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# THE EXISTENCE OF MORSE–SMALE DIFFEOMORPHISMS

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PERHAPS THE simplest smooth discrete time dynamical systems are the Morse–Smale diffeomorphisms. Among structurally stable systems they exhibit the simplest recurrent behavior—a finite set of hyperbolic periodic pionts and no other recurrence. These systems have been the object of considerable research. Palis and Smale[13] proved that they are structurally stable. More recent work has dealt with the relationship of the homotopy class of a diffeomorphism to the kind of dynamics it exhibts[5, 9, 11, 16] and the question of the existence of a Morse–Smale diffeomorphism in a given homotopy class. This latter topic is the subject of this article.

In[16], Shub and Sullivan showed among other things that a necessary condition for the existence of a Morse–Smale diffeomorphism is that all eigenvalues of the induced maps on homology be roots of unity. In the case of simply connected manifolds of dimension greater than five they reduced the question of existence to an algebraic condition on the chain level for the diffeomorphism (1.2, below) and using this condition pionted out the existence of additional obstructions related to the ideal class groups of the cyclotomic fields.

In this article we identify the group in which the obstruction lies, express it in terms of the algebraic $K$-theory of the induced endomorphisms $f_*: H_*(M) \to H_*(M)$ and show that there are no further obstructions. More precisely, we consider the category $QI$ of abelian groups with quasi-idempotent endomorphisms (i.e. having all eigenvalues zero or roots of unity) and show the obstruction lies in the torsion subgroup $G$ of $K_0(QI)$. The obstruction depends in fact only on the elements $\phi([f_{*k}])$ in $G$, where $[f_{*k}]$ denotes the class in $K_0(QI)$ of the quasi-idempotent endomorphism $f_{*k}: H_k(M, Z) \to H_k(M, Z)$ and $\phi: K_0(QI) \to G$ is a projection onto the summand $G$. Of course if $f$ is a diffeomorphism $f_{*k}$ will be quasi-unipotent (having only roots of unity as eigenvalues) since it is an automorphism. Our main result is the following.

THEOREM. *Suppose $f: M \to M$ is a diffeomorphism of a compact manifold and for all $k, f_{*k}: H_k(M; Z) \to H_k(M; Z)$ is quasi-unipotent. Then if $f$ is homologous to a Morse–Smale diffeomorphism, $\chi(f_*) = \Sigma(-1)^k \phi([f_{*k}])$ is zero in $G$. If $M$ is simply connected and of dimension greater than five then $\chi(f_*) = 0$ implies that $f$ is isotopic to a Morse-Smale diffeomorphism.*

The map $\phi$ is defined in §2. We do not know how to identify nonzero elements of $G$ or even if $G$ is non-trivial! We would find either result extremely interesting. If $G = 0$, then the characterization of which isotopy classes of diffeomorphisms of simply connected manifolds of dimension greater than five contain Morse–Smale diffeomorphisms would be particularly simple. If a nonzero element of $G$ were exhibited we would construct an isotopy class of diffeomorphisms of a manifold with

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boundary as in[16] such that $\chi(f_*)$ represents the element. We would then be confronted by the problem of determining the simplest dynamics of diffeomorphisms in this isotopy class. The best description of $G$ that we know are by Bass[2] and Grayson[6].

The group $G$ may be of interest in the study of the monodromy isolated singularities. The monodromy map is quasi-unipotent on homology and may be isotopic to a Morse–Smale diffeomorphism.

## §1. BACKGROUND AND DEFINITIONS

We begin by recalling a few definitions. If $f: M \to M$ is a diffeomorphism and $x \in M$ then $x$ is said to be *chain-recurrent* provided that given any $\varepsilon > 0$ there exist points $x = x_1, x_2, x_3, \dots, x_n = x$ such that $d(f(x_i), x_{i+1}) < \varepsilon$ where $d$ is a fixed metric on $M$. The set of chain recurrent points $\mathcal{R}$ is a compact invariant set (e.g. [3]). If $\mathcal{R}$ is finite it clearly consists of a finite set of periodic orbits. If $x$ is a periodic point of period $p$ then $x$ is called *hyperbolic* if $df_x^p: TM_x \to TM_x$ has no eigenvalues of absolute value one.

If $x$ is a hyperbolic periodic point then its stable manifold $W^s(x)$ and unstable manifold $W^u(x)$ are defined by $W^s(x) = \{y | d(f^n x, f^n y) > 0 \text{ as } n \to \infty\}$ and $W^u(x) = \{y | d(f^n x, f^n y) \to 0 \text{ as } n \to -\infty\}$. They are injectively immersed Euclidean spaces with $\dim(W^u(x)) = \text{index of } x$ (see [17]).

1.1 *Definition.* A diffeomorphism $f: M \to M$ is called *Morse–Smale* provided its chain recurrent set $\mathcal{R}$ consists of a finite set of hyperbolic periodic points and for any $x, y \in \mathcal{R}$ the stable and unstable manifolds $W^s(x)$ and $W^u(y)$ intersect transversely.

The condition that $\mathcal{R}$ consists of hyperbolic points implies that $f$ satisfies Axiom $A$ of[17].

We now quote a result of Shub and Sullivan ([16], see[14] also) which is the basis of our further work. We have specialized the results of[16] to fit our setting.

1.2 THEOREM [16] *Suppose $M$ is a simply connected compact manifold of dimension $n$ greater than five and $f: M \to M$ is a diffeomorphism. Then a necessary and sufficient condition that there exists a Morse–Smale diffeomorphism isotopic to $f$ is that there exists a finitely generated $n$-dimensional free chain complex $\mathcal{C}$ with $C_1 = C_{n-1} = 0$ and an endomorphism $\tau: \mathcal{C} \to \mathcal{C}$ such that:*

(1) *If $\sigma: \mathcal{D} \to \mathcal{D}$ is a chain level representation of $f$ then there is a chain homotopy equivalence $h: \mathcal{C} \to \mathcal{D}$ such that $\sigma \circ h \sim h \circ \tau$, $\sigma \circ h$ is chain homotopic to $h \circ \tau$, and*

(2) *for each $1 \le k \le n$, $\tau_k: C_k \to C_k$ is representable by a virtual permutation matrix.*

1.3 *Definition.* A matrix will be called *virtual permutation* (or V.P. for short) provided it has the form

$$\begin{bmatrix} P_1 & * & & * & \\ 0 & P_2 & * & & \\ & 0 & & & \\ & & & & * \\ 0 & & & 0 & P_r \end{bmatrix}$$

where each $P_i$ is the 0 matrix, $\pm I$, or has the form

$$\begin{bmatrix} 0 & 1 & 0 & & 0 \\ & 0 & 1 & 0 & \\ & & & & \\ 0 & & & & 1 \\ \pm 1 & 0 & & & 0 \end{bmatrix}.$$

A matrix in the above form but with each $P_i$ a companion matrix will be called *companion-like.*

We will also need a fact from [5] concerning similarity classes over the integers of companion matrices. We will denote by $C(p)$ the companion matrix of the monic polynomial $p(x)$ and the $n \times m$ matrix with 1 in $n$th row and 1st column and 0 elsewhere will be denoted by $E(n, m)$.

