}^k; Z)\circlearrowleft$ is nilpotent for $j \ne k$ [1]. Bowen and Franks proved that if M is orientable then $f_{*_k}: H_k(M_i^k, M_{i-1}^k; Z)\circlearrowleft$ is shift equivalent to B_i^k, and nilpotent on the torsion summand of $H_k(M_i^k, M_{i-1}^k; Z)$ [3]. Furthermore, if $0 \to (A, \alpha) \to (B, \beta) \to (C, \gamma) \to 0$ is an exact sequence in $\mathbf{M}$ and α or γ are nilpotent, then the other two maps are shift equivalent. [3, Lemma 3.4].

LEMMA 4. $f_{*_k}: H_k(M_k, M_{k-1}; Z)\circlearrowleft$ *is shift equivalent to a matrix* B_k *which is a nilpotent extension of the* $B_1^k, \ldots, B_{s_k}^k$.

Proof: Consider first the exact sequence of the triple $M_{k-1} \subset M_1^k \subset M_2^k$ (all coefficients are Z, all maps induced by f.)

$$\begin{array}{ccccccccc} H_{k+1}(M_2^k, M_1^k) & \to & H_k(M_1^k, M_{k-1}) & \xrightarrow{i} & H_k(M_2^k, M_{k-1}) & \xrightarrow{j} & H_k(M_2^k, M_1^k) & \to & H_{k-1}(M_1^k, M_{k-1}) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ H_{k+1}(M_2^k, M_1^k) & \to & H_k(M_1^k, M_{k-1}) & \xrightarrow{i} & H_k(M_2^k, M_{k-1}) & \xrightarrow{j} & H_k(M_2^k, M_1^k) & \to & H_{k-1}(M_1^k, M_{k-1}) \end{array}$$

where $\beta \underset{\text{shift}}{\sim} B_1^k$ and $\delta \underset{\text{shift}}{\sim} B_2^k$. α and ε are nilpotent so applying $\bar{\ }$ we obtain

$$\begin{array}{ccccccccc} 0 \to & \bar{H}_k(M_1^k, M_{k-1}) & \xrightarrow{\bar{i}} & \bar{H}_k(M_2^k, M_{k-1}) & \xrightarrow{\bar{j}} & \bar{H}_k(M_2^k, M_1^k) & \to 0 \\ & \downarrow \bar{\beta} & & \downarrow \bar{\gamma} & & \downarrow \bar{\delta} & \\ 0 \to & \bar{H}_k(M_1^k, M_{k-1}) & \to & \bar{H}_k(M_2^k, M_{k-1}) & \to & \bar{H}_k(M_2^k, M_1^k) & \to 0 \end{array}$$

which may fail to be exact, but the map induced by $\bar{\gamma}$ on ker $(\bar{j})/\text{im}(\bar{i})$ is nilpotent. Now the sequence $0 \to \ker(\bar{j}) \overset{\text{inc}}{\to} \bar{H}_k(M_2^k, M_{k-1}) \xrightarrow{\bar{j}} \bar{H}_k(M_2^k, M_1^k) \to 0$ is exact and the right hand term is free so $\bar{\gamma}$ can be represented by a matrix:

$$\begin{pmatrix} \bar{\gamma}/\ker(\bar{j}) & * \\ 0 & \bar{\delta} \end{pmatrix}$$

Also $0 \to \text{im}(\bar{i}) \overset{\text{inc}}{\to} \ker(\bar{j}) \to \ker(\bar{j})/\text{im}(\bar{i}) \to 0$ is exact, so $\bar{\gamma}/\ker(\bar{j}) \underset{\text{shift}}{\sim} \bar{\gamma}/\text{im}(\bar{i}) \underset{z}{\sim} \bar{\beta} \underset{\text{nil}}{\sim} \beta \underset{\text{shift}}{\sim} B_1^k$. Similarly $\bar{\delta} \underset{\text{nil}}{\sim} \delta \underset{\text{shift}}{\sim} B_2^k$. It follows that $\gamma = f_{*k}\colon H_k(M_2^k, M_{k-1})\circlearrowleft$ is shift equivalent to a nilpotent extension of B_1^k and B_2^k. The lemma follows by induction on the number of basic sets of index k. □

Let (D_*, F) be the complex $D_k = \bar{H}_k(M_k, M_{k-1}; Z)$, $F_k = \bar{f}_{*_k}\colon D_k\circlearrowleft$. It follows from the proof of [3 Lemma 3.3] that f_{*_k} is nilpotent on the torsion summand of $H_k(M_k, M_{k-1}; Z)$ so D_k is a free Z-module and $F_k \underset{\text{nil}}{\sim} f_{*k} \underset{\text{shift}}{\sim} B_k$, so $F_k \underset{\text{shift}}{\sim} B_k$. We will prove Theorem 1 by constructing a Z-endomorphism of f from a nilpotent extension of (D_*, F).

