Société Mathématique de France
Astérisque 51 (1978) p.395-413

ALEXANDER COCYCLES AND DYNAMICS

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Differential topology has studied differentiable manifolds and the mappings between them by studying functors from manifolds to algebra. In dynamics we are given a map $f : M \to M$ from a manifold to itself and we study the asymptotic properties of the iterates of f. In principle it should be much easier to study asymptotic algebraic data. Then we must ask: What is the dynamical interpretation of the asymptotic data? Here it is usually easier to give answers in the generic or stable cases, but we may ask for all f as well. One example is the entropy conjecture. This conjecture is true for an open and dense set of homeomorphisms of manifolds, except perhaps in dimension 4, although not true in general for homeomorphisms. It is also true for diffromorphisms in the stable case. In the general case we have the results of Manning which bound the topological entropy from below by the growth rate of the induced map on the first cohomology or homotopy group for continuous f, and the results of Misiurewicz and Przytycki which bound the topological entropy from below by $\log|\text{degree } f|$ for continuously differentiable f. But we are still in the dark for the general $C^1 f$ and all dimensions.

We consider another example where we know even less in general, but where the generic case is clearer. Let M^n be compact and orientable, and let V and W be oriented submanifolds of M^n of dimension k and $n-k$ respectively.

V represents a homology class in $H_k(M)$. We may consider W as a homomorphism $[W] : H_k(M) \to \mathbb{Z}$ by intersection. That is $[W](V)$ is the number of points of intersection of W and V counted with multiplicity and sign; $[W] : H_k(M) \to \mathbb{Z}$ and $f_{k_*} : H_k(M) \to H_k(M)$ are homomorphisms. So in principle at least, the growth rate $\limsup \frac{1}{n} \log|[W](f^n_{k_*}V)|$ is easy to calculate. This number is the growth rate of the homological or algebraic number of points of intersection of W and $f^n(V)$. Now let $N_n(f,V,W)$ be the actual or geometric number of points of intersection of W and $f^n(V)$. We would like to compare the asymptotic algebraic information to the asymptotic geometric information.

Problem .

Suppose that f is C^1 and that $f^n|V$ is an embedding for each n.

a) Is $\limsup \frac{1}{n} \log N_n(f,V,W) \geq \limsup \frac{1}{n} \log |[W](f^n_{*k}V)|$?

b) What is the distribution of the points of intersection in W ?

In the generic case $f^n|V$ is always transverse to W, and $N_n(f,V,W) \geq |[W](f^n_{*k}V)|$ for all n ; so a) is trivially true. Also in case V has dimension 1, any isolated point of intersection of $f^n(V)$ and W has index -1, 0 or $+1$. So once again $N_n(f,V,W) \geq |[W](f^n_{*k}V)|$ for all n. But in neither of these cases do we know the distribution of the points of intersection.

A special case of the problém arises from a C^1 map $g : M \to M$. We let $f : M \times M \to M \times M$ be $\mathrm{Id}_M \times g$ that is $f(x,y) = (x,g(y))$. We let $V = W = \Delta_{M \times M}$ the diagonal of $M \times M$. Then $N_n(f,V,W) = N_n(g)$ the number of periodic points of g of period n, and the Lefschetz trace of

f^n, $L(f^n) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{trace} f^n_{*i}$, is $[W](f^n_{*\dim M}V)$.

Sub-problem .

Suppose $g : M \to M$ is C^1 .

a) Is $\limsup \frac{1}{n} \log N_n(g) \geq \limsup \frac{1}{n} \log |L(f^n)|$?

b) How are the periodic points of g distributed in M ?

Once again a) is true for the generic g and is true in dimension 1 . So our emphasis is on all C^1 maps g . The hypothesis that g is C^1 is already necessary in dimension 2 . For C^1 maps g we know that when the Lefschetz traces $L(f^n)$ are unbounded then there must be an infinite number of periodic points, but we know very little about their growth rate. Nothing is known about the distribution of the periodic points in the general case.

This sub-problem is already interesting for the two sphere, S^2 . Even for polynomials in one complex variable thought of as acting on S^2 the answer is not obvious, but in fact a strong statement may be made for rational maps.

Proposition 1.

If $g = \frac{p}{q} : S^2 \to S^2$ is a rational map of the two sphere of degree d , then there is a constant $K \geq 0$ such that $N_n(g) \geq d^n - K$.

Proof.

Let $d > 1$. According to [Julia,G. -Sur l'iteration des fonctions rationnels- Journal de Mathématiques 1918 p 236] for any non-transversal fixed point p of a rational map g of S^2 of degree bigger than 1 there is a point $x_p \in S^2$ such that $g^n(x_p) \to p$ as $n \to +\infty$ and $Dg(x_p) = 0$. Now it is clear that g can have at most d non-transversal periodic orbits. For any periodic point p which is non-transversal take a power of g, g^k, to make p fixed. There is now a singularity of g^k, x_p, such that $g^{kn}(x_p) \to p$ as $n \to +\infty$. The singularities of g^k are g^{-k} (singularities of g) so there is an original singularity x'_p

such that $g^{kn}(x_p') \to p$ as $n \to +\infty$. As g has at most d singularities, g has at most d non-transversal periodic orbits and we are done.

Returning to our general problem, we may put $f_k^* : H^k(M,\mathbb{C}) \to H^k(M,\mathbb{C})$ into Jordan form. Then the eigenclasses and eigenvalues of f_k^* play a special role, but their relationship to manifolds or to entropy is not clear. We seek a geometric object which represents an eigenclass in cohomology. To find the object we use Alexander cohomology theory.

