THE INTEGRAL HOMOLOGY OF SMALE DIFFEOMORPHISMS

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(Received 28 June 1983)

The dynamics and topology of diffeomorphisms are closely related. In this paper we show how to deduce information about the topology of a large class of diffeomorphisms from local information about the dynamics.

Let \( M \) be a connected closed manifold of dimension \( n \), and \( f: M \to M \) a diffeomorphism. A closed \( f \)-invariant subset \( A \subset M \) is said to be hyperbolic if there exists an invariant splitting of the tangent bundle over \( A \), \( T_AM = E^s \oplus E^u \), and constants \( C > 0, 0 < \lambda < 1 \) such that \( \| Tf^n|E^s \| \leq C\lambda^n \) for all \( n \geq 0 \), and \( \| Tf^n|E^u \| \leq C\lambda^n \) for all \( n \leq 0 \). \( f \) is said to satisfy Axiom A if the non-wandering set, \( \Omega(f) = \{ x \in M : \text{for all neighborhoods } U \text{ of } x, f^n(U) \cap U \neq \emptyset, \text{for some } k > 0 \} \), is hyperbolic and the periodic points of \( f \) are dense in \( \Omega(f) \). For these diffeomorphisms Smale's spectral decomposition theorem says \( \Omega(f) \) is a finite disjoint union of closed \( f \)-invariant subsets called basic sets, \( \Omega(f) = \Omega_1 \cup \ldots \cup \Omega_r \) [14]. The index of \( \Omega_i \) is the fiber dimension of \( E^s/\Omega_i \).

For each \( \Omega_i \) let \( W^s(\Omega_i) = \{ x \in M : d(f^n(x), \Omega_i) \to 0 \text{ as } n \to +\infty \} \) and \( W^u(\Omega_i) = \{ x \in M : d(f^n(x), \Omega_i) \to 0 \text{ as } n \to -\infty \} \). One writes \( \Omega_i \geq \Omega_j \) if \( W^s(\Omega_i) \cap W^u(\Omega_j) \neq \emptyset \). \( f \) is said to have no cycles if \( \geq \) can be extended to a total ordering of the basic sets. If in fact \( W^s(\Omega_i) \cap W^u(\Omega_j) = \emptyset \) whenever index \( (\Omega_i) < \text{index } (\Omega_j) \) we will say \( f \) has an index-compatible ordering. Finally, if \( W^s(x) \) and \( W^u(y) \) have transverse intersection for all \( x, y \in \Omega(f) \), then \( f \) is said to satisfy the strong transversality condition.

A diffeomorphism is said to be Smale if it satisfies Axiom A and strong transversality and has zero dimensional \( \Omega \). Our results hold under somewhat weaker hypotheses. We will write \( f \in \mathcal{S} \) if \( f \) satisfies Axiom A and no cycles, has zero dimensional \( \Omega \) and an index compatible ordering on the basic sets.

Basic sets of Axiom A diffeomorphisms admit Markov partitions. When \( \Omega_i \) is zero-dimensional the partition can be constructed so that \( \Omega_i \) is topologically conjugate to the subshift of finite type \( \Sigma_A \), where \( A_i \) is the \( 0 \)-\( 1 \) intersection matrix of \( f \) on the partition. Furthermore \( E^s/\Omega_i \) is orientable, and if the partition is sufficiently fine the orientation numbers of \( Tf|E^s \) are constant on rectangles, so the intersection numbers of \( f \) can be recorded with signs. The resulting integral matrix \( B_i \) is called a signed representative of \( \Omega_i \) [3].

Let \( C_* \) be a free finitely generated \( \mathbb{Z} \)-complex \( 0 \to C_m \to \ldots \to C_0 \to 0 \). We will say \( C_* \) is a complex of \( M \) if \( H_* (C_*) \cong H_* (M; \mathbb{Z}) \). It follows that \( H_* (C_* ; R) \cong H_* (M; R) \) for all coefficient rings \( R \). Given a chain map \( E: C_* \to C_* \) we will say the pair \( (C_*, E) \) is an \( R \)-endomorphism of \( f \) if there exists an isomorphism which conjugates \( E_\bullet \) and \( f_\bullet \).

\[
\begin{align*}
H_\bullet (C_* ; R) &\xrightarrow{E_\bullet} H_\bullet (C_* ; R) \\
&\xrightarrow{f_\bullet} H_\bullet (M ; R) \\
H_\bullet (C_* ; R) &\xrightarrow{E_\bullet} H_\bullet (C_* ; R)
\end{align*}
\]

† Research partially supported by PSC-CUNY grant 1981-13723.
‡ Research partially supported by an NSF grant.
If \( F_1, \ldots, F_s \) are square integral matrices, we will say an integral matrix \( E \) is a nilpotent extension of \( F_1, \ldots, F_s \) if \( E \) is similar over the integers to a matrix of the form

\[
\begin{pmatrix}
N_1 & F_1 & & \\
& N_2 & F_2 & \\
& & \ddots & F_s \\
& & & N_{s+1}
\end{pmatrix}
\]

where the \( N_i \) are square nilpotent matrices and the above diagonal entries * are arbitrary.

**Theorem 1.** Suppose \( M \) is orientable and \( f \in \mathcal{F} \). For \( 0 \leq k \leq n \) let \( B_1^k, \ldots, B_s^k \) be signed representatives of the basic sets of index \( k \). Then there exists a \( \mathbb{Z} \)-endomorphism of \( f (C_*, E) \) such that for each \( k \) \( E_k \) is a nilpotent extension of the \( B_1^k, \ldots, B_s^k \).

This theorem extends previous results of Bowen and Franks [3] who studied a single basic set on an orientable manifold. If \( f \) is actually a Smale diffeomorphism, Theorem 1 can also be proved using Pixton’s theory of fitted rectangular decompositions, without requiring that \( M \) be orientable [10].

Fitted diffeomorphisms are Smale diffeomorphisms which preserve a handle decomposition of the manifold [13]. They are dense in the \( C^0 \)-topology on \( \text{Diff}^r(M) \) and exhibit a particularly simple structure and a close connection between dynamics and homology theory. Examples of Bowen [2], Newhouse [9], and Pixton [11] show there exist Smale diffeomorphisms which are not fitted. In Pixton’s terminology these diffeomorphisms are dynamically wild. From Theorem 1 and the techniques of [13] we obtain

**Theorem 2.** Let \( M \) be a 2-connected manifold with torsion free homology, and \( \dim M \geq 6 \). If \( f \in \mathcal{F} \) has at least one fixed source and one fixed sink, then \( f \) is isotopic to an omega-conjugate fitted diffeomorphism.

