SUSPENDING SUBSHIFTS*

By Charles C. Pugh and Michael Shub**

1. Introduction. Every full shift on k-symbols is conjugate to a basic set of an Axiom AS diffeomorphism of S^2, Smale [12], but the same is false for a subshift of finite type (ssft), as shown by an example of Franks [5]. The corresponding question on a general compact 2-manifold M^2, instead of S^2, remains open. Partial results may be found in Franks [5] and Batterson [1].(1)

Here we show that the situations for flows is quite different. For Axiom A flows one can pass from basic sets to sub-basic sets on the same manifold, Theorem 6 below. In particular for ssft's we have

THEOREM 1. *The suspension of any basic subshift of finite type (ie a ssft with dense periodic orbits and a dense orbit) is conjugate to a basic set of some Axiom AS flow on S^3 – or any other $M^m, m \geqslant 3$.*

Theorem 6 is proved in §4. We are indebted to R. Mañé for simplifying our proof of Theorem 6. Sections 2 and 3 set the notations we use, define the terms, and put the problem in context.

Remarks. 1) The Axiom AS flow we produce in Theorem 1 has singularities. It is not known if they can be eliminated on S^3.

2) Basic ssft's are identical to those which arise from irreducible matrices. See [4, 11] and §2.

2. Cascades. Before proving Theorem 1, we explain the general problem to which it relates. Background references are Smale [12] and Shub [11]. Let $f: M \to M$ be a diffeomorphism, M compact. Its *cascade* is $\{f^n : n \in \mathbb{Z}\}$. Let Ω be the nonwandering set of f. Then f obeys *Axiom A* if

(Aa) Ω is a hyperbolic set of f

(Ab) The periodic points are dense in Ω.

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(1) P. Blanchard and J. Franks have produced subshifts of finite type which cannot occur on any compact M^2.

If, in addition, all invariant manifolds of Ω meet transversally then f is said to obey Axiom AS and it follows that f is structurally stable. Smale shows that Axiom A implies a finite disjoint decomposition

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_k$$

where the Ω_j are the *basic sets* of Ω. Each Ω_j is compact, f-invariant, contains an f-orbit which is dense in Ω_j, and $f|\Omega_j$ is expansive. "Expansive" means that for some $\delta > 0$ and some metric d, if $d(f^n x, f^n y) \leqslant \delta$ for all $n \in \mathbb{Z}$ then $x = y$.

Embedding Problem. Given a compact metrizable Λ, a homeomorphism $g : \Lambda \to \Lambda$, and a manifold M, when are there an Axiom A (or AS) diffeomorphism $f : M \to M$ and an embedding $i : \Lambda \hookrightarrow M$ onto a basic set of f making

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \uparrow i & & \uparrow i \\ \Lambda & \xrightarrow{g} & \Lambda \end{array}$$

commute? Obviously, necessary conditions are

(1) g is expansive, g has a dense orbit, the periodic points of g are dense in Λ, and (Λ, g) embeds in some diffeomorphism of M.

A well behaved sort of (Λ, g) is furnished by symbolic dynamics. Let $2 \leqslant k < \infty$ be fixed and let $\sum = \sum^k$ denote the bi-infinite sequences of letters $0, 1, \ldots, k-1$, i.e.

$$\sum = \{0, 1, \ldots, k-1\}^{\mathbb{Z}}$$

The general element of $\sum$ is written $\underline{a} = (\ldots a_{-1} \cdot a_0 a_1 a_2 \ldots)$ where each a_i is a letter from 0 to $k-1$. Note the convenient decimal point. Put the product topology on $\sum$. It is compact, zero dimensional, and metrizable. Let $\sigma : \sum \to \sum$ be the map:

$$\underline{a} = (\ldots a_{-1} \cdot a_0 a_1 a_2 \ldots) \mapsto (\ldots a_{-1} a_0 \cdot a_1 a_2 \ldots) = \sigma(\underline{a})$$

which shifts the decimal point to the right, i.e., σ shifts the entries of $\underline{a}$ to the left. σ is an homeomorphism called the *full shift on k symbols*. In [12], Smale shows