Let X be a compact space. Let $\phi : X\underbrace{x \ldots x}_{k+1}X \to \mathbb{C}$ be a function. The support of ϕ, supp ϕ, is the set of points $x \in X$ such that ϕ is not identically zero on any neighborhood of $\underbrace{(x,x,\ldots,x)}_{k+1}$. The boundary of ϕ is the function

$$\delta\phi : \underbrace{Xx\ldots xX}_{k+2} \to \mathbb{C}$$

defined by

$$\delta\phi\,(x_o,\ldots,x_{k+1}) = \sum_{i=o}^{k+1} (-1)^i \phi(x_o,\ldots,\hat{x}_i,\ldots,x_{k+1}) .$$

If $f : Y \to X$ then $f^*\phi$ is the function defined by

$$f^*\phi(y_o,\ldots,y_k) = \phi(f(y_o),\ldots,f(y_k))$$

which I will sometimtes also write $\phi \circ f$. Let F^k be the vector space of functions $\phi : Xx\ldots xX \to \mathbb{C}$ and let $Z^k \subset F^k$ be the subspace of those functions with empty supports. The homology groups of the complex

$$\ldots \to F^k/Z^k \to F^{k+1}/Z^{k+1} \to \ldots$$

are the Alexander cohomology groups of X with complex coefficients. We may also do the same for real coefficients. It makes little difference if I use R or $\mathbb{C}$ in the propositions which follow. I will start with $\mathbb{C}$ and change to R when it is convenient. For various categories we may restrict the functions ϕ considered and still get the same cohomology theory.

Examples .

1) In the category of compact spaces and continuous maps we may restrict to continuous ϕ or alternating continuous ϕ .

2) In the category of compact metric spaces and Lipschitz maps we may restrict to Lipschitz or Hölder ϕ and we may assume that the functions ϕ are alternating.

3) In the category of compact differentiable manifolds and differentiable maps, we may restrict to differentiable ϕ and we may assume they are alternating.

All this results from sheaf theory and is easily derivable in an earlier version from [Borel,A. Cohomology des Espaces Localement Compacts d'après J. Leray 1964 Springer Lecture Notes # 2, 1964].

We give a sample proposition.

Proposition 2.

Suppose that X is a compact, connected metric space and that $f : X \to X$ is Lipschitz. Suppose that $v \in H^k(X,\mathbb{C})$ is a λ eigenclass for

$$f_k^* : H^k(X,C) \to H^k(X,C)$$

with $|\lambda| > 1$, that is $f_k^*(v) = \lambda v$. Then

a) there is a λ alternating eigencocycle ϕ representing v , that is $f^*(\phi) = \lambda\phi + z$ with $z \in Z^k$.

b) if $k = 1$, ϕ is unique mod Z^1 . That is if ϕ' also represents v , $f^*(\phi') = \lambda\phi' + z'$ with $z' \in Z'$ then there is a $z'' \in Z'$ such that $\phi = \phi' + z''$.

c) ϕ is Hölder.

Proof.

a) We start with an alternating cocycle α representing v . Thus $f^*(\alpha) = \lambda\alpha + \delta\gamma + z$ for some alternating $\gamma \in F^{k-1}$ and some $z \in Z^k$. Let

$$\phi = \alpha + \delta\left(\sum_{n=0}^{\infty} \frac{f^{*n}(\gamma)}{\lambda^{n+1}} \right) .$$

Since $|\lambda| > 1$ $\sum_{n=o}^{\infty} \frac{f^{*n}(\gamma)}{\lambda^{n+1}}$ converges uniformly so ϕ is defined and does the trick. For the same z, $f^*(\phi) = \lambda\phi + z$.

b) Suppose ϕ' represents v and $f^*(\phi') = \lambda\phi' + z'$. There are $\mu \in F^o$ and $z'' \in F^1$ such that $\phi = \phi' + \delta\mu + z''$. Thus $f^*(\phi) = f^*(\phi') + \delta f^*(\mu) + f^*(z'')$ and $\lambda\phi + z = \lambda\phi' + z' + \delta f^*(\mu) + f^*(z'')$ so

$$\lambda\phi' + \lambda\delta\mu + \lambda z'' + z = \lambda\phi' + z' + \delta f^*(\mu) + f^*(z'') .$$

Hence $\delta(f^*(\mu) - \lambda\mu) = z_4$ for some $z_4 \in Z'$. This means that $f^*(\mu) - \lambda\mu$ is locally constant and since X is connected $f^*(\mu) - \lambda\mu = c$ for some constant c. But now

$$\mu = \sum_{n=o}^{\infty} \frac{f^{*n}c}{\lambda^{n+1}} = \sum_{n=o}^{\infty} \frac{c}{\lambda^{n-1}}$$

which is constant. Thus $\delta\mu = o$ and $\phi = \phi' + z''$.

c) In a way we may choose α and γ to be Hölder. Let $1 \geq \varepsilon \geq o$ be chosen such that $(\mathrm{Lip}\ f)^{\varepsilon} < |\lambda|$ and such that γ is Hölder of Hölder exponent ε . We denote its Hölder constant by $H(\gamma)$. Then

$$\left| \sum_{n=o}^{\infty} \frac{\gamma f^n(x)}{\lambda^{n+1}} - \sum_{n=o}^{\infty} \frac{\gamma f^n(y)}{\lambda^{n+1}} \right| \leq \sum_{n=o}^{\infty} \frac{|\gamma f^n(x) - \gamma f^n(y)|}{|\lambda^{n+1}|}$$

$$\leq \sum_{n=o}^{\infty} \frac{H(\gamma)\ d(f^n(x), f^n(y))^{\varepsilon}}{|\lambda|^{nH}} \leq H(\gamma) \sum_{n=o}^{\infty} \frac{(\mathrm{Lip}\ f)^{n\varepsilon}}{|\lambda|^{n+1}} d(x,y)^{\varepsilon}$$