In principle this theorem reduces to algebra the problem of determining the omega-conjugacy types of Smale diffeomorphisms in the component of \( f \), using the known machinery for fitted diffeomorphisms.

Let \( A \) and \( B \) be square integral matrices; they are said to be shift equivalent (\( A_{\text{shift}} \equiv B \)) if there exist integral matrices \( R \) and \( S \) and an integer \( k > 0 \) such that \( A^k = RS \), \( B^k = SR \), and \( SA = BS \), \( AR = RB \).

Bowen and Franks [3] proved that for a single basic set \( \Omega_i \) of index \( k \), the signed \( B_i^k \) is shift equivalent to the map induced by \( f \) in the relative \( k \)-homology of a filtration pair for \( \Omega_i \) (see section 2). A main tool in the proof of Theorem 1 is a characterization of shift equivalence of integral matrices.

For square integral matrices \( A \) and \( B \), we will say that \( A \) is nil-equivalent to \( B \) (\( A_{\text{nil}} \equiv B \)) if there are nilpotent extensions

\[
\begin{pmatrix}
N_1 & A & * \\
0 & N_2 & \\
& \ddots & & \ddots \\
& & & & & N_4
\end{pmatrix}
\]

of \( A \) and of \( B \) which are similar over \( \mathbb{Z} \).

**Proposition.** If \( A \) and \( B \) are square integral matrices then \( A_{\text{shift}} \equiv B \) if and only if \( A_{\text{nil}} \equiv B \).

We would like to thank Bob Williams, Dennis Pixton, Ken Kramer and Bruce Kitchens for very helpful conversations.
§1. NILPOTENT EXTENSIONS

Suppose \( L \) is a finitely generated \( \mathbb{Z} \)-module and \( \alpha : L \to L \) a linear map. Let \( \text{Nil}(\alpha) = \{ v \in L : \text{for some } k \geq 0 \quad \alpha^k(v) = 0 \} \). Then \( \text{Nil}(\alpha) \) is invariant under \( \alpha \). Let \( \tilde{L} \) be the quotient module \( L/\text{Nil}(\alpha) \) and \( \bar{\alpha} : \tilde{L} \to \tilde{L} \) the injective quotient map. If \( L \) is free then \( \tilde{L} \) is free as well, so the sequence \( 0 \to \text{Nil}(\alpha) \to L \to \tilde{L} \to 0 \) splits and \( \alpha \) can be represented by a matrix

\[
\begin{pmatrix}
\alpha/\text{Nil}(\alpha) & * \\
0 & \bar{\alpha}
\end{pmatrix}
\]

Let \( \mathcal{M} \) be the category whose objects are pairs \( (L, \alpha) \) and whose morphisms \( i : (L, \alpha) \to (M, \beta) \) are linear maps \( i : L \to M \) such that \( \beta i = i \alpha \). Then \( \cdot \) is a functor from \( \mathcal{M} \) to itself.

\( \cdot \) fails to preserve exactness. Given a short exact sequence in \( \mathcal{M} \)

\[
0 \to L \xrightarrow{i} M \xrightarrow{j} N \to 0
\]

in the quotient sequence \( \tilde{i} \) is 1-1 and \( \tilde{j} \) is onto but in general image \( \tilde{i} \neq \text{kernel } \tilde{j} \). If \( w \in \text{kernal } \tilde{j} \) then for some \( k \geq 0 \), \( 0 = \beta^k(w) \) so there exists \( v \in L \), \( i(v) = \beta^k(w) \). Therefore the map induced by \( \tilde{\beta} \) on kernel \( \text{image } \tilde{i} \) is always nilpotent. If \( \bar{\alpha} \) is onto then exactness is preserved: for if \( \tilde{\alpha}^k(\tilde{v}) = \tilde{0} \) then \( \tilde{\beta}^k(\tilde{i}v) = \tilde{i}(\tilde{0}) = \tilde{\beta}^k(\tilde{w}) \) so \( i(\tilde{v}) = \tilde{w} \). In particular, exactness is preserved in the category of finite dimensional vector spaces. Similarly, if \( \alpha \) is nilpotent then \( \tilde{j} \) is 1-1, and again exactness is preserved. However consider the example

\[
0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j} \mathbb{Z} \to 0
\]

where \( i(v) = (v, 0) \) and \( j(v, w) = w \). The quotient sequence is \( 0 \to \mathbb{Z} \xrightarrow{j} \mathbb{Z} \xrightarrow{j} \mathbb{Z} \to 0 \) which is not exact since \( (0, 1) \notin \text{image } \tilde{i} \). We will need the following fact.

**Lemma 1.** Given two exact sequence in \( \mathcal{M} \)

\[
0 \to (A, \alpha) \xrightarrow{i} (B, \beta) \xrightarrow{j} (C, \gamma) \to 0 \quad \text{and} \quad 0 \to (B, \beta) \xrightarrow{k} (D, \delta) \xrightarrow{l} (E, \epsilon) \to 0,
\]

suppose that \( A \) and \( E \) are finite, \( \alpha \) and \( \epsilon \) are nilpotent, and \( \gamma : C \to C \) is an isomorphism. Then \( \text{Nil}(\delta) \) is finite and \( (D, \delta) \cong (C, \gamma) \).

**Proof:** Observe that \( \text{Nil}(\beta) = i(A) \); therefore \( (C, \gamma) \cong (B, \beta) \), and exactness is preserved by \( \cdot \) in the second sequence. Since \( \epsilon \) is nilpotent \( (B, \beta) \cong (D, \delta) \). If \( \text{Nil}(\delta) \) were infinite, so would kernel \( (l) \cap \text{Nil}(\delta) = \text{im}(k) \cap \text{Nil}(\delta) \cong \text{Nil}(\beta) = i(A) \) but \( A \) is finite. \( \square \)

If \( A \) is an \( (n \times n) \) integral matrix then \( A \) is defined up to similarity over \( \mathbb{Z} \) by allowing \( A \) to act on \( \mathbb{Z}^n \). Recall that in the category of integral matrices shift equivalence coincides with the \textit{a priori} stronger relation of strong shift equivalence. Let \( A \approx B \) if there exist integral matrices \( R \) and \( S \) such that \( A = RS \) and \( B = SR \), i.e. \( A \sim_k B \) with the lag \( k = 1 \). \textit{Strong shift equivalence} \( (A \approx B) \) is the transitive closure of \( \approx_1 \). These relations were introduced by Williams in [16]. The so called "Williams problem" is whether the two relations also coincide in the category of non-negative matrices [17]. In the category of integral matrices similarity over \( \mathbb{Z} \) implies strong shift equivalence; if \( PAP^{-1} = B \) let \( S = PA \) and \( R = P^{-1} \). Therefore shift equivalence is also a relation on linear maps.
Lemma 2. Let $A, B$ be square integral matrices. Then $A \sim Z B \Rightarrow A \equiv_{nil} B \Rightarrow A \equiv_{shift} B \Rightarrow A \sim_{Q} B$.