THEOREM 2. *Each full shift is conjugate to a basic set of some Axiom AS diffeomorphism on S^2, or any other m-manifold $m \geqslant 2$.*

Definition. Let $\sigma : \sum \to \sum$ be the full shift on k symbols. If $\sum' \subset \sum$ is σ-invariant then $\sigma | \sum'$ is a *subshift*. It is of *finite type* if for some fixed N there is a list $L \subset \{0, 1, \ldots, k-1\}^N$ such that

$$\underline{a} \in \textstyle\sum' \text{ iff each } N\text{-string of } \underline{a} \text{ appears in the list } L.$$

An N-string of $\underline{a}$ is just N consecutive entries of $\underline{a}$. All ssft's are compact. This definition of ssft is conjugacy-invariant and is equivalent to the transition matrix definition in [11]. See [4, p. 121].

Definition. A ssft is *basic* if it has a dense orbit and a dense set of periodic points.

In [14] R.F. Williams shows

THEOREM 3. *Each basic ssft is conjugate to a basic set of some Axion AS diffeomorphism on S^3, or any other m-manifold $m \geqslant 3$.*

Thus the Embedding Problem for zero-dimensional sets Λ is fairly well understood. Let $g : \Lambda \to \Lambda$ be a homeomorphism of the compact metrizable space Λ and ask: when does it embed onto a basic set? The necessary condition of expansiveness in (1) implies that (Λ, g) is conjugate to some subshift $\sum' \subset \sum$, provided Λ has dimension zero [10]. By [4, p. 244], for a subshift to be conjugate to a basic set it must be ssft. Thus

(2) A zero-dimensional (Λ, g) is conjugate to a basic set iff it is a basic ssft.

Here, what is left to know is which subshifts embed as basic sets in which 2-manifolds.

Basic sets of higher dimension are grossly messier than shifts, so (2) is far from a full solution to the Embedding Problem for cascades. See the example of Guckenheimer [6, 2] for a basic set that is topologically bizarre.

3. Flows. In a straight forward manner, the notion of Axiom AS generalizes from cascades to flows. The Embedding Problem is the same.

Definition. Let $g : \Lambda \to \Lambda$ be a homeomorphism. The identification space Susp(Λ, g) is the set

$$[0,1] \times \Lambda /_{(1,x)\sim(0,gx)}$$

The *suspension* of g is the flow ϕ on Susp(Λ, g) whose trajectory through $(0,x) \in 0 \times \Lambda$ is

$$\phi_t(0,x) = (t,x) \qquad 0 \leqslant t \leqslant 1$$

The time-one map of ϕ caries $0 \times \Lambda$ onto itself by $g : (0,x) \mapsto (0, gx)$.

Let (S,ϕ) be the suspension of the full k-shift. We have

THEOREM 4. *(S,ϕ) embeds onto a basic set in any $M^m, m \geqslant 3$.*

Proof. See [12] as for Theorem 2.

A result of Bowen essentially solves the embedding problem for 1 dimensional basic sets.

THEOREM 5 [3]. *A one dimensional basic set for a flow is conjugate to the suspension of a subshift of finite type.*

Theorem 3 would allow us to embed a suspension of a basic ssft in any M^4. Here we accomplish the embedding in any M^3.

4. Embedding Sub-objects. Let $\sum$ be the full shift on k symbols, let $\sum'$ be a basic ssft in $\sum$, and let S', S be the suspension of $\sum'$, $\sum$. By Theorem 3, embed S as a basic set of some Axiom A flow ϕ on (any) M^3. Call Λ the image of S' in M^3.