Now $\sum_{n=o}^{\infty} \frac{((\mathrm{Lip}\ f)^{\varepsilon})^n}{|\lambda|^{nH}}$ converges, by the choice of ε . Thus $\sum_{n=o}^{\infty} \frac{f^{*n}\gamma}{\lambda^{nH}}$ is Hölder of exponent ε and the same is true for $\delta(\sum_{n=o}^{\infty} \frac{f^{*n}\gamma}{\lambda^{n+1}})$. As α was Hölder the sum is Hölder.

The process carried out in a), which was the inversion of $(\lambda I - f^*)$ that is $\sum_{n=o}^{\infty} \frac{f^{*n}(\gamma)}{\lambda^{n+1}} = (\lambda I - f^*)^{-1}(\gamma)$, can be carried out for more general subspaces.

Recall that a linear map $A : V \to V$ is hyperbolic if the spectrum of A is disjoint from the unit circle. If the spectrum is outside the unit circle, we say A is expanding. If $f^*: V \to V$ is hyperbolic on an f_k^* invariant subspace $V \subset H^k(X,\mathbb{C})$ then we can invert the appropriate operator on the space of cochains for a homeomorphism f.

Proposition 3.

Suppose that X is a compact space and that $f : X \to X$ is a homeomorphism (continuous). Let $f_k^* : V \to V$ be hyperbolic (expanding) where $V \subset H^k(X,\mathbb{C})$ is a finite dimensional invariant subspace for $f_k^* : H^k(X,\mathbb{C}) \to H^k(X,\mathbb{C})$. Then there is a subspace W of cocycles contained in $F^k(X)$ which projects isomorphically onto V and such that $f^* : W \to W \bmod Z^k$. The isomorphism identifies $f^*: W \to W \bmod Z^k$ with $f_k^* : V \to V$. If $k=1$ then W is unique mod Z^1.

Proof.

Choose a basis $v_1,\dots,v_m$ for V and cocycles $\phi_1,\dots,\phi_m$ in $F^k(X)$ which represent $v_1,\dots,v_m$. We use matrix notation and write

$$f^*\begin{pmatrix}\phi_1\\ \vdots\\ \phi_m\end{pmatrix} = f_k^*\begin{pmatrix}\phi_1\\ \vdots\\ \phi_m\end{pmatrix} + \delta\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} + \begin{pmatrix}z_1\\ \vdots\\ z_m\end{pmatrix}$$

Now we attempt to add boundaries $\delta\alpha_i$ to the ϕ_i in order to eliminate the γ_i. Thus we want to solve the equation

$$f^*\begin{pmatrix}\phi_1+\delta\alpha_1\\ \vdots\\ \phi_m+\delta\alpha_m\end{pmatrix} = f_k^*\begin{pmatrix}\phi_1+\delta\alpha_1\\ \vdots\\ \phi_m+\delta\alpha_m\end{pmatrix} + \begin{pmatrix}z_1\\ \vdots\\ z_m\end{pmatrix}$$

or

$$f^*\begin{pmatrix}\phi_1\\ \vdots\\ \phi_m\end{pmatrix} + \delta f^*\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix} = f_k^*\begin{pmatrix}\phi_1\\ \vdots\\ \phi_m\end{pmatrix} + \delta f_k^*\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix} + \begin{pmatrix}z_1\\ \vdots\\ z_m\end{pmatrix}$$

Since

$$f^*\begin{pmatrix}\phi_1\\ \vdots\\ \phi_m\end{pmatrix} = f_k^*\begin{pmatrix}\phi_1\\ \vdots\\ \phi_m\end{pmatrix} + \delta\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} + \begin{pmatrix}z_1\\ \vdots\\ z_m\end{pmatrix}$$

this amounts to solving

$$\delta\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} = \delta f_k^*\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix} - \delta f^*\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix}.$$

We drop the boundaries and solve the seemingly more difficult equation

$$\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} = f_k^*\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix} - f^*\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix} \text{ for } \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix}.$$

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix} = \left(f_k^* - f^*\right)^{-1}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix}.$$

We indicate why this last equation makes sense. $f_k^* : V \to V$ is hyperbolic. This means that we may express V as the direct sum of two f_k^* invariant subspaces. $V = V^s \oplus V^u$. The map $f_k^* : V^s \to V^s$ is a contraction and $f_k^* : V^u \to V^u$ is an expansion. More precisely $\exists c > o$ and $\Lambda > 1$ such that

$$||f_k^{*n}\,|V^s|| < C\Lambda^n \quad \text{for} \quad n > 0$$

and

$$||f_k^{*-n}|V^u|| < C\Lambda^n \quad \text{for} \quad n > 0.$$

For any point $x \in X$ we may consider $\begin{pmatrix}\gamma_1(x)\\ \vdots\\ \gamma_m(x)\end{pmatrix}$ and $f^*\begin{pmatrix}\gamma_1(x)\\ \vdots\\ \gamma_m(x)\end{pmatrix} = \begin{pmatrix}\gamma_1(f(x))\\ \vdots\\ \gamma_m(f(x))\end{pmatrix}$ as elements of V. Thus we may write

$$\begin{pmatrix}\gamma_1(x)\\ \vdots\\ \gamma_m(x)\end{pmatrix} = (\gamma_s(x),\gamma_u(x)) \text{ and } \begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} = (\gamma_s,\gamma_u).$$