Proof: (i) $A \sim_{Z} \left( \begin{array}{cc} A/Nil(A) & \ast \\ 0 & A \end{array} \right)$ so $A \equiv_{nil} A \sim_{Z} B \equiv_{nil} B$.

(ii) Let $\ast$ represent an arbitrary additional final column. Then

$$(A^\ast)\begin{pmatrix} I_d \\ 0 \end{pmatrix} = A \quad \text{and} \quad \begin{pmatrix} I_d \\ 0 \end{pmatrix}(A^\ast) = \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad \text{so} \quad A \equiv_{1} \begin{pmatrix} A \\ 0 \end{pmatrix}.$$

(“bordering”)

Similarly

$$A \equiv_{1} \begin{pmatrix} 0 & \ast \\ 0 & A \end{pmatrix}.$$

Any nilpotent block $N$ is similar over $Z$ to a strictly upper triangular matrix so by iterating bordering $A \equiv_{shift} \begin{pmatrix} N_1 & \ast \\ 0 & N_2 \end{pmatrix}$ which proves the second implication. (iii) Bowen and Franks [3] proved that if $A \equiv_{shift} B$, then for any abelian group $G$, regarding $A$ and $B$ as maps $G^n \rightarrow G^n$, $\lim_{G} A \equiv_{Q} \lim_{G} B$. Let $G = Q$; taking inverse limits over $Q$ we have $A \equiv_{Q} \lim_{1} A \otimes_{Q} \lim_{1} B \equiv_{Q} B$. 

Lemma 3. Let $B$ be an $(n \times n)$ integral matrix and suppose $L \equiv_{Q} Z^k$ is a $B$-invariant sublattice of $Z^n$ such that $B(Z^n) \subseteq L \subseteq Z^n$. Then $B \equiv_{1} B/L$.

Proof: Let $A$ be a matrix representing $B/L$: $L \rightarrow L$ in the basis inherited from $Z^k$, and let $R$ be the matrix of $B$ regarded as a map $Z^n \rightarrow L \equiv Z^n$, and let $S$ be the matrix of the inclusion $Z^k \rightarrow L \subseteq Z^n$. Then $RS = A$ and $SR = B$.

It follows that the $(k + n) \times (k + n)$ matrices

$$\begin{pmatrix} A \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & R \\ 0 & B \end{pmatrix}$$

and similar over $Z$. For:

$$\begin{pmatrix} I_d & 0 \\ S & I_d \end{pmatrix} \begin{pmatrix} A & R \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & R \\ 0 & B \end{pmatrix} \begin{pmatrix} I_d & 0 \\ S & I_d \end{pmatrix}$$

Example: It seems natural to ask whether the non-singular quotient map $\sim_{Q}$ can be changed by a nilpotent extension, i.e. if $N$ is nilpotent is

$$\begin{pmatrix} A & \ast \\ 0 & N \end{pmatrix} \sim_{Q} A$$

We show this fails as follows. Let $C$ and $D$ be non-singular $(n \times n)$ integral matrices which are similar over $Q$ but not over $Z$. Let $PC = DP$ where $P$ is integral; then $C \sim_{Q} D/P(Z^n)$ and there exists an integer $k$ such that $kD(Z^n) \subseteq P(Z^n)$. Letting $B = kD$ and $L = P(Z^n)$ we have

$$\begin{pmatrix} kC & \ast \\ 0 & 0 \end{pmatrix} \sim_{Q} \begin{pmatrix} 0 & \ast \\ 0 & kD \end{pmatrix} \quad \text{but} \quad kC \not\sim_{Z} kD.$$ 

Thus

$$\begin{pmatrix} kC & \ast \\ 0 & 0 \end{pmatrix} \not\sim_{Q} kC.$$ 

For example, let

$$C = \begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 3 \\ -2 & 1 \end{pmatrix}$$
which represent distinct ideal classes of the ring \( \mathbb{Z}[(\sqrt{5})i] \). Then
\[
\begin{pmatrix}
0 & -15 & -5 & 0 \\
3 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
0 & 0 & -5 & 0 \\
0 & 0 & -1 & 3 \\
0 & 0 & -3 & 9 \\
0 & 0 & -6 & 3
\end{pmatrix}
\sim
\begin{pmatrix}
-3 & 9 \\
-6 & 3
\end{pmatrix}
\]

**Proposition 1.** In the category of integral matrices \( A \sim B \) if and only if \( A \sim_{\text{shift}} B \).

**Proof:** \( \sim_{\text{nil}} \Rightarrow \sim_{\text{shift}} \) was proved in Lemma 2. To prove the converse, suppose that \( A \sim B \) and \( A = RS, B = SR \). Then, as above,
\[
\begin{pmatrix}
A & R \\
0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
0 & R \\
0 & B
\end{pmatrix}
\]
Therefore
\[
A \sim_{\text{nil}} \begin{pmatrix}
A & R \\
0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
0 & R \\
0 & B
\end{pmatrix}
\sim_{\text{nil}} B
\]

### §2. \( \mathbb{Z} \)-Endomorphisms

Let \( f \in \mathcal{F} \). Number the basic sets \( \Omega_i^k, 0 \leq k \leq n, 1 \leq i \leq s_k \), where \( k = \text{index } \Omega_i^k \) and if \( i < j \) then \( W^s(\Omega_i^k) \cap W^s(\Omega_j^k) = \emptyset \). There exist filtrations of \( M \) with one basic set added at each stage, that is, a sequence of submanifolds with boundary \( M_i^k \) such that

(i) \( M_i^{k-1} \subset M_i^k \) (if \( i = 1 \) let \( M_0^k = M_{s_i-1}^k \)).

(ii) \( f(M_i^k) \subset \text{int } M_i^k \)

(iii) \( \bigcap_{n \in \mathbb{Z}} f^n(M_i^k - M_i^{k-1}) = \Omega_i^k \)

If we require in addition that the boundaries of \( M_i^k \) and \( M_i^{k-1} \) meet transversely, then in the language of [3] \( M_i^k, M_i^{k-1} \) are a filtration pair for \( \Omega_i^k \).

We suppose that signed representatives \( B_i^k \) are given for \( \Omega_i^k \). Let \( M_k = \bigcup_{i=1}^{s_k} M_i^k = M_{s_k}^k \). We will show that \( f_{\alpha_i^k} : H_k(M_k, M_{k-1}^k, Z) \) is shift equivalent to a matrix \( B_k \) which is itself a nilpotent extension of the signed representatives \( B_1^k, \ldots, B_{s_k}^k \).