1.4 Proposition [5]. *Let $p(x)$ be a monic polynomial with integer coefficients. If $p(x) = f_1(x)f(x)\ldots f_k(x)$, then $C(p)$ is similar over the integers to the block triangular matrix*

$$\begin{bmatrix} C(f_1) & E(d_1d_2) & 0 & \cdots & 0 \\ 0 & C(f_2) & E(d_2d_3) & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0\cdots C(f_{k-1}) & & E(d_{k-1}d_k) \\ 0 & 0 & 0\cdots\ \ 0 & & C(f_k) \end{bmatrix}$$

*where $d_i = \deg f_i(x)$.*

1.5 Lemma. *Suppose $C$ is a companion-like matrix whose eigenvalues are all roots of unity or zero. Then there are matrices*

$$\begin{bmatrix} Q_1 & * & * \\ 0 & Q_2 & \\ & \ddots & * \\ 0 & 0 & Q_r \end{bmatrix} \text{ and } \begin{bmatrix} P_1 & * & & * \\ 0 & \ddots & * & \\ & & P_s & * \\ 0 & & 0 & C \end{bmatrix}$$

*which are similar over $Z$ where each $P_i$ is a permutation matrix and each $Q_j$ is a permutation matrix or the zero matrix.*

*Proof.* We first consider the case where $C$ is actually a companion matrix, say $C(p(t))$. The characteristic polynomial $p(t)$ of $C$ has the form $t^m\ (\prod_i \Phi_{n_i})$ where the $\Phi_n$ is the $n$th cyclotomic polynomial. A simple induction shows that any polynomial of this form can be written in the form

$$t^m\left((\prod_k(t^{m_k}-1)/\prod_j(t^{p_j}-1)\right)$$

for some sets of positive integers $\{m_i\}$ and $\{p_j\}$. The inductive step uses the fact that the $n$th cyclotomic polynomial $\Phi_n$ equals $(t^n - 1)$ divided by a product of $\Phi_j$'s, $j < n$.

Thus we have

$$p(t)\prod_j(t^{p_j}-1) = t^m\prod_k(t^{m_k}-1).$$

Now according to 1.4, $C(p(t))\prod_j(t^{p_j}-1))$ is similar over $Z$ to a matrix of the form

$$\begin{bmatrix} P_1 & * & & * \\ 0 & \ddots & & \\ & & P_s & * \\ 0 & & 0 & C \end{bmatrix}$$

and $C(t^m\Pi(t^{m_k}-1))$ is similar to a matrix of the form

$$\begin{bmatrix} Q_1 & * & * \\ & \ddots & * \\ 0 & & Q_r \end{bmatrix}.$$

To prove the result for any companion-like $C$ we induct on the number of companion matrix diagonal blocks in $C$. Thus if $C = \begin{pmatrix} C_1 & * \\ 0 & C_2 \end{pmatrix}$, there is by the induction hypothesis a matrix

$$A = \left[\begin{array}{cccc|ccc} P_1 & 0 & & & & & 0 \\ 0 & \ddots & & & & & \\ & & P_s & 0 & & & \\ & & 0 & C_1 & & 0 & * \\ \hline & & & & P_1' & & \\ & & & & & \ddots & \\ & & & & & P_t' & 0 \\ 0 & & & & & 0 & C_2 \end{array}\right]$$

which is similar to a matrix of the form

$$\begin{bmatrix} Q_1 & & * \\ & & \\ 0 & & Q_r \end{bmatrix}.$$

On the other hand $A$ is similar to

$$\left[\begin{array}{cccc|c} P_1 & & & 0 & 0 \\ & \ddots & & & \\ & & P_s & & \\ & & P_1' & & \\ & & & \ddots & \\ 0 & & & P_t' & \\ \hline 0 & & & & \begin{matrix} C_1 & * \\ & C_2 \end{matrix} \end{array}\right],$$

so this completes the proof

## §2. THE SUMMAND $G$ OF $K_*(\overline{QI})$

Let $\overline{QI}$ denote the category whose objects are pairs $(F, e)$ with $F$ a finitely generated free abelian group and $e$ an endomorphism of $F$ which is quasi-idempotent (an endomorphism is called *quasi-idempotent* if all its eigenvalues are roots of unity or zero). A morphism $h: (F_1, e_1) \to (F_2, e_2)$ in this category is a homomorphism $h: F_1 \to F_2$ such that $he_1 = e_2h$.

2.1 *Definition.* Let $QI$ denote the category of finitely generated abelian groups with quasi-idempotent endomorphisms, and let $QU$ (for quasi-unipotent) denote the full subcategory of $QI$ consisting of automorphisms from $QI$.

An object in one of these categories is a pair $(G, e)$ with $G$ a finitely generated abelian group and $e$ an endomorphism (or automorphism) such that the endomorphism (automorphism) induced by $e$ on $G$/torsion is quasi-idempotent. An exact sequence in $QI$ or $\overline{QI}$ is a sequence $0 \to (A, a) \to (B, b) \to (C, c) \to 0$ which is exact on the level of abelian groups.

2.2 PROPOSITION. *The map $K_0(\overline{QI}) \to K_0(QI)$ induced by inclusion is an isomorphism.*

*Proof.* This follows easily using (7.1) of [1] when one observes that each object of $QI$ has a finite resolution by objects of $\overline{QI}$. To demonstrate the latter fact it suffices to consider the case of $(G, e)$ with $G$ finite. Then let $ZG$ be the free abelian group with $G$ as a set of generators and $\alpha: ZG \to G$ the obvious map. If $e_1: ZG \to ZG$ is the endomorphism induced by the action of $e$ on generators then clearly it is quasi-idempotent and $\alpha$ is a morphism in $QI$. Finally, if $K$ is the kernel of $\alpha$, then

$$0 \to (K, e_1|K) \to (ZG, e_1) \to (G, e) \to 0$$

is the desired resolution.

Because of the result above, every element of $K_0(QI)$ can be represented in the form $[e_1] - [e_2]$ where $e_1$ and $e_2$ are endomorphisms of free abelian groups and [ ] denotes their class in $K_0$ (henceforth we will generally write simply the endomorphism $e$ for an object $(G, e)$ in $\overline{QI}$ or $QI$, suppressing the abelian group on which it acts). We can now define a homomorphism $p: K_0(QI) \to \mathcal{P}$ where $\mathcal{P}$ is the multiplicative group of rational functions whose numerators and denominators are monic polynomials with roots either zero or roots of unity. It is given by $p([e_1] - [e_2]) = f_1(t)/f_2(t)$ where $f_i(t)$ is the characteristic polynomial of $e_i$. It is easy to see that $p$ is well defined and a homomorphism.

2.3 PROPOSITION. *The homomorphism $p: K_0(QI) \to \mathcal{P}$ has a right inverse. That is, $\mathcal{P}$ is isomorphic to a direct summand of $K_0(QI)$.*

*Proof.* Define $i: \mathcal{P} \to K_0(QI)$ by $i(f_1(t)/f_2(t)) = [C(f_1(t))] - [C(f_2(t))]$ where $C(f_i(t))$ is the endomorphism induced by the companion matrix of $f_i(t)$. Clearly $p \circ i = id: \mathcal{P} \to \mathcal{P}$.

Two immediate corollaries of 1.5 are:

2.4 PROPOSITION. *If $(C, c)$ is companion-like then there is an exact sequence $0 \to (A, a) \to (B, b) \to (C, c) \to 0$ where $(A, a)$ and $(B, b)$ are V.P.*

2.5 PROPOSITION. *The subgroup of $K_0(QI)$ generated by the classes of permutation endomorphisms and the zero endomorphism is $i(\mathcal{P})$.*

We remark that a similar proof shows that $i(\mathscr{P})$ is generated by companion matrices or by virtual permutations and zero.