This splitting is invariant for f^* and f_k^* that is

$$f^*\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} = \begin{pmatrix}\gamma_1\circ f\\ \vdots\\ \gamma_m\circ f\end{pmatrix} = (\gamma_s\circ f, \gamma_u\circ f)$$

and

$$f_k^*\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} = (f_k^*\,\gamma_s,\ f_k^*\,\gamma_u)\ .$$

Now we may write

$$(f_k^* - f^*)\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_m\end{pmatrix} == (f_k^* - f^*)(\gamma_s,\gamma_u) = ((f_k^* - f^*)\gamma_s,\ (f_k^* - f^*)\gamma_u)\ .$$

To invert this product map it suffices to invert it on each factor. We invert these by the usual series.

$$(f_k^* - f^*)^{-1}(\gamma_s,\gamma_u) = ((f_k^* - f^*)^{-1}\gamma_s,\ (f_k^*-f^*)^{-1}\gamma_u)$$

$$= (-(-f_k^*\circ(f^{-1})^* + \mathrm{Id})\circ f^*)^{-1}\gamma_s,\ (f_*^k\,(\mathrm{Id} - f_k^{*-1}\,f^*))^{-1}\,\gamma_u)$$

$$= (-f^{*-1}\sum_{n=0}^{\infty}(f_k^*\circ(f^{-1})^*)^n\,\gamma_s,\ \sum_{n=0}^{\infty}(f_k^{*-1}\,f^*)^n\,f_k^{*^{-1}}\,\gamma_u)$$

$$= (-\sum_{n=0}^{\infty} f_k^{*^n}\,f^{*^{-n}}\,\gamma_s\ ,\ \sum_{n=0}^{\infty} f_k^{*^{-n-1}}\,f^{*^n}\,\gamma_u)\ .$$

and both series converge uniformly. If $V^s = 0$ then we don't need to invert f. (The second formula is the one which we used when we had a single eigenvalue λ with $|\lambda| > 1$.)

Now we turn to uniqueness in the case $k = 1$. Suppose that W and W' are vector spaces of cocycles which project isomorphically onto V and such that $f^*\colon W \to W \bmod Z'$ and $f^*\colon W' \to W' \bmod Z'$. Choose cocycles $\phi_1,\dots,\phi_m$ in

W and $\phi_1',\ldots,\phi_m'$ in W' which project to the same basis $v_1,\ldots,v_m$ in V.

Thus $\phi_i = \phi_i' + \delta u_i + z_i'$ for some $u_i \in F$ and some $z_i'' \in Z'$.

$$f_k^*\begin{pmatrix} 0 \\ \vdots \\ \phi_i \\ \vdots \\ 0 \end{pmatrix} = f_k^*\begin{pmatrix} 0 \\ \vdots \\ \phi_i' \\ \vdots \\ 0 \end{pmatrix} + f_k^*\begin{pmatrix} 0 \\ \vdots \\ \delta u_i \\ \vdots \\ 0 \end{pmatrix} + f_k^*\begin{pmatrix} 0 \\ \vdots \\ z_i'' \\ \vdots \\ 0 \end{pmatrix}$$

Now

$$f_k^*\begin{pmatrix} 0 \\ \vdots \\ \phi_i \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ z_i \\ \vdots \\ 0 \end{pmatrix} = f^*\big(\phi_i \big) = f^*\big[\phi_i'\big] + f^*\big[\delta u_i\big] + f^*\big[z_i'' \big]$$

$$= f_k^*\begin{pmatrix} 0 \\ \vdots \\ \phi_i' \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ z_i' \\ \vdots \\ 0 \end{pmatrix} + f^*\begin{pmatrix} 0 \\ \vdots \\ \delta u_i \\ \vdots \\ 0 \end{pmatrix} + f^*\begin{pmatrix} 0 \\ \vdots \\ z_i'' \\ \vdots \\ 0 \end{pmatrix}$$

So

$$f_k^*\begin{pmatrix} 0 \\ \vdots \\ \phi_i \\ \vdots \\ 0 \end{pmatrix} + f_k^*\begin{pmatrix} 0 \\ \vdots \\ \delta u_i \\ \vdots \\ 0 \end{pmatrix} + f_k^*\begin{pmatrix} 0 \\ \vdots \\ z_i'' \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ z_i \\ \vdots \\ 0 \end{pmatrix} = f_k^*\begin{pmatrix} 0 \\ \vdots \\ \phi_i' \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ z_i' \\ \vdots \\ 0 \end{pmatrix} + f^*\begin{pmatrix} 0 \\ \vdots \\ \delta u_i \\ \vdots \\ 0 \end{pmatrix} + f^*\begin{pmatrix} 0 \\ \vdots \\ z_i'' \\ \vdots \\ 0 \end{pmatrix}$$

Thus $f_k^*\begin{pmatrix} 0 \\ \vdots \\ \delta u_i \\ \vdots \\ 0 \end{pmatrix} - f^*\begin{pmatrix} 0 \\ \vdots \\ \delta u_i \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} z_1''' \\ \vdots \\ z_m''' \end{pmatrix}$ for some $z_i'' \in Z'$.