We observe first that $H_*(M; Z)$ and $H_*(D_*)$ have isomorphic free summands. Let K be a field, and, for $0 \le j \le n$ let $X_j = \bigcap_{n \ge 0} f^n(M_j) \cong \varprojlim M_j \xrightarrow{f} M_j$. Using Čech theory $\varprojlim \{H_L(M_j, M_{j-1}; K)\circlearrowleft f_{*_L}\} \cong \check{H}_L(\varprojlim (M_j, M_{j-1})\circlearrowleft f; K) \cong \check{H}_L(X_j, X_{j-1}; K)$. Therefore, for $L \ne j$ it follows from the nilpotence of f_{*L} that $\check{H}_L(X_j, X_{j-1}; K) = 0$. Now $X_n = M$ and X_0 is discrete; since K is a field it follows that the complex $\check{C}_*^K$, $\check{C}_j^K = H_j(X_j, X_{j-1}; K)$ carries the K-homology of M [15, p. 205]. On the other hand, by the universal coefficient theorem $H_j(M_j, M_{j-1}; K) \cong (H_j(M_j, M_{j-1}; Z) \otimes K) \oplus (H_{j-1}(M_j, M_{j-1}; Z) * K)$. Since $f_{*_{j-1}}$ is nilpotent on the second term on the right, $\check{C}_j^K \cong \varprojlim (H_j(M_j, M_{j-1}; Z) \otimes k)\circlearrowleft (f_{*_j} \otimes 1_K) \cong \varprojlim (D_j \otimes K)\circlearrowleft (F_j \otimes 1_K)$. Now let $K = Q$; we have $\varprojlim (D_j \otimes Q)\circlearrowleft (F_j \otimes 1_Q) \cong (D_j \otimes Q)\circlearrowleft F_j \otimes 1_Q$. Therefore $H_*(M; Q) \cong H_*(\check{C}_*^Q) \cong H_*(D_* \otimes Q) = H_*(D_*) \otimes Q$. Therefore $H_*(M; Z)$ and $H_*(D_*)$ have the same free summands.

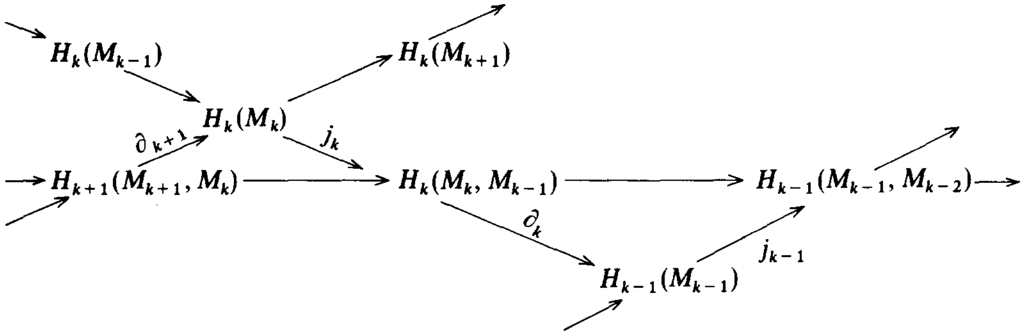
Remark. It follows from the work of Bowen and Franks that for K a field the inverse limit $\varprojlim B_j$: $K^{n_j} \to K^{n_j}$ where $n_j = \text{rank}(B_j)$ is a K-endomorphism of f. For $B_j \underset{\text{shift}}{\sim} f_{*j}\colon H_j(M_j, M_{j-1}; K)\circlearrowleft$ so by [3, 1.1] $\varprojlim B_j \cong \varprojlim f_{*_j} \cong \check{C}_j^k$. However more delicate methods are needed working with Z-coefficients. In particular B_j may be injective but over Z, $\varprojlim B_j = 0$.

The next lemma shows we obtain the integral homology of M and f by applying $\bar{\ }$ again, to $H_*(D_*)\circlearrowleft F_*$.

