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Entropy of a Differentiable Map

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1. INTRODUCTION

The "entropy conjecture" asserts that if T is a differentiable map of a compact manifold X, the topological entropy (cf. [1]) h of T is at least as great as $\log \lambda$, where λ denotes the spectral radius of T^*, the map on de Rham cohomology induced by T. Various results related to the entropy conjecture have been obtained [1, 3, 4–6]. On the other hand, it is known that no such conjecture is true for maps that are Lipschitz rather than differentiable [4, 5].

We have not been able to prove the entropy conjecture in full generality; however, we establish several closely related inequalities. In Section 2, numbers h_2 and h_3 that are related to Hausdorff measure and have properties similar to those of h are defined. It is shown that $h_3 \leqslant h$, and under certain conditions $h_2 = h_3$. In Section 3, another number h_1 is defined in terms of the derivative DT^n and $\log \lambda \leqslant h_1$ is demonstrated. In Section 4 it is proved that $h_1 \leqslant h_2$ and thus we always have $\log \lambda \leqslant h_1 \leqslant h_2$ and $h_3 \leqslant h$. Thus what is lacking to prove the entropy conjecture in full generality is the inequality $h_2 \leqslant h_3$, which we have only been able to prove under special conditions.*

2. h_2 AND h_3

It is assumed that X is a compact manifold of dimension m and that X has a Riemannian metric; d denotes the distance function that defines $d(x, y)$ to be the greatest lower bound of the lengths of the arcs connecting points x and y of X. Define a sequence of distance functions $d_1, d_2, \ldots$ by $d_n(x, y) = \max\{d(T^k(x), T^k(y)): 0 \leqslant k \leqslant n\}$ and let $D_n(x, \epsilon) = \{y \in X: d_n(x, y) \leqslant \epsilon\}$. If $r(n, \epsilon)$ denotes the minimum cardinality of a set $\{x_1, \ldots, x_r\}$ such that $X = \bigcup D_n(x_i, \epsilon)$, the results of Bowen [1] show that

$$h = \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup(1/n) \log r(n, \epsilon). \tag{2.1}$$

* M. Misiurewicz and W. Szlenk have constructed a smooth map for which $h_1 > h$, hence $h_2 > h_3$. ("Entropy of Piecewise Monotone Mappings," University of Warsaw Preprint.)

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Let Ω_m be the volume of a Euclidean sphere of radius 1. For any $\epsilon > 0$ and $n = 1, 2,...$ let $M(n, \epsilon)$ be the greatest lower bound of $\Omega_m \sum_{i=1}^{p} \epsilon_i^m$, where $X = \bigcup_{i=1}^{p} D_n(x_i, \epsilon_i)$ and $\epsilon_i \leqslant \epsilon$ for $i = 1,..., p$. We define

$$h_3 = \lim_{\epsilon\to 0} \limsup_{n\to\infty} (1/n) \log M(n, \epsilon).$$

It is obvious that

$$M(n, \epsilon) \leqslant r(n, \epsilon)\, \Omega_m \epsilon^m;$$

hence (2.1) and the definition of h_3 give our first inequality,

$$h_3 \leqslant h. \tag{2.2}$$

The quantity h_2 is defined by taking limits in the opposite order, that is, $h_2 = \lim_{n\to\infty}(1/n) \lim_{\epsilon\to 0} \log M(n, \epsilon)$. There is no difficulty about the existence of the first limit. In fact, if we denote the m-dimensional Hausdorff measure of a subset A of X, with respect to the metric d_n by $M_n(A)$, then the definition of Hausdorff measure shows that $M_n(X) = \lim_{\epsilon\to 0} M(n, \epsilon)$.

Since $M_n(X) \geqslant M(n, \epsilon) \geqslant M(n, \delta)$ if $\epsilon \leqslant \delta$, one always has $h_2 \geqslant h_3$. One would like to be able to show that $h_2 = h_3$; however, we have only been able to show this under the conditions of the proposition below.