Thus $f_k^*\begin{pmatrix} 0 \\ \vdots \\ u_i \\ \vdots \\ 0 \end{pmatrix} - f^*\begin{pmatrix} 0 \\ \vdots \\ u_i \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_i \\ \vdots \\ c_{im} \end{pmatrix}$ where the c_{ij} are constants.

Finally
$$f_k^*\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} - f^*\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}$$

where the d_i are constants and

$$\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = (f_k^* - f^*)^{-1}\begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}$$

which is a constant vector. Thus u_i is constant for each i, $\delta u_i = o$ and

$\phi_i = \phi_i'' + z_i''$.

Remarks.

1) The same argument and theorem are true for real coefficients, and we will use them.

2) If we worked in the category of compact metric spaces and Lipschitz homeomorphisms with Lipschitz inverses we could produce Hölder cocycles ϕ_i as above. As it is we have continuous ϕ_i in the general case.

We return to $H'(X,R)$ and consider a geometric interpretation of an Alexander cocycle. For convenience I will limit myself to a connected differentiable manifold M . An Alexander 1-cocycle is something like a closed 1-form and consequently defines something like a foliation with a transversal structure. A translational H structure of codimension j on M is a collection of charts U_i which cover M and functions $\phi_i : U_i \to R^j$ such that if $U_i \cap U_k = \phi$ then there is a constant vector $v_{ik} \in R^i$ such the

$$\phi_i \mid U_i \cap U_k = \phi_k \mid U_i \cap U_k + v_{i_k} \circ .$$

Here H is for Haefliger not homology or cohomology. Our functions ϕ_i are to be assumed continuous and in many interesting cases Hölder .We will call the level surfaces of the $\phi_i | U_i$ strokes. Because of the overlap conditions the strokes overlap coherently and form maximal subsets which we call stripes. We call the stripe structure on M induced by a translational H-structure of codimension j a translational H-striation of M of codimension j . Given a translational H-structure of codimension j defined by functions $\phi_i : U_i \to R^j$ and a translational H-structure of codimension 1 defined by functions $\Psi_i : U_i \to R^1$ there is an induced translational H-structure of codimension $(j+1)$ defined by $\phi_i \times \Psi_i : U_i \to R^j \times R^1$. The stripes of this structure are the intersections of the stripes defined by the ϕ_i and the Ψ_i-structures.

Lemma 1 .

An Alexander 1-cocycle $\phi : M \times M \to R$ defines a translational H-structure of codimension 1 on M, $\phi_i : U_i \to R$ such that $\delta(\phi_i) : U_i \times U_i \to R$ equals $\phi : U_i \times U_i \to R$.

Proof.

Cover M by geodesically convex balls U_x for each $x \in M$. For each x $\phi : U_x \times U_x \to R$ is null cohomologous thus there is a function $\phi_x : U'_x \to R$ such that $\delta(\phi_x) = \phi | U'_x \times U'_x$ for some perhaps smaller geodesically convex neighborhood U'_x of x . Cover M by a finite collection U_i of the U'_x . If $U_i \cap U_j \neq \emptyset$ then $\delta(\phi_i) = \delta(\phi_j)$ so $\delta(\phi_i - \phi_j) = 0$ on $U_i \cap U_j$. As $U_i \cap U_j$ is contractible $\phi_i - \phi_j = c_{i_j}$ for some constant c_{i_j} .

On the other hand a translational H-striation of M of codimension 1 defines an Alexander 1-cocycle which induces it. To see this fix the cover U_i and the functions $\phi_i : U_i \to R$ defining the striation. Let b be a Lebesgue number for this cover, that is if $d(x,y) < b$ then x and y are both in the same U_i . Define $\phi(x,y) = \phi_i(x) - \phi_i(y)$ if $d(x,y) < b$. This makes sense since the ϕ_i differ by a constant. Now leave ϕ fixed on a neighborhood of the diagonal in $M \times M$ and extend it to be continuous or Hölder as the case may be .

Given a translational H-striation S of M of codimension j and $f : M \to M$, we say that S is f-invariant if f of every stripe of S is contained in a stripe of S . If we let S_x be the stripe of S containing x then our condition is $f(S_x) \subset S_{f(x)}$. If in addition there are charts U_i on M , functions $\phi_i : U_i \to R$ which define S and a linear map A of R^j such that $\phi_j(f(x)) - \phi_j(f(y)) = A(\phi_i(x) - \phi_i(y))$ whenever $x, y \in U_i$ and $f(x), f(y) \in U_j$ then we say S is transversally transformed by A . In the case that $j = 1$ and $A(x) = \lambda x$ with $|\lambda| > 1$ we say that S is transversally stretched by λ .

Thus we may restate proposition 2 in terms of manifolds and striations. We say that a translational H-striation represents a cohomology class if its corresponding Alexander 1-cocycle does .

Proposition 4.

Suppose that M is a compact connected manifold and that $f : M \to M$ is Lipschitz (continuous). Suppose that $v \in H^k(M,R)$ is a λ eigenclass for $f_k^* : H^k(M,R) \to H^k(M,R)$ with $|\lambda| > 1$, that is $f_k^*(v) = \lambda v$. Then there is a unique Hölder (continuous) translational H-striation S of M of codimension 1 such that S represents v , is f invariant and is transversally stretched by λ .

Proof.

Mod Z^1 there is a unique Alexander cocycle ϕ such that ϕ represents v and $f^*\phi = \lambda\phi$ mod Z^1 . Two cocycles which are equal mod Z^1 define the same striation $\square$.