We now turn to a more categorical description of $i(\mathscr{P})$.

2.6 *Definition.* Let $P$ denote the full subcategory of $QI$ whose objects are objects of $QI$ equivalent in $K_0$ to a linear combination of permutation endomorphisms and the zero endomorphism, or equivalently by 2.5 to a linear combination of companion matrix endomorphisms.

Now we show that in a very strong sense, for each endomorphism $e$ in $\overline{QI}$, There is another endomorphism $e'$ in $\overline{QI}$ which represents an inverse of $e$ in $K_0(QI)$ modulo $K_0(P)$.

2.7 PROPOSITION. *If $(F, e)$ is an object of $QI$ with $F$ free, then there exists $(F', e') \in QI$ with $F'$ free such that $(F \oplus F', e \oplus e')$ is representable by a companion-like matrix, i.e. a matrix of the form*

$$\begin{bmatrix} C_1 & * & & * \\ 0 & \ddots & & \\ & & \ddots & * \\ 0 & & 0 & C_n \end{bmatrix}$$

*with each $C_i$ a companion matrix.*

*Proof.* It is a well known fact (see [12, p. 50]) that we can choose a basis for $F$ such that the matrix $A$ corresponding to $e$ has the form

$$\begin{bmatrix} A_1 & * & & * \\ 0 & \ddots & & \\ & & \ddots & * \\ 0 & & 0 & A_n \end{bmatrix}$$

where each $A_i$ has irreducible characteristic polynomial. It will suffice then to prove the existence of a matrix $A_i'$ such that $\begin{pmatrix} A_i & 0 \\ 0 & A_i' \end{pmatrix}$ is similar to $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ where $C$ is the companion matrix of the characteristic polynomial of $A_i$, since the desired endomorphism $e'$ is then endomorphism induced by the matrix

$$\begin{bmatrix} A_1' & 0 & & 0 \\ 0 & A_2' & & \\ & & \ddots & 0 \\ 0 & & 0 & A_n' \end{bmatrix}.$$

We note that the existence of $A_i'$ is trivial if $A_i$ is the zero matrix, so we may assume the characteristic polynomial of $A_i$ is a cyclotomic polynomial $\Phi_k$.

Now let $\omega$ be a primitive $k$th root of unity and let $\Lambda = Z[\omega]$. Then isomorphism classes of elements of $QI$ with characteristic polynomial $\Phi_k$ are in one-to-one correspondence with the elements of the ideal class group of $\Lambda$ (e.g. [12, p. 53]). The correspondence is given as follows: If $I$ is an ideal in $\Lambda$ then it is a free $Z$ module and multiplication by $\omega$ is a $Z$ linear endomorphism and hence determines a similarity class of matrices with characteristic polynomial $\Phi_k$. It is easy to see that the principal ideal class, i.e. the class of $\Lambda$, corresponds to the companion matrix $C$.

Now $\Lambda$ is a Dedekind domain so the ideal classes form a group and given ideals $I_1$, $I_2$ we have $I_1 \oplus I_2 \cong \Lambda \oplus I_1 I_2$ as $\Lambda$ modules (e.g. [10]). Let $I$ be an ideal representing the similarity class of $A_i$ and $I'$ an ideal representing the inverse element of the ideal class group. Then

$$I \oplus I' = \Lambda I \cdot I' \cong \Lambda \oplus \Lambda \text{ as } \Lambda \text{ modulus.}$$

Let $h$ be the endomorphism of $I \oplus I'$ obtained by multiplication by $\omega$. By choosing a $Z$ basis of $I \oplus I'$ appropriately we see $h$ can be represented by a matrix of the form $\begin{pmatrix} A_i & 0 \\ 0 & A_i' \end{pmatrix}$ for some $A_i'$. On the other hand, since $I \oplus I' \cong \Lambda \oplus \Lambda$, (as $\Lambda$ modules), with a different choice of $Z$ basis, $h$ will be represented by the matrix $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.

2.8 Lemma. *Suppose* $(F_i; e_i) \in \overline{QI}$, $i = 1, 2$, *and* $[(F_1, e_1)] = [(F_2, e_2)]$ *in* $K_0(QI)$. *Then there are companion-like endomorphisms* $(C, c)$ *and* $C', C')$ *and* $x_i$: $C' \to C$, $i = 1, 2$, *such that*

$$(F_1 \oplus M, e_1 \oplus m_1) \cong (F_2 \oplus M, e_2 \oplus m_2)$$

*where* $M = C \oplus C'$ *and*

$$m_i = \begin{pmatrix} c & x_i \\ 0 & c' \end{pmatrix}.$$

*Hence* $(M, m_i)$ *is companion like and* $[(M, m_1)] = [(M, m_2)]$ *in* $K_0(P)$.

*Proof.* Since $K_0(QI) \cong K_0(\overline{QI})$, we may work in $K_0(\overline{QI})$. By [1, §(4.10)], there are exact sequences in $\overline{QI}$

$$0 \to (X, x) \to (N_i, n_i) \to (Y, y) \to 0 \quad i = 1, 2,$$

such that $(F_1 \oplus N_1, e_1 \oplus n_1) \cong (F_2 \oplus N_2, e_2 \oplus n_2)$. Now using 2.7 we take inverses, i.e. we pick $(X', x')$ and $(Y', y')$ such that

$$0 \to (X \oplus X', x \oplus x') \to (N_i \oplus X' \oplus Y', n_i \oplus x' \oplus y') \to (Y \oplus Y', y \oplus y') \to 0$$

is exact for $i = 1, 2$.

Let $(M_i, m_i) = (N_i \oplus X' \oplus Y', n_i \oplus x' \oplus y')$ for $i = 1, 2$. Since $M_i$ is free, $M_1 \cong M_2$ and we denote it simply $M$. Now

$$(F_1 \oplus M, e_1 \oplus m_1) = (F_2 \oplus M, e_2 \oplus m_2)$$

and one checks easily that $m_i$ has the desired form and hence $[M, m_1] = [M, m_2]$ in $K_0(P)$.

2.9 Lemma. *Suppose* $(H, h) \in P$; *then there is a short exact sequence*

$$0 \to (X_1, x_1) \to (X_2, x_2) \to (H, h) \to 0$$

*with* $X_i$ *free*, $i = 1, 2$ *and* $(X_i, x_i) \in P$, $i = 1, 2$.

*Proof.* As in 2.1 we have

$$0 \to (A, a) \to (B, b) \to (H, h) \to 0$$

with $(A, a)$ and $(B, b)$ in $\overline{QI}$. By (2.7) we have $(B', b')$ with $(B \oplus B', b \oplus b')$ companion-like, so

$$0 \to (A, a) \oplus (B', b') \to (B, b) \oplus (B', b') \to (H, h) \to 0$$

is exact. the two right hand elements in the sequence are in $P$ so the left hand one is, too.

We now collect some facts about the category $P$. We will say that a subcategory $\mathscr{A}$ of a category $B$ is closed with respect to short exact sequences if whenever two terms of a short exact sequence in $B$ are in $\mathscr{A}$ so is the third.