LEMMA 5. *Nil*$(F_{*_k}\colon H_k(D_*)\circlearrowleft)$ *is finite for all* k, *and* $(\bar{H}_*(D_*), \bar{F}_*) \cong (H_*(M; Z), f_*)$.

Proof: We show that for $0 \le k \le n$ there exists an F_{*_k}-invariant subgroup $A_k \subset H_k(D_*)$ and an eppimorphism $\emptyset_k\colon A_k \twoheadrightarrow H_k(M, Z)$ conjugating f_{*_k} and F_{*_k}/A_k; furthermore kernel $(\emptyset_k)$ and $H_k(D_*)/A_k$ are finite groups on which F_{*_k} induces nilpotent maps. The result then follows from Lemma 1 above.

Consider the diagram of the exact sequences of (M_{k-1}, M_{k-2}), (M_k, M_{k-1}) and $M_{k+1}, M_k)$. (All coefficients are Z.)



First we prove by induction on L that $f_{*_k}: H_k(M_L)\circlearrowleft$ is nilpotent for $k > L$. When $L = 0$ consider $\emptyset \subset M_1^0 \subset \ldots \subset M_{s_0}^0$. Now $(M_1^0, \emptyset)$ is a filtration pair for Ω_1^0 so in this case the claim follows from [1]. Assume inductively that $f_{*_k}: H_k(M_i^L)\circlearrowleft$ is nilpotent for $k > L$. Then in the exact sequence $\to H_k(M_i^L) \to H_k(M_{i+1}^L) \to H_k(M_{i+1}^L, M_i^L) \to$ the maps induced on the end terms are nilpotent, hence on the middle term as well, which proves the claim. In particular, for all k, $f_{*_k}: H_k(M_{k-1})\circlearrowleft$ is nilpotent so the maps $\bar{j}_k: \bar{H}_k(M_k) \to \bar{H}_k(M_k, M_{k-1})$ are injective.

Next suppose $k < L$ and consider the exact sequence $H_{k+1}(M_{L+1}, M_L) \to H_k(M_L) \xrightarrow{i} H_k(M_{L+1})$. The map induced by f on the left is nilpotent so $\bar{i}: \bar{H}_k(M_L) \to \bar{H}_k(M_{L+1})$ is injective for $k < L$. In addition, for $k \le L$ we will prove by induction that $\bar{i}$ is surjective. First let $L = n-1$. Taking a power of f if necessary we can assume that $f_{*_k}: H_k(M_n, M_{n-1})\circlearrowleft$ is zero for $k < n$. In the diagram

$$\begin{array}{ccc} H_k(M_n, M_{n-1}) & \xrightarrow{f_{*_k}} & H_k(M_n, M_{n-1}) \\ & f_* \searrow \quad \nearrow (\mathrm{inc})_* & \\ & H_k(f(M_n), f(M_{n-1})) & \end{array}$$

the map induced by f on the left is an isomorphism, so $(\mathrm{inc})_*$ is also zero. Consider the diagram

$$\begin{array}{ccccccc} \to H_k(M_{n-1}) & \xrightarrow{i} & H_k(M_n) & \to & H_k(M_n, M_{n-1}) & \to \\ \uparrow \mathrm{inc}_* & & \uparrow \mathrm{inc}_* & & \uparrow \mathrm{inc}_* & \\ \to H_k(f(M_{n-1})) & \xrightarrow{i} & H_k(f(M_n)) & \to & H_k(f(M_n), f(M_{n-1})) & \to \end{array}$$

The right hand $(\mathrm{inc})_*$ is zero while the middle $(\mathrm{inc})_*$ is an isomorphism. It follows that $i: H_k(M_{n-1}) \to H_k(M_n)$ is onto, for $k \le n-1$. To continue the induction, observe that if $k \leqq n-2$ in the left square above both horizontal maps induce isomorphisms under $\bar{\cdot}$. Therefore, for $k \le n-2$ $\overline{(\mathrm{inc})_*}: \bar{H}_k(f(M_{n-1}) \to \bar{H}_k(M_{n-1})$ is an isomorphism, and the induction continues.