THEOREM 6. *There is an Axiom A no cycle flow ψ on M^3 such that Λ is a basic set of $\psi, \phi|\Lambda = \psi|\Lambda$, and $\Omega(\psi) - \Lambda$ is finite.*

In fact, our proof of Theorem 6 is more general and gives

THEOREM 7. *Let Λ be any locally maximal compact invariant set for the flow ϕ on M. Then there is a flow ψ on M such that*

(i) *$\psi \equiv \phi$ on a neighborhood of Λ*

(ii) *$\Omega(\psi) \subset \Lambda \cup P$ where P consists of finitely many hyperbolic fixed points of $\psi, p_1, p_2, \ldots, p_n$*

(iii) *There are no cycles among $p_1, \ldots, p_n, \Lambda$*

(iv) *Moreover, if Λ is hyperbolic for ϕ, has a dense orbit and dense set of periodic points then ψ is Axiom A and one of its basic sets is Λ.*

Note that (iii) and (iv) include Theorem 6. We need the following lemma, referred to in [9], and for which we give a direct proof below, avoiding the subtleties of Wilson [15].

LEMMA 1. *Let Λ be a compact invariant set for the C^r flow ϕ on M and suppose U is a neighborhood of Λ in which Λ is the maximal ϕ-invariant subset. Then there exists a C^∞ Lyapunov function $\lambda : U \to \mathbb{R}$ for ϕ at Λ.*

This means

$$\lambda | \Lambda \equiv 0 \equiv D\lambda | \Lambda$$

$$\dot{\lambda}(x) \stackrel{\text{def}}{=} \frac{d}{dt}\lambda(\phi_t x)\Big|_{t=0} > 0 \qquad x \in U - \Lambda.$$

Remark. The authors cannot agree on the sign of $\dot{\lambda}(x)$ in the definition of Lyapunov function. In this section we have taken it positive.

Proof of Theorem 7. Λ is the maximal ϕ-invariant set in some neighborhood $U \supset \Lambda$. Let $\lambda : U \to \mathbb{R}$ be the Lyapunov function supplied by Lemma 1. Since λ is nonsingular off Λ, it extends smoothly to all of M

$$\lambda : M \to \mathbb{R}$$

such that, except for Λ, λ has only nondegenerate critical points: $\operatorname{sing}(\lambda) = \Lambda \cup P$ when $P = \{p_1, \ldots p_n\}$ and $P \cap U = \emptyset$

Fix a smooth bump function $\beta : M \to [0, 1]$ which has support in U and is identically equal to 1 on a neighborhood of Λ. Set

$$Y = \beta X + (1 - \beta)\operatorname{grad}\lambda$$

where $X = \dot{\phi}$. Let ψ be the Y-flow; we claim that ψ verifies Theorem 7. (i) is clear.

Consider $\langle \beta X, \operatorname{grad}\lambda\rangle$ where $\langle , \rangle$ is the inner product defining the gradient. If $\beta(x) = 0$ or $x \in \Lambda$ then this quantity equals 0. If $\beta(x) \neq 0$ and $x \notin \Lambda$ then $x \in U$, $\beta(x) > 0$, $\operatorname{grad}\lambda(x) \neq 0$, and by Lemma 1, λ increases along the X-trajectory at x, so

$$\langle \beta(x)X(x), \operatorname{grad}\lambda(x)\rangle > 0$$

Thus

$$\langle \beta X, \operatorname{grad} \lambda \rangle \geqslant 0$$

everywhere on M. Adding $(1 - \beta)$ grad λ to βX only improves the inequality and we get

$$\langle Y(x), \operatorname{grad} \lambda(x) \rangle \geqslant 0 \qquad x \in M$$

with equality only at $\operatorname{sing}(\lambda) = \Lambda \cup P$. This proves (iii) and

$$\Omega(\psi) \subset \operatorname{Sing}(\lambda)$$

which gives (ii). For λ is continuous and $\lambda(\psi_t x)$ increases (strictly) monotonically with t if $x \in M - \operatorname{sing}(\lambda)$, so all $x \in M - \operatorname{sing}(\lambda)$ wander under the flow ψ.

(iv) Since $\beta \equiv 1$ on a neighborhood of $\Lambda, D\phi_t|\Lambda = D\psi_t|\Lambda$ for all t and Λ is a hyperbolic set for ψ. Since the periodic orbits are dense in $\Lambda, \Lambda \subset \Omega(\psi)$. Thus, by (ii), ψ is Axiom A and Λ is basic set. Q.E.D.