2.10 Proposition. (a) *The category $P$ is the smallest full subcategory of $QI$ containing companion matrix endomorphisms (or virtual permutation endomorphisms) which is closed with respect to short exact sequences.* (b) *The map $K_0(P) \to K_0(QI)$ induced by inclusion is an isomorphism onto the summand $i(\mathscr{P})$ of $K_0(QI)$.* (c) *If $(F, e) \in P$ and $F$ is free then there is an $(F', e') \in P$ with $F'$ free and $e'$ companion-like such that $e \oplus e'$ is companion-like.*

*Proof.* (a) The category $P$ is closed under short exact sequences and contains companion matrix endomorphisms. To prove $P$ is contained in the category described we take $(F, e) \in P$ (by 2.9 we may in fact assume $F$ is free) and show $(F, e)$ is in this category. Now $[(F, e)] = [(C_1, c_1)] - [(C_2, c_2)]$ in $K_0(QI)$ with $c_i$ companion-like, so by 2.8 there are $(M, m_i)$ with $m_i$ companion-like such that

$$(F, e) \oplus (C_2, c_2) \oplus (M, m_2) \cong (C_1, c_1) \oplus (M, m_1).$$

Thus $e \oplus c_2 \oplus m_2$ is companion-like and

$$0 \to (C_2, c_2) \oplus (M, m_2) \to (F, e) \oplus (C_2, c_2) \oplus (M, m_2) \to (F, e) \to 0$$

is exact and $(F, e)$ is in the aforementioned category. The same is true for virtual permutations by 1.4 and 1.5.

(b) We first show the map $K_0(P) \to K_0(QI)$ induced by inclusion is injective. By (a) any element of $K_0(P)$ can be expressed as $[(C_1, c_1)] - [(C_2, c_2)]$ in $K_0(P)$. If such an element is in the kernel, then $[(C_1, c_1)] = [(C_2, c_2)]$ in $K_0(QI)$. Hence by 2.8 there are $(M, m_i)$ with $m_i$ companion-like such that $(C_1, c_1) \oplus (M, m_1) \cong (C_2, c_2) \oplus (M, m_2)$ and $[(M, m_1)] = [(M, m_2)]$ in $K_0(P)$ so $[(C_1, c_1)] = [(C_2, c_2)]$ in $K_0(P)$.

Now the image of $K_0(P)$ in $K_0(QI)$ and $i(\mathscr{P}) \subset K_0(QI)$ can both be described as the subgroup generated by the classes of companion endomorphisms, so (b) follows.

(c) As in (a), $(F, e) \oplus (C_2, c_2) \oplus (M, m_2) \cong (C_1, c_1) \oplus (M, m_1)$.

2.11 *Definition. Let $G$ be the group $K_0(QI)/K_0(P)$ so that*

$$K_0(QI) \cong G \oplus K_0(P) = G \oplus \mathscr{P}.$$

*Let $\phi$: $K_0(QI) \to G$ be the projection.*

2.12 *Remark.* The group $G$ can be identified with the torsion subgroup of $K_0(QI)$. It is not difficult to show that every element of $G$ has finite order. This uses the arguments of 2.7 together with the fact that the ideal class group of the ring $\Lambda$ from the proof of 2.7 is a finite group. Thus since every element of $\mathcal{P}$ has infinite order, $G$ is isomorphic to the torsion subgroup of $K_0(QI)$.

### §3. REALIZING HOMOLOGY ENDOMORPHISMS BY VIRTUAL PERMUTATION CHAIN MAPS

In this section we will consider the category of finitely generated chain complexes with chain map endomorphisms. In this category a morphism $h: (\mathscr{C}, \tau) \to (\mathscr{C}', \tau')$ is a chain map $h: \mathscr{C} \to \mathscr{C}'$ such that $h \circ \tau \sim \tau' \circ h$, $h \circ \tau$ is chain homotopic to $\tau' \circ h$. Thus when we refer to a chain homotopy equivalence $h$ from $(\mathscr{C}, \tau)$ to $(\mathscr{C}', \tau')$, we assume tacitly that $h \circ \tau \sim \tau' \circ h$.

Most often we will be concerned with $QI$ (*or* $QU$) *chain complexes*, that is, pairs $(\mathscr{C}, \tau)$ where $\mathscr{C} = \{C_i\}$ is a finitely generated complex and $\tau = \{\tau_i\}$ is a chain map such that for each $i$, $(C_i, \tau_i)$ is in $QI$ (or $QU$). Our object is to determine when there is a chain homotopy equivalence $h$: $(\mathscr{C}, \tau) \to (\mathscr{D}, \sigma)$ to another $QI$ chain complex with $\sigma_i: D_i \to D_i$ representable by a virtual permutation matrix for all $i$, and hence with $D_i$ free.

We first recall from 2.11 that we defined the projection

$$\phi: K_0(QI) \to K_0(QI)/K_0(P) = G.$$

If $(F, e) \in QI$, we will abuse notation slightly and refer to $\phi(e)$ when we mean $\phi([e])$.

3.1 *Definition. If* $(\mathscr{C}, \tau)$ *is a QI chain complex we define* $\chi(\tau) = \Sigma(-1)^i \phi(\tau_i)$.

3.2 Proposition. *If* $(\mathscr{C}, \tau)$ *is a QI chain complex and* $\tau_*: H_*(\mathscr{C}) \to H_*(\mathscr{C})$ *is the map induced on homology then* $\chi(\tau) = \chi(\tau_*)$.

*Proof.* The fact that a map $\phi$ is defined on $K_0(QI)$, i.e. depends only on $[e]$ rather than $e$ means that $\phi$ is an Euler–Poincaré mapping that is, additive on short exact sequences. The result claimed is well known for such mappings (see [8, pp. 98, 99]).

3.3 Theorem. *If* $(\mathscr{C}, \tau)$ *is a finitely generated free chain complex with* $(H_k(\mathscr{C}), \tau_{*k}) \in QI$ *for all* $k$, *then a necessary and sufficient condition that* $(\mathscr{C}, \tau)$ *be chain homotopy equivalent to a free QI complex* $(\mathscr{D}, \sigma)$ *with* $\sigma_i$ *representable by virtual permutation matrices is that* $\chi(\tau_*) = 0$, *where*

$$\chi(\tau) = \Sigma(-1)^i \phi(\tau_{i*}) \in G = K_0(QI)/K_0(P).$$

The first step in the proof of this theorem is to show that it suffices to prove the theorem when $\mathscr{C}$ is a free $QI$ chain complex.

3.4 Proposition. *If* $(\mathscr{C}, \tau)$ *is a finitely generated free chain complex with chain endomorphism* $\tau$ *such* $(H_k(\mathscr{C}), \tau_{*k}) \in QI$ *for all* $k$, *then* $(\mathscr{C}, \tau)$ *is chain homotopy equivalent to a QI chain complex* $(\mathscr{C}', \tau')$ *with* $\mathscr{C}'$ *free.*

We will actually prove a slightly more general proposition, which is of interest in light of [16]. First we need two definitions.

3.5 *Definition.* Let $h: G \to G$ be an endomorphism of a finitely generated abelian group. A resolution of $h$ is a collection of finitely generated free abelian groups $F_i$

with endomorphisms $h_i: F_i \to F_i$, $0 \le i \le n$ for some finite $n$ and maps $d_i: F_i \to F_{i-1}$, $\varepsilon: F_0 \to G$ such that

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
F_n & \xrightarrow{h_n} & F_n \\
\downarrow d_n & & \downarrow d_n \\
F_{n-1} & \xrightarrow{h_{n-1}} & F_{n-1} \\
\downarrow d_{n-1} & & \downarrow d_{n-1} \\
\vdots & & \vdots \\
\downarrow d_1 & & \downarrow d_1 \\
F_0 & \xrightarrow{h_0} & F_0 \\
\downarrow & & \downarrow \\
G & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

commutes and the columns are exact.

