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Entropy on Sphere Bundles

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1. INTRODUCTION

Let X and Y be compact metric spaces and $F: Y \to Y$, $f: X \to X$, and $\pi: Y \to X$ continuous maps such that $\pi \circ F = f \circ \pi$. Bowen [1] has proved that if π is surjective then

$$h(f) \leqslant h(F) \leqslant h(f) + \sup\{h(F \mid \pi^{-1}(x)): x \in X\}, \tag{1.1}$$

where h denotes topological entropy. It is often difficult to apply Bowen's inequality (1.1) because, in principle, the last term involves computation of the entropy $h(F \mid \pi^{-1}(x))$ for every x in X. The computation is difficult not only because X has many points in general, but because the orbit of $F \mid \pi^{-1}(x)$ involves the metric on $\pi^{-1}(f^n(x))$ for $n = 0, 1,\ldots$.

Here we show how this difficulty can be overcome for a special class of maps. Let $\pi: E \to X$ be a vector bundle with a Finsler structure, that is, with a norm $\| \; \|$ on the fibers, and let $\pi: S(E) \to X$ be the corresponding unit sphere subbundle of E. If $A: E \to E$ is a vector bundle map that is nonsingular on every fiber, one can define a map $S(A): S(E) \to S(E)$ by

$$S(A)V = A(V)/\| A(V)\|. \tag{1.2}$$

We prove the following:

THEOREM. *Let $\pi: E \to X$ be a locally trivial vector bundle over the compact metric space X with a finite-dimensional fiber and a Finsler structure. Suppose that $A: E \to E$ is a vector bundle endomorphism of E over $f: X \to X$ that is nonsingular on every fiber. Then $h(S(A)) = h(f)$, where $S(A)$ is as defined above.*

Our approach differs from Bowen's in that, whereas he defined the entropy of a map on a (not necessarily invariant) subset, we use a concept of the entropy of a sequence of maps as defined below.

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2. The Entropy of a Sequence

Let X_0, X_1,... be a sequence of metric spaces with metrics d_0, d_1,... and let $\Phi = \{\varphi_i : i = 0, 1, 2,...\}$ (φ_0 = identity of X_0) and $\mathscr{F} = \{f_i : i = 1, 2,...\}$ be sequences of continuous maps where $\varphi_i : X_0 \to X_i$ and $f_i : X_{i-1} \to X_i$. $\mathscr{F}$ is said to be a *compositional representation* of Φ if for every i, $\varphi_i = \prod_{j=1}^{i} f_j$. If the φ_i's are all homeomorphisms, a (unique) compositional representation of Φ can be constructed by taking $f_{i+1} = \varphi_{i+1} \circ \varphi_i^{-1}$.

If Φ is as above, $\delta > 0$, and $K \subset X_0$ is compact, a subset $W \subset K$ is said to *(n, δ)-span* K if for every y in K there is an x in W such that for $0 \leqslant j \leqslant n$, $d_j(\varphi_j(x), \varphi_j(y)) < \delta$. Such a set W is said to be (n, δ) *separated* in K if for all x and y in W with $x \neq y$, $d_j(\varphi_j(x), \varphi_j(y)) > \delta$ for some j, $0 \leqslant j \leqslant n$. Let $r_n(\delta, K)$ denote the minimum cardinality of an (n, δ) spanning set for K and $s_n(\delta, K)$ the maximum cardinality of an (n, δ) separated set. Let $R_\Phi(\epsilon, K) = \limsup (1/n) \log r_n(\epsilon, K)$ as $n \to \infty$; $S_\Phi(\epsilon, K) = \limsup (1/n) \log s_n(\epsilon, K)$ as $n \to \infty$.

The following lemma is proved as in [1].

Lemma 2.1. *$r_n(\epsilon, K) \leqslant s_n(\epsilon, K) \leqslant r_n(\epsilon/2, K)$ and if $\epsilon_1 < \epsilon_2$, $R_\Phi(\epsilon_1, K) \geqslant R_\Phi(\epsilon_2, K)$, and $S_\Phi(\epsilon_1, K) \geqslant S_\Phi(\epsilon_2, K)$.*