For all k we obtain a commutative diagram

$$\begin{array}{ccccc} \bar{H}_k(M_k) & \twoheadrightarrow & \bar{H}_k(M_n) & \cong & H_k(M; Z) \\ \downarrow \bar{f}_{*_k} & & \downarrow \bar{f}_{*_k} & & \downarrow f_* \\ \bar{H}_k(M_k) & \twoheadrightarrow & \bar{H}_k(M_n) & \cong & H_k(M; Z) \end{array}$$

Let $A_k = (\bar{j}_k(\bar{H}_k(M_k))/\bar{j}_k \circ \bar{\partial}_{k+1}(D_{k+1})) \subset H_k(D_*)$. A_k is invariant under $F_{*_k}: H_k(D_*)\circlearrowleft$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} A_k & \xrightarrow{\cong} & \dfrac{\bar{H}_k(M_k)}{\bar{\partial}_{k+1}(D_{k+1})} & \twoheadrightarrow & \dfrac{\bar{H}_k(M_k)}{\mathrm{kernel}\,(\bar{i}_k)} & \xrightarrow{\cong} & \bar{H}_k(M_{k+1}) & \xrightarrow{\cong} & H_k(M; Z) \\ \downarrow F_{*_k} & & \downarrow & & \downarrow & & \downarrow & & \downarrow f_{*_k} \\ A_k & \xrightarrow{\cong} & \dfrac{\bar{H}_k(M_k)}{\bar{\partial}_{k+1}(D_{k+1})} & \twoheadrightarrow & \dfrac{\bar{H}_k(M_k)}{\mathrm{kernel}\,(\bar{i}_k)} & \xrightarrow{\cong} & \bar{H}_k(M_{k+1}) & \xrightarrow{\cong} & H_k(M; Z) \end{array}$$

Let $\emptyset_k\colon A_k \twoheadrightarrow H_k(M, Z)$ be the composition of the rows. We have a diagram of exact sequences

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \text{(kernel } (\bar{i}_k)/\text{image } (\bar{\bar{\partial}}_{k+1}) & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & A_k & \longrightarrow & H_k(D_*) & \longrightarrow & \dfrac{\text{kernel } (\bar{\partial}_k)}{\text{image } (\bar{j}_k)} \to 0 \\ & & \downarrow{\scriptstyle \emptyset_k} & & & & \\ & & H_k(M, Z) & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

By construction F_{*_k} is nilpotent on kernel $(\emptyset_k) \cong$ kernel $(\bar{i}_k)$/image $(\bar{\partial}_{k+1})$ and on $H_k(D_*)/A_k$ $\cong$ kernel $(\bar{\partial}_k)$ image $(\bar{j}_k)$. By the remarks above $H_k(D_*)$ and $H_k(M;\, Z)$ have the same free rank. It follows that kernel $(\emptyset_k)$ and $H_k(D_*)/A_k$ are finite groups. The lemma now follows by Lemma 1 above. □

LEMMA 6. *Suppose (C_*, E) is an endomorphism and $T \subset H_k(C_*)$ is a finite E_*-invariant subgroup on which E_* is nilpotent. There exists an endomorphism (C'_*, E') such that $H_*(C'_*) \cong H_*(C)/T$, where E'_j is a nilpotent extension of E_j for $j = k+1, k+2$ and $E'_j = E_j$ otherwise.*

Proof. We construct a nilpotent resolution of $E_{*_k}\colon T \to T$ as in [5]. Let Z_1 be a free Z-module with one generator for each element of $T - \{0\}$, $\varepsilon\colon Z_1 \to T$ the associated map, and define $N_1\colon Z_1 \to Z_1$ on generators according to E_*/T, so N_1 is nilpotent. Let $Z_2 =$ kernel (ε) and $N_2 = N_1/Z_2$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_2 & \xrightarrow{\text{inc}} & Z_1 & \xrightarrow{\varepsilon} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N_2} & & \downarrow{\scriptstyle N_1} & & \downarrow{\scriptstyle E_{*_k}} & & \\ 0 & \longrightarrow & Z_2 & \xrightarrow{\text{inc}} & Z_1 & \xrightarrow{\varepsilon} & T & \longrightarrow & 0 \end{array}$$

T is also resolved by cycles and boundaries: let $\psi\colon Z_k \to H_k(C_*)$ be the projection and $Z'_k = \psi^{-1}(T)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_k & \xrightarrow{i} & Z'_k & \xrightarrow{\psi} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle E_{k/}} & & \downarrow{\scriptstyle E_{k/}} & & \downarrow{\scriptstyle E_{*_k}} & & \\ 0 & \longrightarrow & B_k & \xrightarrow{i} & Z'_k & \xrightarrow{\psi} \pi & I & \longrightarrow & 0 \end{array}$$