3.6 *Definition.* If $h: G \to G$ is an endomorphism of a finitely generated abelian group we denote by $\gamma(h)$ the largest modulus of an eigenvalue of the induced endomorphism on $G$/torsion. Given a finitely generated chain complex with endomorphism $(\mathscr{C}, \mathscr{E})$ we define $\gamma(\mathscr{E}) = \max \gamma(E_i)$ where $\mathscr{E} = \{E_i\}$.

3.7 LEMMA. *If $h: G \to G$ is an endomorphism then there is a resolution*

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
F_1 & \xrightarrow{h_1} & F_1 \\
\downarrow & & \downarrow \\
F_0 & \xrightarrow{h_0} & F_0 \\
\downarrow & & \downarrow \\
G & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

*such that $\gamma(h_i) \le \max(\gamma(h), 1)$ for $i = 0, 1$. Moreover, if $A: G \to H$ is an isomorphism and $h': H \to H$ is an endomorphism such that*

$$\begin{array}{ccc}
G & \xrightarrow{h} & G \\
A \downarrow & & \downarrow A \\
H & \xrightarrow{h'} & H
\end{array}$$

*commutes then there is a resolution*

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ F_1 & \xrightarrow{h_1'} & F_1 \\ \downarrow & & \downarrow \\ F_0 & \xrightarrow{h_0'} & F_0 \\ \downarrow & & \downarrow \\ H & \xrightarrow{h} & H \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

*such that* $\gamma(h_0') = \gamma(h_0)$, $\gamma(h_1') = \gamma(h_1)$.

The proof is very similar to the proof of 2.1 and hence we omit it.

Given a finitely generated free chain complex with chain endomorphism $(\mathscr{C}, \mathscr{E})$, we can always write $C_i \approx Z_i \oplus B_{i-1}$, the $i$-cycles plus the $i-1$ boundaries. With this decomposition $E_i$ has a matrix representation which we denote

$$\begin{pmatrix} \mathscr{Z}_i & \mathscr{D}_i \\ 0 & \mathscr{B}_i \end{pmatrix}.$$

3.8 LEMMA. *Let* $(\mathscr{C}, \mathscr{E}^1)$ *and* $(\mathscr{C}, \mathscr{E}^2)$ *be a free chain complex with two chain endomorphisms. Suppose that* $H_*(\mathscr{E}^1) = H_*(\mathscr{E}^2)$ *and that in the matrix representation*

$$\mathscr{E}_i^1 = \begin{pmatrix} \mathscr{Z}_i^1 & \mathscr{D}_i^1 \\ 0 & \mathscr{B}_i^1 \end{pmatrix}, \quad \mathscr{E}^2 = \begin{pmatrix} \mathscr{Z}_i^2 & \mathscr{D}_i^2 \\ 0 & \mathscr{B}_i^2 \end{pmatrix}$$

*we have* $\mathscr{D}_i^1 = \mathscr{D}_i^2$ *for all i. Then* $\mathscr{E}^1$ *and* $\mathscr{E}^2$ *are chain homotopic.*

*Proof.* Given $x \in Z_i$, $\mathscr{E}_i^1(x)$ is homologous to $\mathscr{E}_i^2(x)$ so $\mathscr{E}_i^1(x) - \mathscr{E}_i^2(x) = \partial v$ for a unique $v$ in $B_{i-1} \subset C_i$. Let $D_i(x) = v$ and let $D_i : C_i \to C_{i+1}$ be $D_i \oplus 0 : Z_i \oplus B_{i-1} \to Z_{i+1} \oplus B_i$. Then the $D_i$ define a chain homotopy $\mathscr{E}^1 - \mathscr{E}^2 = \partial D + D\partial$.

3.9 PROPOSITION. *If* $(\mathscr{C}, \tau)$ *is a finitely generated free chain complex with chain endomorphism* $\tau$, *then* $(\mathscr{C}, \tau)$ *is chain homotopy equivalent to a finitely generated free chain complex* $(\mathscr{C}', \tau')$ *with* $\gamma(\tau') \leq \max(\gamma(\tau_*), 1)$, *where* $\tau_*: H_*(\mathscr{C}) \to H_*(\mathscr{C})$.

*Proof.* We will argue inductively. Let $N$ be the dimension of the chain complex $\mathscr{C}$. Assuming that $(\mathscr{C}, \tau)$ is chain homotopy equivalent to $(\mathscr{C}', \tau')$ with $C_n' = 0$ for all $n \geq N$ and $\gamma(\tau_i) \leq \gamma(\tau_*)$ for all $i \leq k < N-1$, we will produce $(\mathscr{C}'', \tau'')$ chain homotopy equivalent to $(\mathscr{C}', \tau')$ with $C_n'' = 0$ for all $n \geq N$ and $\gamma(\tau_i'') \leq \max(\gamma(\tau_*), 1)$ for all $i \leq k+1$. If $k = N-1$ we produce $(\mathscr{C}'', \tau'')$ chain homotopy equivalent to $(\mathscr{C}', \tau')$ with $C'' = 0$ for all $n \geq N+1$ and $\gamma(\tau_i'') \leq \max(\gamma(\tau_*), 1)$. We have two resolutions

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
F_1 \arrow[r, "h_1"] \arrow[d, "d_1"] & F_1 \arrow[d, "d_1"] \\
F_0 \arrow[r, "h_0"] \arrow[d] & F_0 \arrow[d] \\
H_k(\mathscr{C}) \arrow[r, "\tau_{k*}"] \arrow[d] & H_k(\mathscr{C}) \arrow[d] \\
0 & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
B_k \arrow[r, "\tau_k"] \arrow[d, "\partial_{k+1}"'] & B_k \arrow[d, "\partial_{k+1}"] \\
Z_k \arrow[r, "\tau_k"] \arrow[d] & Z_k \arrow[d] \\
H_k(\mathscr{C}) \arrow[r, "\tau_{k*}"] \arrow[d] & H_k(\mathscr{C}) \arrow[d] \\
0 & 0
\end{tikzcd}
$$

with $\gamma(h_i) \leq \max(\gamma(\tau_{k*}), 1)$. We could add contractible free chain complex $\mathscr{M}$ with $M_i = 0$ for $i \neq k+1, k$, $M_k = M_{k+1} = K_1$ for some finitely generated free abelian group and $\partial_{k+1} = id$ with chain endomorphism 0 to $(\mathscr{C}, \tau)$. Our new boundary, cycle exact sequence would then be

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
B_k \oplus K_1 \arrow[r, "\tau_k \oplus 0"] \arrow[d, "\partial_{k+} \oplus id"'] & B_k \oplus K_1 \arrow[d, "\partial_{k+} \oplus id"] \\
Z_k \oplus K_1 \arrow[r, "\tau_k \oplus 0"] \arrow[d] & Z_k \oplus K_1 \arrow[d] \\
H_k(\mathscr{C}) \arrow[r, "\tau_{k*}"] \arrow[d] & H_k(\mathscr{C}) \arrow[d] \\
0 & 0.
\end{tikzcd}
$$

Similarly, if $K_2$ is a free abelian group we have the resolution

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
F_1 \oplus K_2 \arrow[r, "h_1 \oplus 0"] \arrow[d, "d_1 \oplus id"'] & F_1 \oplus K_2 \arrow[d, "d_1 \oplus id"] \\
F_0 \oplus K_2 \arrow[r, "h_1 \oplus 0"] \arrow[d, "\varepsilon \oplus 0"'] & F_0 \oplus K_2 \arrow[d, "\varepsilon \oplus 0"] \\
H_k(\mathscr{C}) \arrow[r, "\tau_{k*}"] \arrow[d] & H_k(\mathscr{C}) \arrow[d] \\
0 & 0
\end{tikzcd}
$$

By the appropriate choice of $K_1$ and $K_2$ we may assume in $(I)$ that rank $(F_0) =$ rank $(Z_k)$, which we now do. By the structure theorem for finitely generated abelian groups there is an isomorphism $A: F_0 \to Z_k$ such that $A(F_1) = B_k$.