Remark. The function λ may be chosen so that $\lambda(p_j)$ are distinct from one another and from $0 = \lambda(\Lambda)$, $j = 1, \ldots, n$. Thus, there is a $c > 0$ such that for $M_c = \lambda^{-1}[-c, c]$

(v) $M_c \cap \Omega(\psi) \subset \Lambda$

(vi) Y is transverse to $\partial M_c = \lambda^{-1}(-c) \cup \lambda^{-1}(c)$

(vii) On $M - M_c$, Y is a gradient-like Morse smale flow.

Moreover by the Kupka-Smale Theorem, all the stable and unstable manifolds of $p_1, \ldots, p_n$ may be assumed transverse: $W^u(p_i) \pitchfork W^s(p_j)$, $1 \leqslant i, j \leqslant n$. To get Theorem 1 we must deal with $W^u(\Lambda), W^s(\Lambda)$ as well. This we do in §5.

Now we return to our construction of Lyapunov functions.

Proof of Lemma 1. If $x \in U$ then the orbit of x exits U in at least one direction of time or else $x \in \Lambda$. If $\phi_t x \in U$ for all $t \geqslant 0$ then $\omega(x)$ is a ϕ-invariant subset of U, which is contained in Λ by maximality of Λ. Similarly for $t \leqslant 0$. Thus, each $x \in U$ has an orbit O_x of precisely one of the four types

(a) $O_x \subset \Lambda$

(b) O_x exits U in forward and reverse time

(c) O_x exits U in forward time and has $\alpha(x) \subset \Lambda$

(d) O_x exits U in reverse time and has $\omega(x) \subset \Lambda$.

The points obeying (c) form the local unstable set for Λ, W^u, and those obeying (d) form the local stable set W^s. Note that all orbits off Λ cross ∂U in at least one direction of time, and that by proper choice of U, $(W^u \cap \partial U)$, $(W^s \cap \partial U)$ are compact disjoint subsets. Denote by B the set of points with orbits of type (b); B is open in U. Let $x \in B$. Choose $x_0 = \phi_{t_0} x \in M - U$ for some $t_0 < 0$, choose a smooth compact $(m - 1) -$ disc D at x_0 transverse to ϕ, suppose $\phi_\tau x_0 \in M - U$ for some $\tau > -t_0$, and consider the flowbox

$$F = \{\phi_t y : y \in D \text{ and } 0 \leqslant t \leqslant \tau\}$$

Choose D small enough so that D and $\phi_\tau D$ lie outside U, see Figure 1. This is a flowbox of type (b) around x.

Let $x \in W^u$. Choose $x_1 = \phi_{t_1} x \in M - U$ for some $t_1 > 0$, choose a smooth compact $(m - 1)$disc D at x_1 transverse to ϕ, choose $\tau > t_1$, and consider the flowbox

$$F = \{\phi_t y : y \in D \text{ and } -\tau \leqslant t \leqslant 0\}$$

F is a flowbox whose forward endface is D and reverse endface is $\phi_{-\tau}(D)$. Choose D small enough so that $D \subset M - U$. This is a flowbox of type (c) around x.

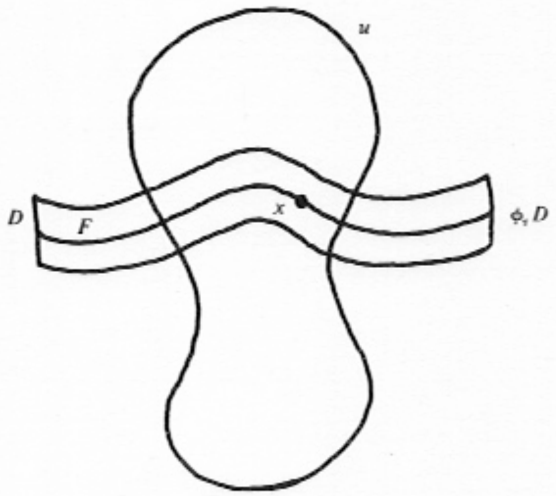

Figure 1. A flowbox of type (b).