It follows from Lemma 2.1 that for any compact K, $\lim_{\epsilon \to 0} R_\Phi(\epsilon, K) = \lim_{\epsilon \to 0} S_\Phi(\epsilon, K) = h(\Phi, K)$ and we define the entropy of Φ by $h(\Phi) = \sup_{K \text{ compact}} h(\Phi, K)$ so that $0 \leqslant h(\Phi) \leqslant \infty$. If $\mathscr{F}$ is a compositional representation of Φ we define $h(\mathscr{F}) = h(\Phi)$. In particular, if $X_i \equiv X$ and $f_i \equiv f$, $h(f) = h(\mathscr{F})$, where $h(f)$ is the topological entropy of f as defined by Bowen [1]. Note, however, that in general $h(\mathscr{F})$ depends on the metrics d_i whereas $h(f)$ is topologically invariant. On the other hand, $\mathscr{F}$ does have the following invariance property:

Lemma 2.2. *Let Φ be as above and suppose that $\theta_i : X_i \to X_i$ ($i = 0, 1,...$) is an isometry. Then if $\Psi = \{\psi_i : i = 0, 1,...\}$, where $\psi_i = \theta_i \circ \varphi_i$, $h(\Psi) = h(\Phi)$.*

The proof is immediate from the definitions involved.

Rematk. The following variant of Lemma 2.2 is also easily proved. Suppose that there are metrics ρ_i on X_i ($i = 0, 1,...$) such that for the positive constants c_1 and c_2,

$$c_1 d_i(z_i, w_i) \leqslant \rho_i(z_i, w_i) \leqslant c_2 d_i(z_i, w_i)$$

holds for all z_i and w_i in X_i and $i = 0, 1, 2,...$. Then $h(\Phi)$ is the same for the sequence of metrics ρ_0, ρ_1,... as for d_0, d_1,... .

In the next lemma $B_\delta(x)$ denotes the open ball of radius δ centered at a point x of a metric space.

LEMMA 2.3. *Let (Y, ρ) and (X, d) be compact metric spaces and let $\pi: Y \to X$ be a surjective map. Then for any $\epsilon > 0$, there is a $\gamma = \gamma(\epsilon) > 0$ such that if $y_1, \ldots, y_p$ are ϵ-dense in $\pi^{-1}(x)$ then*

$$\pi\left(\bigcup_{i=1}^{p} B_{2\epsilon}(y_i)\right) \supset B_\gamma(x).$$

The proof is a standard compactness argument, which we omit.

3. MAPS OF SPHERES

Let $X_i = S^1 \cong R^1/Z^1$ for $i = 0, 1, \ldots$, and suppose that $f_i: S^1 \to S^1$ is a homeomorphism for every i.

LEMMA 3.1. *Let $\mathscr{F} = \{f_i: i = 1, 2, \ldots\}$ be as above. Then for any $\delta > 0$, $r_n(\delta, S^1) \leqslant 2n\delta^{-1}$ for large n, hence $h(\mathscr{F}) = 0$.*

Proof. Let $k = [2\delta^{-1}]$ and let the "intervals" $[x_1, x_2], \ldots, [x_k, x_0]$ on S^1 be of length k^{-1}. Let $Q_n = \bigcup_{j=0}^{n-1} \varphi_j^{-1}(\{x_1, \ldots, x_k\})$, where, as above, $\varphi_j = \prod_{i=1}^{j} f_i$. The set Q_n has at most $nk = n[2\delta^{-1}] \leqslant 2n\delta^{-1}$ elements and the centers of the intervals in $S^1 - \{Q_n\}$ form an (n, δ) spanning set with at most nk elements.

LEMMA 3.2. *Let $\pi: Y \to X$ be a locally trivial S^1 bundle over a compact base space X. Suppose that there are continuous maps $g_i: Y \to Y$ and $f_i: X \to X$ such that $\pi g_i = f_i \pi$ for $i = 0, 1, \ldots$, where every g_i restricted to a fiber $\pi^{-1}(x)$ is a homeomorphism. Assume the maps g_i are equicontinuous, that is, for every $\gamma > 0$ there is a $\delta = \delta(\gamma)$ independent of i such that $d(g_i(x), g_i(y)) < \gamma$ whenever $d(x, y) < \delta$. Then $h(\mathscr{F}) = h(G)$, where $\mathscr{F} = \{f_i\}$, $G = \{g_i\}$.*

The proof is an imitation of Bowen's argument for proving (1.1) using Lemmas 2.3 and 3.1. We omit the details.