We write $(\mathscr{C}', \tau')$ with respect to cycles and boundaries in the appropriate dimensions

$$\begin{tikzcd} \cdots \arrow[r] & C'_{k+2} \arrow[r, "\partial_{k+2}"] \arrow[d, "\tau'_{k+2}"] & Z_{k+1} \oplus B_k \arrow[r, "\partial_{k+1}"] \arrow[d, "\alpha"] & Z_k \oplus B_{k-1} \arrow[r, "\partial_k"] \arrow[d, "\beta"] & C'_{k-1} \arrow[r] \arrow[d, "\tau'_{k+1}"] & \cdots \\ \cdots \arrow[r] & C'_{k+2} \arrow[r, "\partial_{k+2}"] & Z_{k+1} \oplus B_k \arrow[r, "\partial_{k+1}"] & Z_k \oplus B_{k-1} \arrow[r, "\partial_k"] & C'_{k-1} \arrow[r] & \cdots \end{tikzcd}$$

where

$$\alpha = \begin{pmatrix} Z_{k+1} & \mathscr{D}_{k+1} \\ 0 & \mathscr{B}_{k+1} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \mathscr{Z}_k & \mathscr{D}_k \\ 0 & \mathscr{B}_k \end{pmatrix}.$$

This chain complex with endomorphism is isomorphic to

$$\begin{tikzcd} \cdots \arrow[r] & C'_{k+2} \arrow[r, "\partial_{k+2}"] \arrow[d, "\alpha_1"] & Z_{k+1} \oplus F_1 \arrow[r, "A^{-1}\partial_{k+1}A"] \arrow[d, "\beta_1"] & F_0 \oplus B_{k-1} \arrow[r, "\partial_k"] \arrow[d, "\tau'_k"] & C'_{k-1} \arrow[r] \arrow[d, "\tau'_{k+2}"] & \cdots \\ \cdots \arrow[r] & C'_{k+2} \arrow[r, "\partial_{k+2}"] & Z_{k+1} \oplus F_1 \arrow[r, "A^{-1}\partial_{k+1}A"] & F_0 \oplus B_{k-1} \arrow[r, "\partial_k"] & C'_{k+1} \arrow[r] & \cdots \end{tikzcd}$$

where

$$\alpha_1 = \begin{pmatrix} \mathscr{Z}_{k+1} & \mathscr{D}_{k+1}A \\ 0 & A^{-1}\mathscr{B}_{k+1}A \end{pmatrix} \text{ and } \beta_1 = \begin{pmatrix} A^{-1}\mathscr{Z}_k A & A^{-1}\mathscr{D}_{k-1} \\ 0 & \mathscr{B}_{k-1} \end{pmatrix}$$

which in turn is isomorphic to

$$\begin{tikzcd} \cdots \arrow[r] & C'_{k+2} \arrow[r, "\partial_{k+2}"] \arrow[d, "\tau'_{k+2}"] & Z_{k+1} \oplus F_1 \arrow[r, dash, "d_1"] \arrow[d, "\alpha_2"] & F_0 \oplus B_{k-1} \arrow[r, "\partial_k"] \arrow[d, "\beta_2"] & C'_{k-1} \arrow[r] \arrow[d, "\tau'_{k-1}"] & \cdots \\ \cdots \arrow[r] & C'_{k+2} \arrow[r] & Z_{k+1} \oplus F_1 \arrow[r, "d_1"] & F_0 \oplus B_{k-1} \arrow[r, "\partial_k"] & C'_{k-1} \arrow[r] & \cdots \end{tikzcd}$$

where

$$\alpha_2 = \begin{pmatrix} \mathscr{Z}_{k+1} & \mathscr{D}'_{k+1} \\ 0 & \mathscr{B}'_{k+1} \end{pmatrix} \text{ and } \beta_2 = \begin{pmatrix} \mathscr{Z}'_k & \mathscr{D}'_k \\ 0 & \mathscr{B}_{k-1} \end{pmatrix}.$$

And now by Lemmas 3.7 and 3.8 this chain complex with chain endomorphism is chain homotopy equivalent to

$$\begin{tikzcd} \cdots \arrow[r] & C'_{k+2} \arrow[r, "\partial_{k+2}"] \arrow[d, "\tau'_{k+2}"] & Z_{k+1} \oplus F_1 \arrow[r, "d_1"] \arrow[d, "\alpha_3"] & F_1 \oplus B_{k-1} \arrow[r, "\partial_k"] \arrow[d, "\beta_3"] & C'_{k-1} \arrow[r] \arrow[d, "\tau'_{k-1}"] & \cdots \\ \cdots \arrow[r] & C'_{k+2} \arrow[r, "\partial_{k+2}"] & Z_{k+1} \oplus F_1 \arrow[r, "d_1"] & F_0 \oplus B_{k-1} \arrow[r, "\partial_k"] & C'_{k-1} \arrow[r] & \cdots \end{tikzcd}$$

where

$$\alpha_3 = \begin{pmatrix} \mathscr{X}_{k+1} & \mathscr{D}'_{k+1} \\ 0 & h'_1 \end{pmatrix} \text{ and } \beta_3 = \begin{pmatrix} h'_0 & \mathscr{D}'_k \\ 0 & \mathscr{B}_k \end{pmatrix},$$

with $\gamma(h'_0) \le \max(\gamma(\tau_{*k}), 1)$ and since $\gamma(B_k) \le \gamma(\tau'_{k-1})$ this completes the proof of 3.9.

*Proof of* 3.4. By the above we obtain a chain complex $(\mathscr{C}', \tau')$ with $\gamma(\tau') \le 1$. This implies (e.g. [15, p. 149]) that all the eigenvalues of the $\tau'_i$ are roots of unity or zero and hence that the $\tau'_i$ are $QI$.