Proposition 3 translates as follows:

Proposition 5.

Suppose that M is a compact connected manifold and that $f : M \to M$ is a homeomorphism (continuous). Let $f_1^* : V \to V$ be hyperbolic (expanding) where $V \subset H^1(X,R)$ is a finite dimensional invariant subspace of dimension j for $f_1^* : H'(X,R) \to H'(X,R)$. Then there is a unique translational H-striation S of M of codimension j which satisfies the following properties.

1) There is a basis $v_1,\dots,v_j$ of V and cocycles $\phi_1,\dots,\phi_j \in F^1(M)$ which represent the v_i such that S is the intersection of the striations defined by the ϕ_i,

2) S is f invariant.

3) S is transversally transformed by $f_1^* : V \to V$.

Proof.

This is just Proposition 3 with the uniqueness mod Z^1.

Examples and construction.

Suppose that $\tilde{A}$ is an $n \times n$ matrix with integer entries. Then $\tilde{A} : R^n \to R^n$ and maps the integer lattice $\mathbb{Z}^n \subset R^n$ into itself. Thus $\tilde{A}$ induces a map of the n-torus, T^n, which is $R^n/\mathbb{Z}^n$. We denote this map by A. We have the following commutative diagram.

$$\begin{array}{ccc} R^n & \xrightarrow{\tilde{A}} & R^n \\ \downarrow & & \downarrow \\ T^n & \xrightarrow{A} & T^n \end{array}$$

R^n is the universal covering space of T^n and $\tilde{A}$ is the unique lifting of A which sends 0 to 0. The first homology group of the n-torus with integer coefficients, $H_1(T^n,\mathbb{Z})$, may be identified with $\mathbb{Z}^n$ and $A_{*1}: H_1(T^n,\mathbb{Z}) \to H_1(T^n,\mathbb{Z})$ is then identified with $\tilde{A} : \mathbb{Z}^n \to \mathbb{Z}^n$. Thus on real homology $A_{*1}: H_1(T^n,R) \to H_1(T^n,R)$ is identified with $\tilde{A} : R^n \to R^n$. On real cohomology $A_1^* : H^1(T^n,R) \to H^1(T^n,R)$ is identified with the transpose of $\tilde{A}$, $\tilde{A}^t : R^n \to R^n$. There is an invariant subspace $V^t \subset R^n$ of dimension j for $\tilde{A}^t$ if and only if V^t is the dual space of an invariant quotient V of $\tilde{A}$ of dimension j. $\tilde{A}_j^t = \tilde{A}^t|V^t$ is hyperbolic or expanding if and only if $\tilde{A}_j : V \to V$ is. In diagram notation we have

$$\begin{array}{ccc} V^t & \xrightarrow{\tilde{A}_j^t} & V^t \\ \Big\downarrow{\scriptstyle \text{incl}} & & \Big\downarrow{\scriptstyle \text{incl}} \\ R^n & \xrightarrow{\tilde{A}^t} & R^n \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R^n & \xrightarrow{\tilde{A}} & R^n \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ V & \xrightarrow{\tilde{A}_j} & V \end{array}$$

where incl is an inclusion and π is a surjection.

If $\tilde{A}_j^t : V^t \to V^t$ is hyperbolic and $A : T^n \to T^n$ is an isomorphism or if $\tilde{A}_j^t$ is expanding we are in the circumstances of propositions 2 through 5. If $v_1{}^t,\dots,v_j{}^t$ is a basis for V^t then $v_i{}^t : R^n \to R$ and $\mathrm{Ker}(v_i{}^t)$ is a subspace of R^n of dimension $n - 1$. The parallel translates of $\mathrm{Ker}(v_i{}^t)$ foliate R^n and the projection of this foliation is a linear foliation of T^n

which we will denote $\mathcal{F}_{v_i^t}$. The intersection $\bigcap_{i=1}^{j} \mathcal{F}_{v_i^t} = \mathcal{F}$ is the invariant foliation for A given by Proposition 5. The intersection $\bigcap_{i=1}^{j} \mathrm{Ker}(v_i^t) = \mathrm{Ker}\pi$. The translates of $\mathrm{Ker}\,\pi$ foliate R^n and the projection of this foliation to T^n is the same foliation $\mathcal{F}$. Here we use the word foliation instead of striation because of the regularity of $\mathcal{F}$. Let x and y be any two close by points in T^n. We may lift x and y to $\tilde{x},\tilde{y} \in R^n$ such that $d(\tilde{x},\tilde{y}) = d(x,y)$ i.e. the distances are unchanged. $\Psi_i(x,y)$ is then defined by the formula $\Psi_i(x,y) = v_i^t(\tilde{x}) - v_i^t(\tilde{y})$. Extend Ψ_i to a global function which we call Ψ_i again and which we may assume to be $C\infty, \Psi_i : T^n \times T^n \to R$. These Ψ_i represent the v_i^t and define the foliation $\mathcal{F}_{v_i^t}$. By definition $\mathcal{F}$ is transversally transformed by $\tilde{A}_j^t$.