Combining these sequences we obtain a resolution of Z'_k as in [4, V.2]. Since Z_1 is free and ψ is onto there exists $\rho\colon Z_1 \to Z'_k$ such that $\psi\rho = \varepsilon$. Let $K =$ kernel $(i \oplus \rho : B_k \oplus Z_1 \to Z'_k)$. We obtain a diagaram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & K & & Z_2 & & \\ & & & & \downarrow{\scriptstyle \text{inc}} & & \downarrow{\scriptstyle \text{inc}} & & \\ & & & & B_k \oplus Z_1 & & Z_1 & & \\ & & & & \downarrow{\scriptstyle i \oplus \rho} & & \downarrow{\scriptstyle \varepsilon} & & \\ 0 & \longrightarrow & B_k & \xrightarrow{i} & Z'_k & \xrightarrow{\psi} & T & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where $i \oplus \rho$ is onto Z_k' and for all $z \in Z_2$ there exists a unique $b \in B_k$ such that $(b, z) \in K$. Also, for $z \in Z_1$ there exists a unique $b \in B_k$ such that $\rho N_1(z) - E_k\rho(z) = i(b)$; let $*(z)$ be that b. We obtain a resolution of E_k/Z_k'

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & B_k \oplus Z_1 & \longrightarrow & Z_k' & \longrightarrow & 0 \\ & & \downarrow H & & \downarrow E_k \;\; {\scriptstyle *}\!\swarrow \;\; \downarrow N_1 & & \downarrow E_{k/} & & \\ 0 & \longrightarrow & K & \longrightarrow & B_k \oplus Z_1 & \longrightarrow & Z_k' & \longrightarrow & 0 \end{array}$$

where $H = \begin{pmatrix} E_k & * \\ 0 & N_1 \end{pmatrix}$ restricted to K. Since N_1 is nilpotent so is H.

The lemma follows by splicing in this resolution of Z_k' to kill T. Since B_k is free we have

$$\begin{array}{ccc} C_{k+1} & \cong & Z_{k+1} \oplus B_k \\ \Big| E_{k+1} & & \Big| E_{k+1} \;\swarrow\; \Big| E_k \\ C_{k+1} & \cong & Z_{k+1} \oplus B_k \end{array}$$

Let inc: $Z_k' \to C_k$ be the inclusion. Then (C_*', E') is the following endomorphism

$$\begin{array}{l} C_{k+3} \longrightarrow C_{k+2} \xrightarrow{\partial_{k+2}} Z_{k+1} \oplus B_k \oplus Z_1 \xrightarrow{\text{inc}\circ(i\oplus\rho)} C_k \xrightarrow{\partial_k} C_{k-1} \\ \qquad\qquad\qquad\;\; \oplus \\ \qquad\qquad\qquad\;\; K \xrightarrow{\text{inc}} \end{array}$$

with maps $E_j' = E_j$ for $j \neq k+1, k+2$ and

$$E_{k+2}' = \begin{pmatrix} E_{k+2} & 0 \\ 0 & H \end{pmatrix} \quad E_{k+1} = \begin{pmatrix} E_{k+1} & * \\ 0 & N_1 \end{pmatrix} \qquad \square$$

Remark. If $H_k(C_*)/T$ is free then (C_*', E') can be constructed so that $E_j' = E_j$ except for $j = k, k+1$. For Z_k' is free so the resolution of E_k/Z_k' can be folded to obtain:

$$\begin{array}{ccc} B_k \oplus Z_1 & \underset{\alpha}{\cong} & K \oplus Z_k' \\ E_k \Big\downarrow \;{\scriptstyle *}\!\swarrow\; \Big\downarrow N_1 & & H \Big\downarrow \;{\scriptstyle *'}\!\swarrow\; \Big\downarrow E_k \\ B_k \oplus Z_1 & \underset{\alpha}{\cong} & K \oplus Z_k' \end{array}$$