3.10 LEMMA (splicing). *Suppose $C$ is a free chain complex with chain map $\tau$ and*

$$0 \to (A, a) \to (B, b) \to (C_k, \tau_k) \to 0$$

*is a short exact sequence of elements of $\overline{QI}$. Then $(C, \tau)$ is chain homotopy equivalent to $(C', \tau')$ with $(C'_j, \tau'_j) = (C_j, \tau_j)$ for $j \ne k, k+1$, $(C'_k, \tau'_k) = (B, b)$, and $C'_{k+1} = C_{k+1} \oplus A$,*

$$\tau'_{k+1} = \begin{pmatrix} a & * \\ 0 & \partial_{k+1} \end{pmatrix}.$$

*Proof.* We define $C'_j$ as in the statement of the lemma and let $\partial'_j \colon C'_j \to C'_{j-1}$ be equal to $\partial_j \colon C_j \to C_{j-1}$ if $j \ne k+2, k+1, k$. Define $\partial'_{k+2} \colon C'_{k+2} \to C'_{k+1}$ by

$$C'_{k+2} = C_{k+2} \xrightarrow{(0,\, \partial_{k+2})} A \oplus C_{k+1} = C'_{k+1},$$

and $\partial'_{k+1}$ by

$$C'_{k+1} = A \oplus C_{k+1} \xrightarrow{(id,\, \partial_{k+1})} A \oplus C_k \cong B = C'_k,$$

and $\partial_k$ by the composition

$$C'_k = A \oplus C_k \to C_k \xrightarrow{\partial_k} C_{k-1} = C'_{k-1}.$$

Now the short exact sequence $0 \to (A, a) \to (B, b) \to (C_k, \tau_k) \to 0$ implies $B \cong A \oplus C_k$ and that $b$ corresponds to an endomorphism of the form $\begin{pmatrix} a & e \\ 0 & \tau_k \end{pmatrix}$ for some $e \colon C_k \to A$. Thus if we define $\tau'_k \colon C'_k \to C'_k$ by

$$C'_k = B \cong A \oplus C_k \xrightarrow{\begin{pmatrix} a & e \\ 0 & \tau_k \end{pmatrix}} A \oplus C_k = C'_k$$

and $\tau'_k \colon C'_{k+1} \to C'_{k+1}$ by

$$C'_{k+1} = A \oplus C_{k+1} \xrightarrow{\begin{pmatrix} A & e \circ \partial_{k+1} \\ 0 & \tau_{k+1} \end{pmatrix}} A \oplus C_{k+1} = C'_{k+1}$$

it follows that $\partial'_{k+1} \circ \tau'_{k+1} = \tau'_k \circ \partial'_{k+1}$. One also checks easily that if we define $\tau'_j = \tau_j$, $j \ne k, k+1$ then $\partial'_j \circ \tau'_j = \tau'_{j-1} \circ \partial'_j$ for all $j$ so $\tau'$ is a chain map of $\mathscr{C}'$.

Finally the inclusion $i \colon \mathscr{C} \to \mathscr{C}'$ is a chain map which induces an isomorphism on homology and hence is a chain homotopy equivalence.

We now continue with the proof of 3.3 henceforth assuming that $\mathscr{C}$ is free. We first alter $(\mathscr{C}, \tau)$ by adding contractible $QU$ chain complexes in such a way as to achieve a new $QI$ complex $(\mathscr{C}', \tau')$ with $(C_i', \tau_i')$ representable by a companion-like matrix for every $i$. We do this using 2.7. Thus if $(F', e')$ is the inverse of $(C_i, \tau_i)$ given by 2.7 we form the complex $\mathscr{B}$ with $(B_i, b_i) = (B_{i-1}, b_{i-1}) = (F', e')$, $\partial_i = id$, and all other $B_j = 0$. Then $\mathscr{C} \oplus \mathscr{B}$ is chain homotopy equivalent to $\mathscr{C}$ and $(C_i \oplus B_i, \tau_i \oplus b_i)$ is representable by a companion-like matrix. We have, however, altered $C_{i-1}$. Alternatively we could have defined $(B_{i+1}, b_{i+1}) = (B_i, b_i) = (F', e')$ and altered $C_{i+1}$. Thus we can start at the top of $\mathscr{C}$ and work down creating companion-like endomorphisms on each level, or start at the bottom and work up, or both. In any case we ultimately arrive at a $QI$ complex $(\hat{\mathscr{C}}, \hat{\tau})$ with a single $(\hat{\mathscr{C}}_k, \hat{\tau}_k)$ remaining that cannot be altered by the previous method without possibly disturbing other levels. However, by 3.2, $\phi(\hat{\tau}_k) = \chi(\hat{\tau}) = \chi(\tau) = 0$, so $(\hat{C}_k, \hat{\tau}_k) \in P$. Thus we can apply 2.10 and choose $(F', e')$ with $e'$ companion-like and $\hat{\tau}_k \oplus e'$ companion-like. Defining $\mathscr{B}$ as before, we see that $(\hat{\mathscr{C}} \oplus \mathscr{B}, \hat{\tau} \oplus b)$ has endomorphisms on all levels representable by companion matrices.

Thus we may assume that $(\mathscr{C}, \tau)$ has the property and proceed to alter it further to achieve virtual permuation endomorphisms.

The proof now follows from repeated applications of 2.4 and splicing 3.10. This completes the proof in one direction. The other direction is immediate from the definitions of $\phi$ and $P$ and 3.2.

3.11 *Remark.* If we assume $C_m = C_0 = Z$, $C_{m-1} = C_1 = 0$, $m \geq 6$ and $C_k = 0$ for $k < 0$ or $k > m$, then we can construct $\mathscr{D}$ satisfying the conclusion of 3.3 and in addition with $D_m = D_0 = Z$ and $D_{m-1} = D_1 = 0$.

This can be done using the technique of *folding* (see Appendix A of [16]). Or alternatively we could use a little more care in the proof of 3.3. The last stage of that proof—the splicing argument—could have been done equally well by changing $C_k$ and $C_{k-1}$ to $C_k'$ and $C_{k-1}'$ instead of making the change on levels $k$ and $k+1$.

In that case we use a short exact sequence

$$0 \to (C_k, \tau_k) \to (B, b) \to (A, a) \to 0$$

where $(B, b)$ and $(A, a)$ are V.P. which obtained by applying 3.10 to the transpose of $(C_k, \tau_k)$ and taking the transpose of the sequence thus reversing the arrows. The definition of $\tau_{k-1}'$ is then $\begin{pmatrix} \tau_{k-1} & \partial_{k^\circ e} \\ 0 & a \end{pmatrix}$. By using both of these changes we can preserve the dimension of $\mathscr{C}$ and the property that $C_1 = C_{n-1} = 0$.

We now have all the elements of our main result.

3.12 THEOREM. *Suppose $f\colon M \to M$ is a diffeomorphism of a compact manifold and $f_{*k}\colon H_k(M; Z)$ is quasi-unipotent. Then if $f$ is homologous to a Morse–Smale diffeomorphism $\chi(f_*) = \Sigma(-1)^i\phi(f_{*k})$ is zero in $G = K_0(QI)/K_0(P)$. If $M$ is simply connected and of dimension greater than five then $\chi(f_*) = 0$ implies that $f$ is isotopic to a Morse–Smale diffeomorphism.*

*Proof.* If $f$ is homologous to a Morse–Smale diffeomorphism then by 1.2 it can be represented on the chain level by $(\mathscr{D}, \sigma)$ where each $\sigma_k\colon D_k \to D_k$ is representable by a virtual permutation matrix. Thus by 2.11 $\phi(\sigma_k) = 0$ for all $k$ and by 3.2, $\chi(f_*) = \chi(\sigma)$. Hence $\chi(f_*) = 0$.

Conversely, suppose $\chi(f_*) = 0$; then if $M$ is simply connected and has dimension $n$ greater than five we can assume (see Appendix A of [16]) that it is represented on the chain level by $(\mathscr{C}, \tau)$ with $C_n = C_0 = Z$, and $C_{n-1} = C_1 = 0$. It now follows from 3.3 and 3.4 that we can satisfy the hypothesis of 1.2 which guarantees a Morse–Smale diffeomorphism isotopic to $f$.