Let $x \in W^s$ and construct a flowbox of type (d) around x symmetric to one of type (c).

On each type of flowbox there is a C^∞ function $\lambda_F : F \to \mathbb{R}$ such that

$$\dot{\lambda}_F > 0 \quad \text{on Interior}(F)$$

$$\lambda_F \equiv 0 \quad \text{on} \quad \partial F \cap U \tag{3}$$

$$\max|\lambda_F(x)| \leq 1 \qquad \max \dot{\lambda}_F(x) \leq 1.$$

For consider the three graphs of λ_F over the straightened-out flowbox in Figure 2. Set $\lambda_F \equiv 0$ on $U - F$, so $\lambda_F : F \cup U \to [-1, 1]$. The flowbox chart is C^r, so λ_F is C^r. [If we want more differentiability we can C^r-approximate λ_F by a C^∞ map $\tilde{\lambda}_F : F \cup U \to [-1, 1]$ such that (3) holds for $\tilde{\lambda}_F$ also. This requires a little care near ∂F.]

Cover $U - \Lambda$ by the interiors of countably many such flowboxes $F_1, F_2, \ldots$ and set

$$\lambda(x) = \sum_{n=1}^{\infty} \epsilon_n \lambda_{F_n}(x) \qquad x \in U \tag{4}$$

where $0 < \epsilon_n < 1/2^n$. Observe that λ is continuous, $\dot{\lambda}$ is continuous, $\dot{\lambda} > 0$ for all $x \in U - \Lambda$, and $\lambda|\Lambda \equiv 0 \equiv D\lambda|\Lambda$. To make $\lambda \in C^r$ we choose $\epsilon_n \to 0$ rapidly.

$$\epsilon_n |\lambda_{F_n}|_r < \frac{1}{2^n} \qquad n = 1, 2, \ldots \tag{5}$$

where $|\ |_r$ denotes the C^r size of a function. Then λ is C^r. [To get $\lambda \in C^\infty$, even if ϕ is only C^1, we replace λ_{F_n} with $\tilde{\lambda}_{F_n}$ in (4) and instead

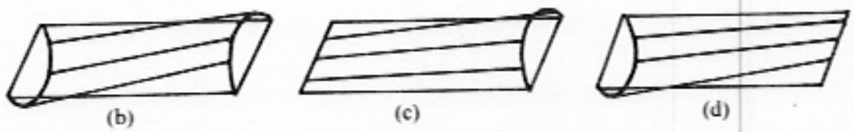

Figure 2. Lyapunov functions on flowboxes.

of (5), we choose $\epsilon_n \to 0$ so rapidly that

$$\epsilon_n |\tilde{\lambda}_{F_n}|_r < \frac{1}{2^n} \qquad n \geqslant r$$

for each $r = 1, 2, 3, \ldots$. Then λ is C^∞.] Q.E.D.

We also state the one-sided result

LEMMA 2. *Suppose A is a compact forward invariant set for ϕ which has a neighborhood U in M such that*

$$(6) \qquad O_-(x) \subset U \Rightarrow O_-(x) \subset A \qquad x \in U$$

where O_- denotes the reverse ϕ-orbit. Then ϕ has a Lyapunov function $\lambda : U \to \mathbb{R}$, and $\lambda < 0$ on $U - A$.

Remark. (6) implies A attracts in U.

Proof. Each $x \in U - A$ has $O_-(x) \subset U$. Forward invariance of A then says $O_-(x) \cap A = \emptyset$, so we get a flowbox F around each $x \in U - A$ of type (d) above. Cover $U - A$ with the interiors of $F_1, F_2, F_3, \ldots$ and set $\lambda = \sum \epsilon_n \lambda_{F_n}$ as above. Q.E.D.