Now $0 \to Z_k' \to Z_k \to H_k(C_*)/T \to 0$ is exact so if $H_k(C_*)/T$ is free then Z_k' is a direct summand of Z_k and the map $*'\colon Z_k' \to K$ extends to a map $*''\colon C_k \to K$. The new complex C_*' is

$$\begin{array}{ccc} \longrightarrow C_{k+2} \longrightarrow C_{k+1} \oplus Z_1 & \longrightarrow & C_k \oplus K \longrightarrow \\ \wr\| & \circlearrowleft & \wr\| \\ (Z_{k+1} \oplus B_k \oplus Z_1) & \longrightarrow & (B_{k-1} \oplus Z_k \oplus K) \\ {\scriptstyle 0\oplus\alpha}\searrow & \circlearrowleft & \nearrow{\scriptstyle \text{inc}\oplus 1_K} \\ & Z_k' \oplus K & \end{array}$$

with maps

$$E_{k+1}' = \begin{pmatrix} E_{k+1} & * \\ 0 & N_1 \end{pmatrix} \quad \text{and} \quad E_k' = \begin{pmatrix} H & *'' \\ 0 & E_k \end{pmatrix},$$

where both N_1 and H are nilpotent. $\square$

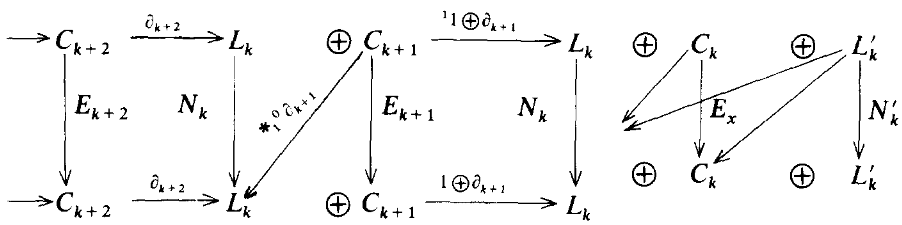
THEOREM 1. *Suppose M is orientable and $f \in \mathscr{F}$. Given signed representatives B_i^k for the basic sets of f, there exists a Z-endomorphism of f (C_*, E) such that for all k, E_k is a nilpotent extension of the signed representatives of index k.*

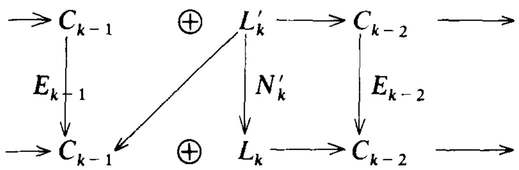
Proof: We construct a nilpotent extension of (D_*, F). By Lemma 5 Nil $(F_{*_0}: H_0(D_*) \circlearrowleft)$ is finite; using Lemma 6 there exists a nilpotent extension $(D'_* F')$ such that $(H_0(D'_*), F'_{*0}) \cong (\bar{H}_0(D_*), \bar{F}_{*_0}) \cong (H_0(M; Z), f_{*_0})$. Continuing inductively we obtain a Z-endomorphism of f (C_*, E) such that for all k, E_k is a nilpotent extension of F_k. Thus $E_k \underset{\text{nil}}{\sim} F_k \underset{\text{nil}}{\sim} B_k$, where B_k is the nilpotent extension of the signed representatives of index k of Lemma 4.

It follows there exist nilpotent blocks N_k, N'_k such that the matrix

$$\begin{pmatrix} N_k & *_1 & *_2 \\ 0 & E_k & *_3 \\ 0 & 0 & N'_k \end{pmatrix}$$

is a nilpotent extension of B_k. Let L_k, L'_k be free Z-modules, of dim L_k = rank N_k, dim L'_k = rank N'_k. At the k-th stage we adjoin contractible pairs in dimension $k+1$, k and, $k-1$.





Since the homology of (C_*, E) is unchanged we obtain at the n-th stage the desired Z-endomorphism of f.

When $k = 0$ the last step could introduce (-1) chains. Each component of M_0 either contains a periodic sink or else its orbit eventually wanders into such a component. We can absorb the wandering components into $M_1 - M_0$ and still have a filtration for f. Therefore we can assume M_0 has one component for each periodic sink so $f_{*_0}: H_0(M_0; Z) \circlearrowleft$ is a permutation and $B_0 = f_{*_0} = F_0 = E_0$, and no (-1) chains are introduced. Similarly, when $k = n$, the last step in the proof need not introduce $(n+1)$ chains. However, $(n+1)$ chains could be introduced when we kill Nil$(F_{*_{n-1}}: H_{n-1}(D_*) \circlearrowleft)$ using Lemma 6. If $H_{n-1}(M; Z) \simeq H_{n-1}(D_*)/\text{Nil}\,(F_{*_{n-1}})$ is free we can use the folded technique above and preserve the dimension of the complex. □