The main lemma needed in the proof of the theorem is:

LEMMA 3.3. *Let $A_i: R^n \to R^n$ ($i = 1, 2, \ldots$; $n \geqslant 2$) be linear isomorphisms such that for some $\lambda > 0$, $\| A_i \| \leqslant \lambda$ and $\| A_i^{-1} \| \leqslant \lambda$. Then if $S(A_i): S^{n-1} \to S^{n-1}$ is defined by (1.2) and $\Sigma = \{S(A_i): i = 1, 2, \ldots\}$, $h(\Sigma) = 0$.*

Proof. For $n = 2$ the lemma follows from Lemma 3.1. Now suppose that it has been proved for all $2 \leqslant k \leqslant n - 1$. Let $e_1, \ldots, e_n$ be the standard orthonormal basis for R^n. Let $\varphi_i = \prod_{j=1}^{i} S(A_j)$ and let θ_i be an orthogonal transformation of R^n such that $\theta_i(\varphi_i(e_1)) = a_i e_1$ ($a_i > 0$). If $\Phi = \{\varphi_i: i = 0, 1, \ldots\}$ and $\Omega = \{\omega_i = \theta_i \circ \varphi_i: i = 1, \ldots\}$, $h(\Phi) = h(\Omega)$ by Lemma 2.2. Thus the extra notation can be dropped and it can be assumed that $A_i(e_1) = a_i e_1$, where $a_i > 0$.

Let $P: R^n \to R^{n-1}$ be the orthogonal projection from R^n to the subspace spanned by $e_2, ..., e_n$, so that for any $x \in R^{n-1}$

$$A_i(x) = (I - P) A_i(x) + B_i(x),$$

where $B_i: R^{n-1} \to R^{n-1}$ is linear. Assume that $x \in R^{n-1}$, $\| x \| = 1$, and define $y = x - a_i^{-1}(I - P)A_i(x)$. Then $A_i(y) = PA_i(x) = B_i(x)$ and

$$\| y \|^2 = \| x \|^2 + a_i^{-2} \|(I - P) A_i(x)\|^2 \geqslant \| x \|^2 = 1.$$

Now

$$\lambda \| x \| \geqslant \| A_i(x)\| \geqslant \| B_i(x)\| = \| A_i(y)\| \geqslant \lambda^{-1} \| y \| \geqslant \lambda^{-1} \| x \|,$$

so $\| B_i^{-1} \| \leqslant \lambda$; $\| B_i \| \leqslant \lambda$. Therefore $B = \{B_i: i = 1,...\}$ satisfies the hypotheses of the lemma and consequently the conclusion by the induction hypothesis.

Let S^{n-2} and S^{n-1} be the unit spheres in R^{n-1} and R^n, let $k: S^1 \times S^{n-2} \to S^{n-2}$ be a projection onto the second factor, and let $g: S^1 \times S^{n-2} \to S^{n-1}$ be defined by $g(e^{2\pi it}, x) = (\cos 2\pi t, x \sin 2\pi t)$, where $S^1 = \{e^{2\pi it}\}$, $x = (x_1, ..., x_{n-1}) \in S^{n-2}$. Then since $S(A_i)$ maps great circles through e_1 to other such circles, there is a map $f_i: S^1 \times S^{n-2} \to S^1 \times S^{n-2}$ such that $g \circ f_i = S(A_i) \circ g$ and $k \circ f_i = S(B_i) \circ k$. Now we know that $h(\{S(B_i)\}) = 0$, so the conclusion follows from Lemma 3.2.

4. Conclusion

Now the theorem stated in Section 1 follows from Lemma 3.3 and Bowen's inequality (1.1). To verify this one only needs to observe that $h(S(A) \mid \pi^{-1}(x)) = h(S(A), \pi^{-1}(x)) = 0$, where on the left is meant the entropy of the sequence consisting of the iterate of $S(A) \mid \pi^{-1}(x)$, and on the right is meant the entropy of $S(A)$ relative to the subset $\pi^{-1}(x)$.

The theorem was motivated by an attempt to prove the entropy conjecture (cf. [2]) for immersions of compact manifolds to themselves. There one would take $E = \Lambda''(TM)$ or $\Lambda^k(T^*M)$. We were not able to find a proof using this method, but it seems that the theorem is of independent interest.

References

1. R. Bowen, Entropy for group endomorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.* **153** (1971), 401–414.
2. R. Sacksteder and M. Shub, Entropy of a differentiable map, *Advances in Math.* **28** (1978), 000–000.