PROPOSITION 2.1. *Suppose that there is a sequence* $a_1, a_2,...$ *of positive constants such that* $\lim(a_n)^{1/n} = 1$ *as* $n \to \infty$ *and for all* x *in* X,

$$M_n(D_n(x, \epsilon)) \leqslant a_n \epsilon^m.$$

Then

$$h_2 = h_3.$$

The proof just depends on observing that if $X = \bigcup_{i=1}^{p} D_n(x_i, \epsilon_i)$, then

$$M_n(X) \leqslant \sum_{i=1}^{p} M_n(D_n(x_i, \epsilon_i)) \leqslant a_n \Omega_m \sum_{i=1}^{p} \epsilon_i^m;$$

hence

$$M_n(X) \leqslant a_n M(n, \epsilon) \qquad \text{for every } n \text{ and } \epsilon.$$

The conclusion follows easily.

3. THE MAPS T^* AND E_T

Let ω_1 and ω_2 be complex-valued p-forms defined on X. We define for x in X, $(\omega_1, \omega_2)(x) = *(\omega_1 \wedge *\bar{\omega}_2)(x)$, where $*$ is the Hodge operator and $^-$ denotes complex conjugation. Also,

$$|\omega|(x) = (\omega, \omega)(x)^{1/2}.$$

Then (cf. [7, Chap. 6])

$$\langle \omega_1 , \omega_2 \rangle = \int_X (\omega_1 , \omega_2)(x)\, dV$$

defines an inner product on the space of p-forms and the norm

$$\| \omega \|_q = \left(\int_X | \omega(x)|^q \, dV \right)^{1/q}$$

is defined for $q \geqslant 1$. Also define $\| \omega \|_\infty = \sup\{| \omega |(x) : x \in X\}$. Here dV is just the volume element associated to the metric, that is, $dV = *1$. The completion of the space of forms (of dimension $p = 0,..., m$) with respect to $\| \; \|_q$ is denoted by L^q, where it is understood that the inner product in L^2 is defined to make homogeneous forms of different dimensions orthogonal.

Let H denote the orthogonal projection from L^2 to the harmonic forms (cf. [7, Chap. 6]). Since $L^2 \subset L^1$ it makes sense to ask if H is bounded with respect to $\| \; \|_1$ and as is well known, the answer is yes.

PROPOSITION 3.1. *H is bounded with respect to $\| \; \|_1$.*

Proof. Let $\alpha_1 ,..., \alpha_r$ be an orthonormal basis for the harmonic forms. Then $H\omega = \sum_{j=1}^r \langle \omega, \alpha_j \rangle \alpha_j$, so

$$| H\omega | (x) \leqslant \sum_{j=1}^r |\langle \omega, \alpha_j \rangle| \, | \alpha_j | (x) \quad \text{and} \quad \| H\omega(x) \|_1 \leqslant \sum_{j=1}^r |\langle \omega, a_j \rangle| \, \| a_j \|_1 .$$

But,

$$|\langle \omega \;\; \alpha_j \rangle| \leqslant \int_X | \omega | (x) \, | \alpha_j | (x)\, dV \leqslant \| \alpha_j \|_\infty \| \omega \|_1 ,$$

so

$$\| H \|_1 \leqslant \sum_{j=1}^r \| \alpha_j \|_\infty \| \alpha_j \|_1 .$$

Now let E_T denote the map on forms and DT the tangent map induced by T. Then if the map on antisymmetric p-tuples of tangent vectors corresponding to DT is denoted by D_pT, one has

$$(E_T\omega)(X_1 ,..., X_p) = \omega(DT(X_1),..., DT(X_p) = \omega(D_pT(X_1 ,..., X_p)).$$

Define $\theta_n{}^p(x) = \| D_pT^n(x) \|$, where the norm of D_pT on the antisymmetric p-tuples of tangent vectors is determined by the Riemannian metric on X. Then $|(E_T{}^n\alpha)| (x) \leqslant \theta_n{}^p(x) \, | \alpha | (x)$; hence

$$\| E_T{}^n\alpha \|_1 \leqslant \int_X \theta_n{}^p(x) \, | \alpha | (x)\, dV \leqslant \Lambda_n \| \alpha \|_\infty , \tag{3.1}$$

where

$$\Lambda_n = \max \left\{ \int_X \theta_n{}^p(x)\, dV, 0 \leqslant p \leqslant m \right\} .$$

Let $T^*: H^*(X, R) \to H^*(X, R)$ denote the induced map on de Rham cohomology. If α is a harmonic from corresponding to the eigenvalue β of T^*, $E_T{}^n\alpha = \beta^n\alpha + d\gamma_n$, so by (3.1)

$$|\beta|^n \|\alpha\|_1 = \|HE_T{}^n\alpha\|_1 \leqslant \|H\|_1 \|E_T{}^n\alpha\|_1 \leqslant \Lambda_n \|H\|_1 \|\alpha\|_\infty . \qquad (3.2)$$

Now if $h_1 = \limsup_{n\to\infty} \log \Lambda_n^{1/2}$, (3.2) gives $\log \lambda \leqslant h_1$, where λ is the spectral radius of T^*.