We will see that these Ψ_i and F are universal in the sense that all others are pull backs of these. Let M be a compact connected manifold and let $f : M \to M$ be a homeomorphism (continuous). The map $f_1^* : H'(M,R) \to H'(M,R)$ is induced by $f_1^* : H'(M,Z) \to H'(M,\mathbb{Z})$. Thus if $\dim(H'(M,R)) = n$ we may identify $H'(M,R)$ with R^n and f_1^* with an integral matrix $\tilde{A}^t : R^n \to R^n$. If $V' \subset H'(M,R)$ is a j-dimensional invariant subspace for f_1^* such that $f_1^* | V'$ is hyperbolic (expanding), then we may identify V' with an $\tilde{A}^t$ invariant subspace $V^t \subset R^n$. We are now in the situation of our example. We keep all the notation and hypotheses from above. We call the identifying map $i^t : R^n \to H'(M,R)$ and $i : H_1(M,R) \to R^n$ its transpose. Recall that R^n is simultaneously $R^n, H^1(T^n,R)$ and $H_1(T^n,R)$. Given two spaces X and Y, continuous map $h : X \to Y$ and a striation S of Y the induced striation h^*S of X is defined by $(h^*S)_x = h^{-1}(S_x)$. We use $\sim$ to denote universal covering spaces and maps lifted to universal covering spaces.

Proposition 6.

Let M be a compact, connected manifold, and let $f : M \to M$ be a homeomorphism (continuous). Suppose that $V' \subset H^1(M,R)$ is a j-dimensional invariant subspace for f_1^*, that $f_1^* : V' \to V'$ is hyperbolic (expanding) and that dimension $H'(M,R) = n$. Let $\tilde{f} : \tilde{M} \to \tilde{M}$ be a lifting of f. Then there is a continuous map $\tilde{h} : \tilde{M} \to R^n$ such that $\tilde{A}_j \pi\tilde{h} = \pi\tilde{h}\tilde{f}$ and $\tilde{h}$ lifts a continuous map $h : M \to T^n$ such that $h_1^* = i^t$.

Proof.

Let $h_1 : M \to T^n$ be a continuous map such that $h_1^* = i^t$. Let $\tilde{h}_1 : \tilde{M} \to R^n$ be a lifting of h_1 to the universal covering spaces. We would like to find a map h, homotopic to h_1 and a lifting $\tilde{h}$ of h such that

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\pi\tilde{h}} & V \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{A}_j} \\ \tilde{M} & \xrightarrow{\pi\tilde{h}} & V \end{array} \qquad \text{commutes.}$$

V splits as a direct sum $E^s \oplus E^u$ with $\tilde{A}_j|E^s$ contracting and $\tilde{A}_j|E^u$ expanding. Let R^s and $R^u \subset R^n$ be chosen such that $\pi : R^s \to E^s$ and $\pi : R^u \to E^u$ isomorphically. We may write R^n as the direct sum of three spaces $R^s \oplus R^u \oplus \mathrm{Ker}(\pi)$. We use coordinates (x,y,z) for this splitting. In this splitting $\tilde{A} : R^n \to R^n$ is represented by

$$(x,y,z) \to (\tilde{A}_j(x), \tilde{A}_j(y), \tilde{A}(z) + B(x,y))$$

where $\tilde{A}_j$ on the first two coordinates comes from the identification with V and $B : R^s \oplus R^u \to \mathrm{Ker}\,\pi$ is linear. Now we may write

$$\pi\tilde{h}_1\tilde{f} - \tilde{A}_j\pi\tilde{h}_1 = (\phi_s,\phi_u)$$

and

$$\tilde{h}_1\tilde{f} - \tilde{A}\tilde{h}_1 = (\phi_s, \phi_u, \phi_{\mathrm{Ker}\pi}) .$$

We would like to change h, such that ϕ_s and ϕ_u are 0. We write $\tilde{h}_1 = (\tilde{h}_{1s},\tilde{h}_{1u},\tilde{h}_{1\,\mathrm{Ker}\pi})$. We will make ϕ_u zero, the case for ϕ_s is similar as in Proposition 5 by adding a function to $\tilde{h}_{1u}$. We would like to solve

$$\pi\,(\tilde{h}_1 + (0,\gamma_u,0))\,\tilde{f} - \tilde{A}_j\pi\,(\tilde{h}_1 + (0,\gamma_u,0)) = 0$$

$$\pi\,(\tilde{h}_1 + (0,\gamma_u,0))\,\tilde{f} - \tilde{A}_j\pi\,(\tilde{h}_1 + (0,\gamma_u,0)) =$$

$$\pi\,\tilde{h}_1\tilde{f} - \tilde{A}_j\pi\tilde{h}_1 + \pi(0,\gamma_u,0)\,\tilde{f} - \tilde{A}_j\,\pi(0,\gamma_u,0) = (\phi_s,\phi_u) + (0,\gamma_u\circ f) - (0,\tilde{A}_j\gamma_u)$$

So we would like to solve the equation

$$\tilde{A}_j\gamma_u - \gamma_u\circ\tilde{f} = \phi_u\ .$$

As usual $\gamma_u = \sum_{k=0}^{\infty} \tilde{A}_j^{-k-1}\,\phi_u\circ\tilde{f}^k$ which converges uniformly. Now we need to be sure that $\tilde{h}_1 + (0,\gamma_u,0)$ lifts a function h homotopic to h_1 . Let $\rho : \tilde{M} \to M$ be the projection. Since $(\phi_s\phi_u,\phi_{\mathrm{Ker}\pi})$ is null homotopic on deck transformations there is a function $(\psi_s,\psi_u,\psi_{\mathrm{Ker}\pi}) : M \to R^s \oplus R^u \oplus \mathrm{Ker}\,\pi$. such that

$$(\phi_s,\phi_u,\phi_{\mathrm{Ker}\ \pi}) = (\psi_s,\psi_u,\psi_{\mathrm{Ker}\ \pi})\circ\rho\ .$$

Thus

$$(0,\phi_u,0) = (0,\psi_u,0)\circ\rho$$

also is null homotopic on deck transformations and the same is true for

$$(0,\gamma_u,0) = (0,\ \sum_{R=o}^{\infty} \tilde{A}_j^{-k-1}\psi_u\circ f^k,0)\circ\rho\ .$$

Thus $\tilde{h}_1 + (0,\gamma_u,0)$ is the lift of a function h homotopic to h_1 .