We recall the following facts: Bowen proved that \( f_{\alpha_j} : H_j(M_i^k, M_{i-1}^k, Z) \) is nilpotent for \( j \neq k \) [1]. Bowen and Franks proved that if \( M \) is orientable then \( f_{\alpha_i} : H_k(M_i^k, M_{i-1}^k, Z) \) is shift equivalent to \( B_i^k \), and nilpotent on the torsion summand of \( H_k(M_i^k, M_{i-1}^k, Z) \) [3]. Furthermore, if \( 0 \rightarrow (A, \alpha) \rightarrow (B, \beta) \rightarrow (C, \gamma) \rightarrow 0 \) is an exact sequence in \( M \) and \( \alpha \) or \( \gamma \) are nilpotent, then the other two maps are shift equivalent. [3, Lemma 3.4].

**Lemma 4.** \( f_{\alpha_i^k} : H_k(M_k, M_{k-1}^k, Z) \) is shift equivalent to a matrix \( B_k \) which is a nilpotent extension of the \( B_1^k, \ldots, B_{s_k}^k \).

**Proof:** Consider first the exact sequence of the triple \( M_{k-1} \subset M_1^k \subset M_2^k \) (all coefficients are \( Z \), all maps induced by \( f \)).

\[
\begin{array}{cccccccc}
H_{k+1}(M_2^k, M_1^k) & \rightarrow & H_k(M_2^k, M_{k-1}) & \rightarrow & H_k(M_2^k, M_1^k) & \rightarrow & H_{k-1}(M_2^k, M_{k-1}) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\
H_{k+1}(M_2^k, M_1^k) & \rightarrow & H_k(M_2^k, M_{k-1}) & \rightarrow & H_k(M_2^k, M_1^k) & \rightarrow & H_{k-1}(M_2^k, M_{k-1})
\end{array}
\]
where \( \beta \text{shift}B_1 \) and \( \delta \text{shift}B_2 \), \( \alpha \) and \( \epsilon \) are nilpotent so applying \( \sim \) we obtain

\[
0 \to \tilde{H}_k(M_1^2, M_{k-1}^2) \xrightarrow{\sim} \tilde{H}_k(M_2^2, M_{k-1}^2) \xrightarrow{\sim} \tilde{H}_k(M_3^2, M_{k-1}^2) \to 0
\]

\[
0 \to \tilde{H}_k(M_1^2, M_{k-1}^2) \xrightarrow{\beta} \tilde{H}_k(M_2^2, M_{k-1}^2) \xrightarrow{\delta} \tilde{H}_k(M_3^2, M_{k-1}^2) \to 0
\]

which may fail to be exact, but the map induced by \( \gamma \) on \( \ker (j)/\text{im}(i) \) is nilpotent. Now the sequence \( 0 \to \ker (j) \xrightarrow{\sim} \tilde{H}_k(M_2^2, M_{k-1}^2) \to \tilde{H}_k(M_3^2, M_{k-1}^2) \to 0 \) is exact and the right hand term is free so \( \gamma \) can be represented by a matrix:

\[
\begin{pmatrix}
\tilde{\gamma}/\ker (j) \\
0
\end{pmatrix}
\]

Also \( 0 \to \text{im}(i) \xrightarrow{\sim} \ker (j) \to \text{im}(i) \to 0 \) is exact, so \( \tilde{\gamma}/\ker (j) \sim \tilde{\gamma}/\text{im}(i) \sim \tilde{\beta} \sim \tilde{\beta} \sim \beta \text{shift}B_1 \). Similarly \( \delta \sim \delta \sim \delta \text{shift}B_2 \). It follows that \( \gamma = f_{nk}; \tilde{H}_k(M_2^2, M_{k-1}^2) \) is shift equivalent to a nilpotent extension of \( B_1^2 \) and \( B_2^2 \). The lemma follows by induction on the number of basic sets of index \( k \).

Let \((D_*, F)\) be the complex \( D_k = \tilde{H}_k(M_k, M_{k-1}; Z), F_k = f_{nk}; D_k \rightarrow \). It follows from the proof of [3 Lemma 3.3] that \( f_{nk} \) is nilpotent on the torsion summand of \( H_k(M_k, M_{k-1}; Z) \) so \( D_k \) is a free \( Z \)-module and \( F_k \text{nil} f_{nk} \text{shift}B_k \). We will prove Theorem 1 by constructing a \( Z \)-endomorphism of \( f \) from a nilpotent extension of \((D_*, F)\).

We observe first that \( H_k(M; Z) \) and \( H_k(D_*) \) have isomorphic free summands. Let \( K \) be a field, and, for \( 0 < j < n \) let \( X_j = \bigotimes_{\geq 0} \tilde{H}_j(M_j, M_{j-1}; K) \). Using \( \text{Čech theory} \)

\[
\lim_{\rightarrow} [H_k((M_j, M_{j-1}; K) \xrightarrow{\sim} f_{nk})] \cong \tilde{H}_k((M_j, M_{j-1}; K) \xrightarrow{\sim} f_{nk}) \cong \tilde{H}_k(X_j, X_{j-1}; K).
\]

Therefore, for \( L \neq j \) it follows from the nilpotence of \( f_{nk} \) that \( \tilde{H}_k(X_j, X_{j-1}; K) = 0 \). Now \( X_n = M \) and \( X_0 \) is discrete; since \( K \) is a field it follows that the complex \( C_*^k; \tilde{C}_k^f = H_k(X_j, X_{j-1}; K) \) carries the \( K \)-homology of \( M \) [15, p. 205]. On the other hand, by the universal coefficient theorem \( H_j(M_j, M_{j-1}; K) \cong (H_j(M_j, M_{j-1}; Z) \otimes K) \oplus (H_{j-1}(M_j, M_{j-1}; Z) \circ K). \) Since \( f_{nk} \) is nilpotent on the second term on the right, \( C_k^f = \lim_{\rightarrow} (H_j(M_j, M_{j-1}; Z) \otimes K) \circ (f_j \otimes 1_k) \cong \lim_{\rightarrow} (D_j \otimes K) \circ (F_j \otimes 1_k). \) Now let \( K = Q \); we have \( \lim_{\rightarrow} (D_j \otimes Q) \circ (F_j \otimes 1_Q) \cong (D_j \otimes Q) \circ (F_j \otimes 1_Q). \) Therefore \( H_*^k(M; Q) \cong H_*^k(C_*^f) \cong H_*^k(D_*^k \otimes Q) = H_*^k(D_*^k) \otimes Q. \) Therefore \( H_*^k(M; Z) \) and \( H_*^k(D_*) \) have the same free summands.