Remark. The argument also gives the stronger result that for any form α (closed or not), $\limsup(1/n) \log \|E_T{}^n\alpha\|_1 \leqslant h_1$.

4. Comparison of h_1 and h_2

If V_x denotes the tangent space at a point x in X, $\| DT^n(x)\langle\xi\rangle\|^2$ and $\|\xi\|^2$ can be viewed as quadratic forms on V_x. The first is positive semidefinite and the second is positive definite, so the eigenvalues $\tau_1{}^2, \tau_2{}^2,..., \tau_m{}^2$ of the first with respect to the second are defined and can be assumed to satisfy $\tau_1(x) \geqslant \tau_2(x) \geqslant \cdots \tau_m(x) \geqslant 0$. It is well known that $\theta_n{}^p(x)$ as defined above is given by

$$\theta_n{}^p(x) = \prod_{j=1}^{p} \tau_j(x).$$

If g denotes the original Riemannian metric on X and g_n the metric defined by $g_n(\xi, \eta) = g(\xi, \eta) + g(DT^n\langle\xi\rangle, DT^n\langle\eta\rangle)$, the volume element dV_n associated to g_n is given by

$$\frac{dV_n}{dV}(x) = \prod_{j=1}^{m} (1 + \tau_j(x)).$$

If the right side is multiplied out, one of the terms is $\theta_n{}^p(x)$, hence

$$V(X)\Lambda_n \leqslant V_n(X). \qquad (4.1)$$

Now if $\gamma_n(\xi, \eta) = \sum_{j=0}^{n} g(DT^j\langle\xi\rangle, DT^j\langle\eta\rangle)$ is still another Riemannian metric on X, with associated volume element dU_n, one has $dU_n(x)/dV_n(x) \geqslant 1$ and by (4.1)

$$V(x)\Lambda_n \leqslant U_n(X). \qquad (4.2)$$

It is known [2, Sect. 6] that if a distance function is defined on X corresponding to the metric γ_n, then dU_n is the corresponding Hausdorff measure. This, of course is not the same as the Hausdorff measure M_n defined in Section 2; however, it will be shown that

$$\limsup_{n\to\infty}(1/n) \log U_n(X) = \limsup_{n\to\infty}(1/n) \log M_n(X) = h_2 . \qquad (4.3)$$

To prove (4.3), first observe that if the definition in Section 2 is made with the distance function

$$\delta_n(x, y) = \left(\sum_{j=0}^{n} d(T^j(x), T^j(y))^2 \right)^{1/2},$$

in place of d_n, the quantity h_2 is unaffected. This follows from

$$(1/n)\, \delta_n(x, y) \leqslant d_n(x, y) \leqslant \delta_n(x, y)$$

and

$$M(n, \epsilon) \leqslant M^*(n, \epsilon) \leqslant n^m M(n, \epsilon/n),$$

where $M^*(n, \epsilon)$ is the analog for δ_n of $M(n, \epsilon)$. Moreover, the Riemannian metric γ_n defined above is related to δ_n by

$$\gamma_n(\xi, \xi)^{1/2} = \lim_{t \to 0}(1/t)\, \delta_n(x(0), x(t)),$$

where ξ is the tangent vector to $x(t)$ at $x = 0$. It is then easy to verify that the Hausdorff measure corresponding to δ_n is nothing but dU_n. This proves (4.3), which together with (4.2) shows that $h_1 \leqslant h_2$.

We collect our results:

THEOREM. *Let T be a C^1 map of a compact manifold. Let λ be the spectral radius of the induced map on cohomology and let h be the entropy of T. Then the quantities h_1, h_2, h_3 defined above satisfy* $\log \lambda \leqslant h_1 \leqslant h_2 \geqslant h_3$ *and* $h_3 \leqslant h$. *Moreover under the assumptions of Proposition* 2.1, $h_2 = h_3$, hence $\log \lambda \leqslant h$.

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