Remark. If f is a Smale diffeomorphism, Theorem 1 can be proved using Pixton's theory of fitted rectangular decompositions, without the assumption that M is orientable. Pixton proves that for Smale diffeomorphisms there exist filtrations $M_k, 0 \le k \le n$ where $\overline{M_k - M_{k-1}}$ is a finite disjoint union of rectangles $R_i^- \times R_i^+$ [10]. R_i^-, R_i^+ are submanifolds with boundary, which embed in Euclidean spaces of dimensions $k, n-k$, but are not necessarily discs. After choosing orientations the partition gives a choice of signed representatives, and it follows from the Künneth formula that $f_{*_k}: H_k(M_k, M_{k-1}; Z) \circlearrowleft$ is a nilpotent extension of the signed representatives of index k. Any two signed representatives of the same basic set are shift equivalent [3], so this is true independent of the choice of signed representatives. The rest of the proof of Theorem 1 is unchanged.

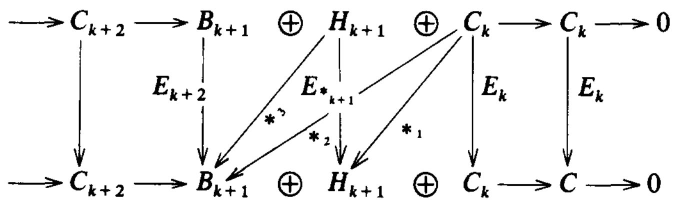
§3. GEOMETRIC REALIZATION

Let (C_*, E) be a Z-endomorphism of f. The geometric realization techniques of [13] require the stronger condition of chain homotopy equivalence. Suppose $\dim M \geq 6$, $\Pi_1(M) = 0$, and let $C_*(M)$ be the complex of a given handle decomposition of M and $f_*: C_*(M)\circlearrowleft$ the induced map. (C_*, E) is realized by a handle decomposition of M and a diffeomorphism isotopic to f if and only if there exists a chain homotopy equivalence $h: C_* \to C_*(M)$ such that $hE \simeq f_* h$. This will be satisfied provided (C_*, E) is also an R-endomorphism of f for $R = Z/n$ all n. (The problem is the off diagonal term in $(E \oplus 1_R)_*$ on $H_k(C_*; R) \cong (H_k(C_*)\otimes R)\oplus(H_{k-1}(C_*) * R)$. In case $H_*(M; Z)$ is torsion free it suffices that (C_*, E) be a Z-endomorphism of f.

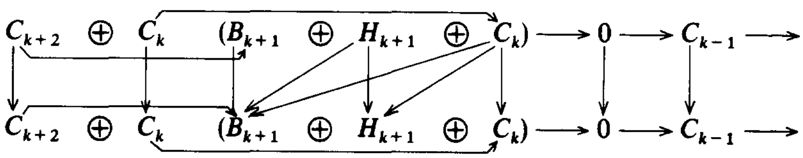
THEOREM 2. *Let M be a 2-connected manifold with torsion free homology and $\dim M \geq 6$. If $f \in \mathscr{F}$ has at least one source and one sink which are fixed then f is isotopic to an omega-conjugate fitted diffeomorphism.*

Proof: Let (C_*, E) be a Z-endomorphism of f, as in Theorem 1. If $C_1 = C_{n-1} = 0$ then by [13] there exists a handle decomposition of M and a fitted diffeomorphism g isotopic to f whose chain map realizes E. g may be chosen so its geometric intersection numbers agree up to sign with the entries of E. If the nilpotent blocks in the E_k are put in upper triangular form they give rise to wandering handles in g. If the signed representatives B_i^k of f arose from Markov partitions with 0-1 geometric intersection numbers, then $\Omega(f)$ is topologically conjugate to $\Omega(g)$. Therefore the result follows provided $C_1 = C_{n-1} = 0$.

We use *folding*, as in [13, Appendix A] to eliminate 1 and $(n-1)$ chains. Suppose (C_*, E) is given, $H_k(C_*) = 0$ and $\partial_k: C_k \to C_{k-1}$ is zero. Since all boundaries are free we have a splitting of C_{k+1}



Folding C_k up into $(k+2)$ we obtain:



The off-diagonal term $*_2: C_k \to B_{k+1}$ in E_{k+1} can be balanced by an off-diagonal term $(\partial_{k+2})^{-1} \circ *_2: C_k \to C_{k+2}$ in dimension $(k+2)$ after folding. However a term $*_1: C_k^{\prime} \to H_{k+1}$ in E_{k+1} cannot be compensated for this way. We assume M is 2-connected so this problem does not arise in folding C_1 and C_{n-1}. .