Remark. It follows from the work of Bowen and Franks that for \( K \) a field the inverse limit \( \lim_{\rightarrow} B_j: K^n \rightarrow K^n \), where \( n_j = \text{rk}(B_j) \) is a \( K \)-endomorphism of \( f \). For \( B_j \text{shift}f_{nk}: H_j(M_j, M_{j-1}; K) \cong \lim_{\rightarrow} B_j \cong \lim_{\rightarrow} f_{nk} \cong C_j^k \). However more delicate methods are needed working with \( Z \)-coefficients. In particular \( B_j \) may be injective but over \( Z \), \( \lim_{\rightarrow} B_j = 0 \).

The next lemma shows we obtain the integral homology of \( M \) and \( f \) by applying \( \sim \) again, to \( H_*^k(D_*) \cong F_*^k \).

\[ \text{Lemmma 5.} \text{ Nil}(F_*^k; H_k(D_*^k) \cong ) \text{ is finite for all } k, \text{ and } (H_*^k(D_*^k), F_*^k) \cong (H_*^k(M; Z), f_*^k). \]

\[ \text{Proof:} \text{ We show that for } 0 \leq k \leq n \text{ there exists an } F_*^k \text{-invariant subgroup } A_k \subset H_k(D_*^k) \\text{ and an epimorphism } 0: A_k \rightarrow H_k(M; Z) \text{ conjugating } f_*^k \text{ and } f_*^k/A_k; \text{ furthermore kernel } (0_k) \text{ and } H_k(D_*^k)/A_k \text{ are finite groups on which } F_*^k \text{ induces nilpotent maps. The result then follows from Lemma 1 above.} \]

Consider the diagram of the exact sequences of \( (M_{k-1}, M_{k-2}), (M_k, M_{k-1}) \) and \( M_{k+1}, M_k). \) (All coefficients are \( Z \).)
First we prove by induction on L that \( f_*: H_k(M_L) \to H_k(M_{L+1}) \) is nilpotent for \( k > L \). When \( L = 0 \) consider \( 0 < M^0_1 < \ldots < M^0_n \). Now \((M^0_1, \emptyset)\) is a filtration pair for \( \Omega^0_1 \) so in this case the claim follows from [1]. Assume inductively that \( f_{L+1}*: H_k(M^0_{L+1}) \to H_k(M^0_{L+1}, M^0_{L+1}) \) is nilpotent for \( k > L \). Then in the exact sequence \( H_k(M^0_L) \to H_k(M^0_{L+1}) \to H_k(M^0_{L+1}, M^0_{L+1}) \to H_k(M^0_{L+1}) \) the maps induced on the end terms are nilpotent, hence on the middle term as well, which proves the claim. In particular, for all \( k, f_{L+1}: H_k(M_{L+1}) \to H_k(M_{L+1}, M_{L+1}) \) is nilpotent so the maps \( j_k*: H_k(M_{L+1}) \to H_k(M_{L+1}, M_{L+1}) \) are injective.

Next suppose \( k < L \) and consider the exact sequence \( H_k(M_{L+1}) \to H_k(M_L) \to H_k(M_L+1) \). The map induced by \( f \) on the left is nilpotent so \( i: H_k(M_L) \to H_k(M_{L+1}) \) is injective for \( k < L \). In addition, for \( k \leq L \) we will prove by induction that \( i \) is surjective. First let \( L = n-1 \). Taking a power of \( f \) if necessary we can assume that \( f_{n-1}: H_k(M_{n-1}) \to H_k(M_{n-1}) \) is zero for \( k < n \). In the diagram

\[
H_k(M_n, M_{n-1}) \xrightarrow{f_*} H_k(M_{n-1}) \xrightarrow{(\text{inc})_*} H_k(M_n, M_{n-1})
\]

the map induced by \( f \) on the left is an isomorphism, so \((\text{inc})_*\) is also zero. Consider the diagram

\[
\begin{array}{ccc}
H_k(M_{n-1}) & \xrightarrow{(\text{inc})_*} & H_k(M_n) \\
| & & | \\
\downarrow{\text{inc}\_k} & & \downarrow{\text{inc}\_k} \\
\downarrow{\text{inc}\_k} & & \downarrow{\text{inc}\_k} \\
H_k(f(M_{n-1})) & \xrightarrow{(\text{inc})_*} & H_k(f(M_n)) \\
\end{array}
\]

The right hand \((\text{inc})_*\) is zero while the middle \((\text{inc})_*\) is an isomorphism. It follows that \( i: H_k(M_{n-1}) \to H_k(M_n) \) is onto, for \( k \leq n-1 \). To continue the induction, observe that if \( k \leq n-2 \) in the left square above both horizontal maps induce isomorphisms under \( \cdot \).

Therefore, for \( k \leq n-2 \) \((\text{inc})_*: H_k(f(M_{n-1})) \to H_k(M_{n-1}) \) is an isomorphism, and the induction continues.

For all \( k \) we obtain a commutative diagram

\[
\begin{array}{ccc}
\bar{H}_k(M_n) & \xrightarrow{f_*} & \bar{H}_k(M_{n-1}) \\
\downarrow{\text{inc}\_k} & & \downarrow{\text{inc}\_k} \\
\bar{H}_k(M_n) & \xrightarrow{f_*} & \bar{H}_k(M_{n-1}) \\
\end{array}
\]

Let \( A_k = (\bar{f}_k(\bar{H}_k(M_k))) = \bar{f}_k(\bar{H}_k(D_k))) \subset H_k(D_k) \). \( A_k \) is invariant under \( F_*: H_k(D_*) \to H_k(D_*) \).