We return for a moment to one of our original questions. Given a finitely generated free chain complex with endomorphism $(\mathscr{C}, \tau)$ when is there a chain homotopy equivalence $h: (\mathscr{C}, \tau) \to (\mathscr{D}, \sigma)$, where $\mathscr{D}$ is a finitely generated free chain complex with $\sigma_i$ representable by a virtual permutation matrix? We let $\mathscr{H}$ be the category of finitely generated free chain complexes with chain endomorphisms. An object in our category is a pair $(\mathscr{C}, \tau)$ where $\mathscr{C}$ is a finitely generated free chain complex and $\tau$ is a chain endomorphism of $\mathscr{C}$. A morphism $h: (\mathscr{C}, \tau) \to (\mathscr{D}, \sigma)$ is a chain map $h: \mathscr{C} \to \mathscr{D}$ such that $h \circ \tau = \sigma \circ h$. An exact sequence

$$0 \to (\mathscr{C}_1, \tau_1) \xrightarrow{h_1} (\mathscr{C}_2, \tau_2) \xrightarrow{h_2} (\mathscr{C}_3, \tau_3) \to 0$$

is a sequence such that $0 \to \mathscr{C}_1 \xrightarrow{h_1} \mathscr{C}_2 \xrightarrow{h_2} \mathscr{C}_3 \to 0$ is exact. We will say that $(\mathscr{C}, \tau)$ is V.P. if there is a matrix representatives for $\tau$ such that $\tau_i$ is V.P. for each $i$. Our question then is: When is $(\mathscr{C}, \tau)$ chain homotopy equivalent to $(\mathscr{D}, \sigma)$ with $(\mathscr{D}, \sigma)$ V.P.?

Let $\mathscr{V}$ be the full subcategory of $\mathscr{H}$ whose objects are all $(\mathscr{C}, \tau)$ where $(\mathscr{C}, \tau)$ is chain homotopy equivalent to $(\mathscr{D}, \sigma)$ with $(\mathscr{D}, \sigma)$ V.P.

3.13 Proposition. *If*

$$0 \to (\mathscr{C}_1, \tau_1) \to (\mathscr{C}_2, \tau_2) \to (\mathscr{C}_3, \tau_3) \to 0$$

*is an exact sequence and two of the $(\mathscr{C}_i, \tau_i)$ are chain homotopy equivalent to $(\mathscr{D}_i, \sigma_i)$ with the $(\mathscr{D}_i, \sigma_i)$ V.P. then so is the third. Thus $\mathscr{V}$ is closed under short exact sequences.*

*Proof.* Since two of the $\tau_{i*}$ are *Q.I.* it follows from the long exact homology sequence that the third is *Q.I.* as well.

Now since $\chi(\tau_{i*}) + \chi(\tau_{3*}) = \chi(\tau_{2*})$ if two of these are zero the third is as well, so by 3.3 we are done.

We now give another description of the category $P$.

3.14 *Definition.* Let $P'$ be the full subcategory of $QI$ consisting of elements which have V.P. resolutions, i.e. $(G, h) \in P'$ iff there is a resolution such that each $h_i: F_i \to F_i$ is V.P.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
F_n & \xrightarrow{h_n} & F_n \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
F_0 & \xrightarrow{h_0} & F_0 \\
\downarrow & & \downarrow \\
G & \xrightarrow{h} & G \\
\downarrow & & \downarrow \\
0 & . & 0
\end{array}$$

*Remark.* By "folding" we could always assume that $n = 1$.

3.15 PROPOSITION. *$P'$ is closed with respect to short exact sequences that is, if* $0 \to (G_1, h_1) \xrightarrow{i_1} (G_2, h_2) \xrightarrow{i_2} (G_3, h_3) \to 0$ is exact with $(G_i, h_i) \in QI$, i = 1, 2, 3, and if two elements of the sequence are in $P'$, so is the third.

*Proof.* By [4, (Chap. 5, Propositions 1.1 and 2.3]) we may first resolve $(G_1, h_1)$ and $(G_3, h_3)$ and fill into a resolution of $(G_2, h_2)$. Now by assumption two of the unaugmented resolutions are in $\mathcal{V}$, therefore the third is as well and this third resolution is chain homotopy equivalent to a V.P. resolution.

3.16 COROLLARY. $P = P'$.

*Proof.* By definition elements of $P'$ are equivalent in $K_0(QI)$ to a linear combination of virtual permutations and hence permutations, so $P' \subset P$. On the other hand, $P'$ contains the permutations and is closed under short exact sequences so by 2.10 $P' \supset P$.

Proposition 3.16 besides giving another characterization of $P$ which might be useful in determining $G$ provides another insight into the proof of our main theorem. It is not very difficult to prove Proposition 3.13 directly and thus to deduce 3.15 and 3.16.

More categorically, our proof is more less the following:

(1) Let V.P. be the full subcategory of $QI$ consisting of V.P. endomorphisms of finitely generated free abelian groups, i.e. they have a V.P. matrix representation. Let $P'$ be the full subcategory of $QI$ consisting of objects with V.P. resolutions. Then $P'$ is closed under short exact sequence and in particular extensions, i.e. if $0 \to (G_1, h_1) \to (G_2, h_2) \to (G_3, h_3) \to 0$ is exact in $QI$ and $(G_1, h_1)$, $(G_3, h_3) \in P'$, then so is $(G_2, h_2)$ (this is actually very simple using [4, Chap. 5, Proposition 2.3] and if $(G_1, h_1) \in P'$, $(G_2, h_2) \in QI$ and $(G_1, h_1) \oplus (G_2, h_2) \in P'$ then $(G_2, h_2) \in P'$).

(2) Given $(G_1, h_1) \in QI$ there exist $(G_2, h_2) \in QI$ such that $(G_1 \oplus G_2, h_1 \oplus h_2) = (G_1, h_1) \oplus (G_2, h_2) \in P'$. This follows from 2.7, the proof of 2.1 and 2.4.

(3) Elements of $QI$ have resolutions by elements of $\overline{QI}$. This is 2.1 again. The identity or zero endomorphism is in $QI$.

3.17 Proposition. *We claim that* (1) *and* (2) *imply: If* $(G_1, h_1) \in P'$ *and* $(G_2, h_2) \in QI$ *and* $[(G_1, h_1)] = [(G_2, h_2)]$ *in* $K_0(QI)$, *then* $(G_2, h_2) \in P'$ *and* $[(G_2, h_2)] = [(G_1, h_1)]$ *in* $K_0(P')$.

The proof is essentially the same as the proof of 2.8 or 2.10.

Thus $K_0(P')$ injects in $K_0(QI)$. The obstruction group $G$ is $K_0(QI)/K_0(P')$. Now (3) allows us to prove 3.4 by splicing the unaugmented resolution of $\tau_{k*}: H_k(\mathscr{C}) \to (\mathscr{C}))$ into $\mathscr{C}$ in place of $0 \to (B_k, \tau_k) \to (Z_k, \tau_k)$.

Once we have 3.4 we add the inverse mod $P'$ as in the proof of 3.3 to give us a chain complex with endomorphism $(\mathscr{C}, \tau)$ such that $(C_i, \tau_i) \in P'$ except perhaps in one dimension, say $(C_k, \tau_k)$. But if $\chi(\tau_*)$ is zero in $G$, then $[(C_k, \tau_k)] \in K_0(P')$ and $(C_k, \tau_k) \in P'$. Thus each $(C_i, \tau_i) \in P'$. By splicing the unaugmented V.P. resolutions of the $(C_i, \tau_i)$ into the chain complex in place of the $(C_i, \tau_i)$ we produce our virtual permutation complex. The splicing is all possible since the $C_i$ are free.

*Added in Proof.* It has come to our attention that some of these arguments appear in $[G_2]$.

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