5. Proof of Theorem 1. We suppose that M has dimension 3 and return to the construction of the flow ψ in Theorem 7 and the remark after its proof. We must make the stable manifolds of the fixed points transverse to the unstable manifolds of the orbits in Λ. There are infinitely many of the latter so we must go beyond the Kupka-Smale Theorem. Each orbit O in Λ has $W^u(O)$ and $W^s(O)$ of dimension 2. Since the flow is transverse to $\lambda^{-1}(c)$ so are the $W^u(O)$. Thus $W^u(\Lambda) \cap \lambda^{-1}(c)$ is "laminated" [7, §7] by the individual 1-dimensional curves $W^u(O) \cap \lambda^{-1}(c)$ where O ranges over all ψ-orbits in Λ. Since dim $\Lambda = 1$, $W^u(\Lambda) \cap \lambda^{-1}(c)$ has empty interior.

Suppose q is a fixed point of ψ.

If dim $W^s(q) = 1$ then $W^s(q) \cap \lambda^{-1}(c)$ is at most two points, q_1 and q_2 and we perturb ψ near q to make q_1, q_2 lie off $W^u(\Lambda) \cap \lambda^{-1}(c)$.

If dim $W^s(q) = 3$ then q is a sink and $W^s(q)$ is transverse to all $W^u(O)$ for any orbit O.

If dim $W^s(q) = 2$ and $W^s(q) \cap \lambda^{-1}(c) \neq \emptyset$, note first that $q \in \lambda^{-1}(c, \infty)$. As the flow is Morse-Smale on $\lambda^{-1}(c, \infty)$, we see that $W^s(q) \cap \lambda^{-1}(c)$ is a one-dimensional manifold which can accumulate only at points of $W^s(q') \cap \lambda^{-1}(c)$ where $\dim W^s(q') = 1$. These points already lie off the compact set $W^u(\Lambda) \cap \lambda^{-1}(c)$. Thus, generally, this 1-manifold can be tangent to the one dimensional lamination $W^u(q) \cap \lambda^{-1}(c)$ in isolated points, so we can push these tangencies off $W^u(\Lambda) \cap \lambda^{-1}(c)$ since $W^u(\Lambda) \cap \lambda^{-1}(c)$ has no interior in $\lambda^{-1}(c)$. See Figure 3.

In each case, we get transversality, so working one at a time on the fixed points of ψ we make ψ satisfy Axiom AS. Do the same for unstable manifolds of fixed points, $W^s(\Lambda)$ and $\lambda^{-1}(-c)$. This completes the proof in dimension 3.

Next, suppose $m = \dim(M) \geqslant 4$ and that we have embedded our suspended subshift Λ as a basic set of an Axiom AS flow ψ on S^{m-1}. We assume ψ has some point sinks by induction.

Let B^m be the ball of radius 2 in $\mathbb{R}^m$ and extend ψ from $S^{m-1} \subset B^m$ to a flow ψ on B^m such that

S^{m-1} is an attractor, ∂B^m is a repellor, the origin is a point source, $\psi | \partial B^m$ is a north-pole south-pole flow.

Let q_0, q_1 be the source and sink of $\psi | \partial B^m$. Clearly ψ is Axiom A. The only transversality in question for Axiom AS is $W^u(q_1) \cap W^s(\Omega_i)$ where Ω_i are the basic sets of $\psi | S^{m-1}$. Since $W^u(q_1)$ has dimension 1, we push it into the basin of attraction of one of the point sinks of $\psi | S^{m-1}$. This gives Axiom AS on B^m and leaves Λ as a basic set.

In M^m, glue a copy of ψ on B^m onto any m-ball B and extend ψ to M-B as a Morse-Smale gradient flow. The resulting flow obeys Axiom AS on M. Q.E.D.

Clearly, we wanted to prove Theorem 6 with Axiom AS instead of Axiom A. Forcing transversality between a single stable manifold and a

Figure 3. Removing tangency.

lamination is no easy task when both have high dimension. Consider Smale's hook [13] and Newhouse's hooked horseshoe [8]. In our case, we are willing to make large perturbations to get transversality, so we retain hope for an Axiom AS Theorem 6.

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