We obtain a commutative diagram

\[
\begin{array}{ccc}
A_k & \xrightarrow{\bar{f}_k(\bar{H}_k(M_k))} & A_k \\
\downarrow{\text{inc}\_k} & & \downarrow{\text{inc}\_k} \\
\bar{H}_k(M_k) & \xrightarrow{\text{kernel}(\text{inc})} & \bar{H}_k(M_{k+1}) \xrightarrow{f_*} H_k(M; Z) \\
\end{array}
\]

\[
\begin{array}{ccc}
A_k & \xrightarrow{\bar{f}_k(\bar{H}_k(M_k))} & A_k \\
\downarrow{\text{inc}\_k} & & \downarrow{\text{inc}\_k} \\
\bar{H}_k(M_k) & \xrightarrow{\text{kernel}(\text{inc})} & \bar{H}_k(M_{k+1}) \xrightarrow{f_*} H_k(M; Z) \\
\end{array}
\]
Let \( \theta_k: A_k \to H_k(M, Z) \) be the composition of the rows. We have a diagram of exact sequences

\[
\begin{array}{ccccccccc}
0 & \to & (\text{kernel } \ell_k)/\text{image } (\ell_{k+1}) & \to & A_k & \to & H_k(D_*) & \to & \text{kernel } \ell_j & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_k & \to & H_k(D_*) & \to & \text{image } \ell_j & \to & 0 & \to & 0 \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & \text{by construction } F_* \text{ is nilpotent on } \text{kernel } (\theta_k) \approx (\text{kernel } \ell_k)/\text{image } (\ell_{k+1}) \text{ and on } H_k(D_*)/A_k & & & & & & \text{and } H_k(D_*)/A_k & & \text{are finite groups}. \\
& & & & & & & & & & \text{The lemma now follows by} \\
& & \text{Lemma 1 above}. & & & & & & & & \square \\
\end{array}
\]

**Lemma 6.** Suppose \((C_*, E)\) is an endomorphism and \(T \subset H_k(C_*)\) is a finite \(E_*\)-invariant subgroup on which \(E_*\) is nilpotent. There exists an endomorphism \((C'_*, E')\) such that \(H_*(C'_*) \approx H_*(C)/T\), where \(E'_j\) is a nilpotent extension of \(E_j\) for \(j = k + 1, k + 2\) and \(E'_j = E_j\) otherwise.

**Proof.** We construct a nilpotent resolution of \(E_*/T \to T\) as in [5]. Let \(Z_1\) be a free \(Z\)-module with one generator for each element of \(T \setminus \{0\}\); \(e: Z_1 \to T\) the associated map, and define \(N_1: Z_1 \to Z_1\) on generators according to \(E_*/T\), so \(N_1\) is nilpotent. Let \(Z_2 = \text{kernel } (\varepsilon)\) and \(N_2 = N_1/Z_2\).

\[
\begin{array}{ccccccccc}
0 & \to & Z_2 & \overset{\text{inc}}{\to} & Z_1 & \overset{\varepsilon}{\to} & T & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_2 & \overset{\text{inc}}{\to} & Z_1 & \overset{\varepsilon}{\to} & T & \to & 0 \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & \text{T is also resolved by cycles and boundaries: let } \psi: Z_k \to H_k(C_*) \text{ be the projection and } Z'_k = \psi^{-1}(T). & & & & & & \text{Since } Z_1 \text{ is free and } \psi \text{ is onto there exists } \rho: Z_1 \to Z'_k \text{ such that } \psi \rho = \varepsilon. \\
\end{array}
\]

Combining these sequences we obtain a resolution of \(Z'_k\) as in [4, V.2]. Since \(Z_1\) is free and \(\psi\) is onto there exists \(\rho: Z_1 \to Z'_k\) such that \(\psi \rho = \varepsilon\). Let \(K = \text{kernel } (i \oplus \rho: B_k \oplus Z_1 \to Z'_k)\). We obtain a diagram

\[
\begin{array}{ccccccccc}
0 & \to & Z_2 & \overset{\text{inc}}{\to} & Z_1 & \overset{\psi}{\to} & T & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \overset{\text{inc}}{\to} & Z_1 & \overset{\psi}{\to} & T & \to & 0 \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & \text{Combining these sequences we obtain a resolution of } Z'_k \text{ as in [4, V.2]. Since } Z_1 \text{ is free and } \psi \text{ is onto there exists } \rho: Z_1 \to Z'_k \text{ such that } \psi \rho = \varepsilon. \\
\end{array}
\]
where \( i \oplus \rho \) is onto \( \mathbb{Z}_k \) and for all \( z \in \mathbb{Z}_2 \) there exists a unique \( b \in B_k \) such that \((b, z) \in K\). Also, for \( z \in \mathbb{Z}_1 \) there exists a unique \( b \in B_k \) such that \( \rho N_1(z) - E_k \rho(z) = i(b) \); let \( *'(z) \) be that \( b \). We obtain a resolution of \( E_k / \mathbb{Z}'_k \)

\[
\begin{array}{ccccccc}
0 & \rightarrow & K & \rightarrow & B_k & \oplus & \mathbb{Z}_1 & \rightarrow & \mathbb{Z}_k & \rightarrow & 0 \\
& & \downarrow H & \searrow E_{k-1} & \downarrow N_1 & \nearrow E_k & & \downarrow & \searrow & \downarrow & \\
0 & \rightarrow & K & \rightarrow & B_k & \oplus & \mathbb{Z}_1 & \rightarrow & \mathbb{Z}_k & \rightarrow & 0 \\
\end{array}
\]

where \( H = \begin{pmatrix} E_k & * \\ 0 & N_1 \end{pmatrix} \) restricted to \( K \). Since \( N_1 \) is nilpotent so is \( H \).

The lemma follows by splicing in this resolution of \( \mathbb{Z}'_k \) to kill \( T \). Since \( B_k \) is free we have

\[
\begin{array}{c}
C_{k+1} \cong Z_{k+1} \oplus B_k \\
\end{array}
\]

Let \( \text{inc} : \mathbb{Z}_k \rightarrow C_k \) be the inclusion. Then \((C'_*, E'_*)\) is the following endomorphism

\[
\begin{array}{cccccc}
C_{k+3} & \rightarrow & C_{k+2} & \rightarrow & Z_{k+1} & \rightarrow & C_k \\
\oplus & & \oplus & & \oplus & & \oplus \\
K & \rightarrow & B_k & \rightarrow & \mathbb{Z}_1 & \rightarrow & C_{k-1} \\
\text{inc} & & & & & \text{inc} & & \text{inc} \oplus (\text{inc} \oplus 0) \\
\end{array}
\]

with maps \( E'_j = E_j \) for \( j \neq k + 1, k + 2 \) and

\[
E'_{k+2} = \begin{pmatrix} E_{k+2} & 0 \\ 0 & H \end{pmatrix}, \quad E_{k+1} = \begin{pmatrix} E_{k+1} & * \\ 0 & N_1 \end{pmatrix} \]

Remark. If \( H_k(C_k) / T \) is free then \((C'_*, E'_*)\) can be constructed so that \( E'_j = E_j \) except for \( j = k, k + 1 \). For \( \mathbb{Z}'_k \) is free so the resolution of \( E_k / \mathbb{Z}'_k \) can be folded to obtain:

\[
\begin{array}{cccccccc}
B_k & \oplus & \mathbb{Z}_1 & \cong & K & \oplus & \mathbb{Z}_k & \cong \\
\downarrow * & & \downarrow \alpha & & \downarrow H & & \downarrow *' & \\
E_k & \oplus & \mathbb{Z}_1 & \cong & K & \oplus & \mathbb{Z}_k \\
\end{array}
\]