Recall that when $k = 0$ the endomorphism E_0 and the signed representative B_0 coincided in the proof of Theorem 1. Therefore if f has a fixed sink the corresponding component of M_0 represents an invariant Z in $H_0(M; Z)$ and hence in C_0; the components do not represent boundaries in $\partial_1: H_1(M_1, M_0; Z) \to H_0(M_0, Z)$ so we obtain an invariant splitting of E_0

$$\begin{array}{ccccc} C_0 & \cong & (B_0 & \oplus & Z) \\ \downarrow{\scriptstyle E_0} & & \downarrow{\scriptstyle E_0} & & \downarrow{\scriptstyle 1} \\ C_0 & \cong & (B_0 & \oplus & Z) \end{array}$$

We first fold $B_0 \xrightarrow{E_0} B_0$ up into dimension $k = 2$ so $\partial_1: C_1 \to C_0$ is zero. Then we fold $C_1 \xrightarrow{E_1} C_1$ up into dimension $k = 3$. $(n-1)$ chains can be folded similarly. The theorem follows since all the non-wandering information in the E_k is preserved although the index is not. □

Remark. "Folding" was introduced in [13] where it was described for complexes. Here we will elaborate further the method for v.p. endomorphisms. Suppose C_* is a free Z complex $0 \to C_m \to \ldots \to C_L \to 0$ which has the homology of a simply connected manifold of dimension n and $E: C_* \to C_*$ is represented by virtual permutation (v.p.) matrices. Then we claim (C_*, E) is chain homotopy equivalent to a v.p. endomorphism concentrated in dimensions $0 \le k \le n$ with $C_1 = C_{n-1} = 0$. As above, off-diagonal terms can present problems in folding. Let $E'_k = E_k$ except delete these off-diagonal terms in $k = n, n-2$ and 0, 2. 1: $(C_*, E) \to (C_*, E')$ is a chain homotopy equivalence and E' is still quasi-unipotent. Folding we obtain a new endomorphism (C_*, E) where C_* has the required form and E is quasi-unipotent. By [5] we can add inverses on contractible pairs in adjacent dimensions so that all C_k, $2 \le k \le (n-3)$ have 2-step v.p. resolutions i.e. exact sequences $0 \to (D_1, F_1) \to (D_0, F_0) \to (C_k, E_k) \to 0$ where the D_i are free and the F_i are v.p. The Euler characteristic $\chi(C_*, E)$ is unchanged so (C_{n-2}, E_{n-2}) has a resolution as well. Now splice in the resolutions for $2 \le k \le n-3$ as in [5]. To avoid re-introducing $(n-1)$ chains we need a transposed v.p. resolution of (C_{n-2}, E_{n-2}), i.e. a sequence $0 \to (C_{n-2}, E_{n-2}) \to (D'_0, F'_0) \to (D'_1, F'_1) \to 0$. Now modules (M, E) which have v.p. resolutions are closed under short exact sequences [5, see also 8]. By duality so are free modules with transposed v.p. resolutions and a v.p. object trivially has a transposed v.p. resolution. Therefore a free object which has a v.p. resolution also has a transposed v.p. resolution. Splicing in a transposed v.p. resolution of (C_{n-2}, E_{n-2}) we obtain the desired v.p. endomorphism. □

It would be good to relax the restrictive conditions of Theorem 2. Our arguments are a continuation of algebraic ideas that appeared briefly in [13] as the Čech theory "going up" proof that Morse–Smale diffeomorphisms can be represented by v.p. matrices, and of the related algebraic methods of [1] and [3]. One would also want to prove Theorem 2 geometrically, possibly in the spirit of the "going down" proof of [13] using the filtration by open manifolds and simple homotopy theory. In a forthcoming paper Pixton proves that a Smale diffeomorphism is fitted provided it satisfies a condition he calls dynamically tame. A natural approach would be to modify a Smale diffeomorphism by an isotopy to make it dynamically tame. This would also give more information about the original diffeomorphism.

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Department of Mathematics
Queens College and The
Graduate School of CUNY