Now \( 0 \rightarrow \mathbb{Z}'_k \rightarrow \mathbb{Z}_k \rightarrow H_k(C_k) / T \rightarrow 0 \) is exact so if \( H_k(C_k) / T \) is free then \( \mathbb{Z}'_k \) is a direct summand of \( \mathbb{Z}_k \) and the map \( *' : \mathbb{Z}'_k \rightarrow K \) extends to a map \( *'' : C_k \rightarrow K \). The new complex \( C'_* \) is

\[
\begin{array}{cccccccc}
\rightarrow & C_{k+2} & \rightarrow & C_{k+1} & \oplus & \mathbb{Z}_1 & \rightarrow & C_k & \oplus & K & \rightarrow \\
\downarrow \| & & \| & & \| & & \| & & \| & \| \\
(Z_{k+1} & \oplus & B_k & \oplus & \mathbb{Z}_1 & \rightarrow & (B_{k-1} & \oplus & \mathbb{Z}_k & \oplus & K) \\
\downarrow 0 \oplus & \| & \| & \| & \| & \| & \| & \| & \| & \| \\
Z'_{k} & \oplus & K & \rightarrow & \mathbb{Z}_k & \oplus & K \\
\end{array}
\]

with maps

\[
E'_{k+1} = \begin{pmatrix} E_{k+1} & * \\ 0 & N_1 \end{pmatrix}, \quad E'_k = \begin{pmatrix} H & *'' \\ 0 & E_k \end{pmatrix}
\]

where both \( N_1 \) and \( H \) are nilpotent.
**Theorem 1.** Suppose $M$ is orientable and $f \in \mathcal{F}$. Given signed representatives $B^+_k$ for the basic sets of $f$, there exists a $Z$-endomorphism of $f(C_*, E)$ such that for all $k$, $E_k$ is a nilpotent extension of the signed representatives of index $k$.

**Proof:** We construct a nilpotent extension of $(D_*, F)$. By Lemma 5 $\text{Nil} \left( F_0; H_0(D_*) \cong \right)$ is finite; using Lemma 6 there exists a nilpotent extension $(D^*_k F^*_k)$ such that $(H_0(D^*_k), F^*_k) \cong (H_0(M; Z), f_0)$. Continuing inductively we obtain a $Z$-endomorphism of $f(C_*, E)$ such that for all $k$, $E_k$ is a nilpotent extension of $F_k$. Thus $E_k \sim F_k \sim B_k$, where $B_k$ is the nilpotent extension of the signed representatives of index $k$ of Lemma 4.

It follows there exist nilpotent blocks $N_k, N'_k$ such that the matrix

$$
\begin{pmatrix}
N_k \\
0 \\
0
\end{pmatrix}
$$

is a nilpotent extension of $B_k$. Let $L_k, L'_k$ be free $Z$-modules, of dim $L_k = \text{rank } N_k, \text{dim } L'_k = \text{rank } N'_k$. At the $k$-th stage we adjoin contractible pairs in dimension $k + 1, k$ and, $k - 1$.

Since the homology of $(C_*, E)$ is unchanged we obtain at the $n$-th stage the desired $Z$-endomorphism of $f$.

When $k = 0$ the last step could introduce $(-1)$ chains. Each component of $M_0$ either contains a periodic sink or else its orbit eventually wanders into such a component. We can absorb the wandering components into $M_1 - M_0$ and still have a filtration for $f$. Therefore we can assume $M_0$ has one component for each periodic sink so $f_0^*: H_0(M_0; Z) \cong$ is a permutation and $B_0 = f_0^* F_0 = E_0$, and no $(-1)$ chains are introduced. Similarly, when $k = n$, the last step in the proof need not introduce $(n + 1)$ chains. However, $(n + 1)$ chains could be introduced when we kill $\text{Nil} (F_0^*; H_{n-1}(D_*) \cong)$ using Lemma 6. If $H_{n-1} (M; Z) \cong H_{n-1}(D_b)/\text{Nil} (F_{n-1}^*)$ is free we can use the folded technique above and preserve the dimension of the complex.

**Remark.** If $f$ is a Smale diffeomorphism, Theorem 1 can be proved using Pixton’s theory of fitted rectangular decompositions, without the assumption that $M$ is orientable. Pixton proves that for Smale diffeomorphisms there exist filtrations $M_k, 0 \leq k \leq n$ where $M_k - M_{k-1}$ is a finite disjoint union of rectangles $R^-_i \times R^+_j [10]$. $R^-_i, R^+_j$ are submanifolds with boundary, which embed in Euclidean spaces of dimensions $k, n - k$, but are not necessarily discs. After choosing orientations the partition gives a choice of signed representatives, and it follows from the Künneth formula that $f_0^*: H_k(M_k, M_{k-1}; Z) \cong$ is a nilpotent extension of the signed representatives of index $k$. Any two signed representatives of the same basic set are shift equivalent $[3]$, so this is true independent of the choice of signed representatives. The rest of the proof of Theorem 1 is unchanged.
§3. GEOMETRIC REALIZATION

Let \((C_\ast, E)\) be a \(Z\)-endomorphism of \(f\). The geometric realization techniques of [13] require the stronger condition of chain homotopy equivalence. Suppose \(\dim M \geq 6\), \(\partial_1(M) = 0\), and let \(C_\ast(M)\) be the complex of a given handle decomposition of \(M\) and \(f_\ast: C_\ast(M) \cong \) the induced map. \((C_\ast, E)\) is realized by a handle decomposition of \(M\) and a diffeomorphism isotopic to \(f\) if and only if there exists a chain homotopy equivalence \(h: C_\ast \to C_\ast(M)\) such that \(hE \cong f_\ast h\). This will be satisfied provided \((C_\ast, E)\) is also an \(R\)-endomorphism of \(f\) for \(R = \mathbb{Z}/n\) all \(n\). (The problem is the off diagonal term in \((E \oplus 1_R)\) on \(H_k(C_\ast; R) \cong (H_k(C_\ast) \otimes R) \oplus (H_{k-1}(C_\ast) \otimes R)\). In case \(H_k(M; Z)\) is torsion free it suffices that \((C_\ast, E)\) be a \(Z\)-endomorphism of \(f\).

**Theorem 2.** Let \(M\) be a 2-connected manifold with torsion free homology and \(\dim M \geq 6\). If \(f \in \mathcal{F}\) has at least one source and one sink which are fixed then \(f\) is isotopic to an omega-conjugate fitted diffeomorphism.

**Proof:** Let \((C_\ast, E)\) be a \(Z\)-endomorphism of \(f\), as in Theorem 1. If \(C_1 = C_{n-1} = 0\) then by [13] there exists a handle decomposition of \(M\) and a fitted diffeomorphism \(g\) isotopic to \(f\) whose chain map realizes \(E\). \(g\) may be chosen so its geometric intersection numbers agree up to sign with the entries of \(E\). If the nilpotent blocks in the \(E_k\) are put in upper triangular form they give rise to wandering handles in \(g\). If the signed representatives \(B_k\) of \(f\) arose from Markov partitions with 0-1 geometric intersection numbers, then \(\Omega(f)\) is topologically conjugate to \(\Omega(g)\). Therefore the result follows provided \(C_1 = C_{n-1} = 0\).

We use folding, as in [13, Appendix A] to eliminate 1 and \((n - 1)\) chains. Suppose \((C_\ast, E)\) is given, \(H_1(C_\ast) = 0\) and \(\delta_1: C_k \to C_{k-1}\) is zero. Since all boundaries are free we have a splitting of \(C_{k+1}\)

\[
\begin{array}{ccccccccc}
C_{k+2} & \to & C_k+1 & \oplus & H_{k+1} & \oplus & C_k & \to & C_k & \to & 0 \\
E_{k+2} & & \downarrow e' & & \downarrow & & \downarrow E_k & & \downarrow E_k & & \\
C_{k+2} & \to & C_k+1 & \oplus & H_{k+1} & \oplus & C_k & \to & C_k & \to & 0 \\
\end{array}
\]

Folding \(C_k\) up into \((k + 2)\) we obtain:

\[
\begin{array}{ccccccccc}
C_{k+2} & \oplus & C_k & \oplus & (B_{k+1} & \oplus & H_{k+1} & \oplus & C_k) & \to & 0 & \to & C_{k-1} \\
C_{k+2} & \oplus & C_k & \oplus & (B_{k+1} & \oplus & H_{k+1} & \oplus & C_k) & \to & 0 & \to & C_{k-1} \\
\end{array}
\]

The off-diagonal term \(*_{z}: C_k \to B_{k+1}\) in \(E_{k+1}\) can be balanced by an off-diagonal term \((\delta_{k+2})^{-1} \circ *_{2}: C_k \to C_{k+2}\) in dimension \((k + 2)\) after folding. However a term \(*_{1}: C_k \to H_{k+1}\) in \(E_{k+1}\) cannot be compensated for this way. We assume \(M\) is 2-connected so this problem does not arise in folding \(C_1\) and \(C_{n-1}\).

Recall that when \(k = 0\) the endomorphism \(E_0\) and the signed representative \(B_0\) coincided in the proof of Theorem 1. Therefore if \(f\) has a fixed sink the corresponding component of \(M_0\) represents an invariant \(Z\) in \(H_0(M; Z)\) and hence in \(C_0\); the components do not represent boundaries in \(\delta_1: H_1(M_1, M_0; Z) \to H_0(M_0, Z)\) so we obtain an invariant splitting of \(E_0\)

\[
\begin{array}{ccc}
C_0 & \cong & (B_0 \oplus Z) \\
E_0 & \downarrow & E_0 \\
C_0 & \cong & (B_0 \oplus Z) \\
\end{array}
\]

We first fold \(B_0 \xrightarrow{E_0} B_0\) up into dimension \(k = 2\) so \(\delta_1: C_1 \to C_0\) is zero. Then we fold \(C_1 \xrightarrow{E_1} C_1\) up into dimension \(k = 3\). \((n - 1)\) chains can be folded similarly. The theorem
follows since all the non-wandering information in the $E_k$ is preserved although the index is not.

Remark. "Folding" was introduced in [13] where it was described for complexes. Here we will elaborate further the method for v.p. endomorphisms. Suppose $C_\bullet$ is a free $\mathbb{Z}$ complex $0 \to C_m \to \ldots \to C_L \to 0$ which has the homology of a simply connected manifold of dimension $n$ and $E: C_\bullet \to C_\bullet$ is represented by virtual permutation (v.p.) matrices. Then we claim $(C_\bullet, E)$ is chain homotopy equivalent to a v.p. endomorphism concentrated in dimensions $0 \leq k \leq n$ with $C_1 = C_{n-1} = 0$. As above, off-diagonal terms can present problems in folding. Let $E'_k = E_k$ except delete these off-diagonal terms in $k = n, n-2$ and $0, 2$. 1: $(C_\bullet, E) \to (C_\bullet, E')$ is a chain homotopy equivalence and $E'$ is still quasi-unipotent. Folding we obtain a new endomorphism $(C_\bullet, E)$ where $C_\bullet$ has the required form and $E$ is quasi-unipotent. By [5] we can add inverses on contractible pairs in adjacent dimensions so that all $C_k$, $2 \leq k \leq (n-3)$ have 2-step v.p. resolutions i.e. exact sequences $0 \to (D_1, F_1) \to (D_0, F_0) \to (C_k, E_k) \to 0$ where the $D_i$ are free and the $F_i$ are v.p. The Euler characteristic $\chi(C_\bullet, E)$ is unchanged so $(C_{n-2}, E_{n-2})$ has a resolution as well. Now splice in the resolutions for $2 \leq k \leq n-3$ as in [5]. To avoid re-introducing $(n-1)$ chains we need a transposed v.p. resolution of $(C_{n-2}, E_{n-2})$, i.e. a sequence $0 \to (C_{n-2}, E_{n-2}) \to (D'_0, F'_0) \to (D'_1, F'_1) \to 0$. Now modules $(M, E)$ which have v.p. resolutions are closed under short exact sequences [5, see also 8]. By duality so are free modules with transposed v.p. resolutions and a v.p. object trivially has a transposed v.p. resolution. Therefore a free object which has a v.p. resolution also has a transposed v.p. resolution. Splicing in a transposed v.p. resolution of $(C_{n-2}, E_{n-2})$ we obtain the desired v.p. endomorphism.

It would be good to relax the restrictive conditions of Theorem 2. Our arguments are a continuation of algebraic ideas that appeared briefly in [13] as the Čech theory “going up” proof that Morse-Smale diffeomorphisms can be represented by v.p. matrices, and of the related algebraic methods of [1] and [3]. One would also want to prove Theorem 2 geometrically, possibly in the spirit of the “going down” proof of [13] using the filtration by open manifolds and simple homotopy theory. In a forthcoming paper Pixton proves that a Smale diffeomorphism is fitted provided it satisfies a condition he calls dynamically tame. A natural approach would be to modify a Smale diffeomorphism by an isotopy to make it dynamically tame. This would also give more information about the original diffeomorphism.

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