Corollary 1.

The striation S of M given by Proposition 5 is $h^*\mathcal{J}$.

Proof.

Let $v_1^t,\ldots,v_j^t$ be a basis for V^t and $i^t(v_1^t) = v_1',\ldots,i^t(v_j^t) = v_j'$ be a basis for V' . The $h^*\psi_i$ represent the v_i' and define S . Thus S is the intersection of the striations defined by the $h^*\psi_i$ which are the $h^*\mathcal{J}_{v_i^t}$ and $h^*\mathcal{J} = S$.

Remarks.

1) The estimates and calculations we have done are quite standard in the theory of Anosov maps and expanding maps.

2) The striations we construct are usually far from smooth. It would be interesting to know how regular they can be made in the homotopy class of a given $f : M \circlearrowleft M$. In the case of a two manifold M^2 of genus $g > 0$ the situation is already interesting. $H'(M,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^g \oplus \mathbb{Z}^g$. If A is any $g \times g$ integral matrix then

$$\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} : \mathbb{Z}^g \oplus \mathbb{Z}^g \to \mathbb{Z}^g + \mathbb{Z}^g$$

is realized by a homotopy class of homeomorphisms of M^2. Thus we can have a very rich eigenvalue structure and many invariant striations for every element of the homotopy class. In special cases these are sometimes Thurston's measured foliations for pseudo Anosov diffeomorphisms, but they exist more generally and there are many more of them.

3) Uniqueness for H' in Proposition 3 and 5 made it convincing that there would be a classifying object for the Alexander cocycles and striations produced. Since H' is represented by maps to S^1 it was natural to consider linear maps of T^n to represent endomorphisms of H'. Hindsight relates these to Anosov maps. It would be interesting to find a universal object which represents the hyperbolic behaviour on the higher cohomology groups, if it exists.

4) Alexander cochains give new dynamic invariants, for example their supports are invariant subsets for f. These invariants are not well understood.

We return to Proposition 6 and Corollary 1. Since S is $h^*\mathcal{F}$ and is transversally transformed by A we may expect the topological entropy $h(f)$ to be bounded from below by $\Sigma \log|\lambda_i|$ where the sum is taken over all eigenvalues of A of modulus bigger than one, but this will not be true unless we have some independence conditions on the striations $h^*\mathcal{F}_{v_i t}$. This condition is expressed in terms of cup products. Suppose the space V' has a basis $v'_1,\dots,v'_j$

such that the cup product $v'_1 \cup \ldots \cup v'_j \neq 0$. This implies that for the map $\tilde{h}$ of Proposition 6, $\pi\tilde{h} : \tilde{M} \to V$ is surjective, by the following simple topological argument. There are real cohomology classes $v_1, \ldots, v_j$ in $H'(T^n, R)$ such that $h^*(v_i) = v'_i$ and $h^*(v_1 \cup \ldots \cup v_j) = v'_1 \cup \ldots \cup v'_j \neq 0$. Thus there are integral classes $w_1, \ldots, w_j$ in $H'(T^n, \mathbb{Z})$ such that $h^*(w_1 \cup \ldots \cup w_j) \neq 0$ and $w_1, \ldots, w_j$ are induced from a quotient torus T^j of T^n and the foliation of T^n induced by the $w_1, \ldots, w_j$ is as close as we want to the foliation induced by the $v_1, \ldots, v_j$. If $\pi\tilde{h} : \tilde{M} \to V$ is not surjective then the image of $\tilde{h}$ misses a plane transverse to the foliation induced by the $v_1, \ldots, v_j$. Thus if the $w_1, \ldots, w_j$ are well chosen the line is transverse to the foliation induced by the $w_1, \ldots, w_j$.

This contradicts the fact that $h^*(w_1 \cup \ldots \cup w_j) \neq 0$.

This argument is incomplete

Since $\tilde{h}$ lifts h , $\tilde{h}$ is uniformly continuous and the same is true for $\pi\tilde{h}$. The topological entropy of $\tilde{h}$ is the same as that of h and it is easy to establish that the topological entropy of $\tilde{h}$ is bigger than or equal to the topological entropy of $\tilde{A}_j$. Manning carries out some of these arguments more carefully in his article in these proceedings where he proves a similar proposition. Also there is my seminar talk in the Orsay Thurston seminar. Altogether we have established the following proposition.

Proposition 7.

Let M be a compact, connected manifold. Let $f : M \to M$ be a homeomorphism (continuous). Suppose that $V' \subset H^1(M, R)$ is a j-dimensional invariant subspace for f^*_1 , that there is a basis $v'_1, \ldots, v'_j$ of V' with $v'_1 \cup \ldots \cup v'_j \neq 0$ and that $f^1_*: V' \to V'$ is hyperbolic (expanding) . Then $h(f) \geq \Sigma \log|\lambda|$ where the sum is taken over all eigenvalues λ of $f^*_1|V$ with $|\lambda| > 1$.

open

It seems that Gromov has a more general proposition. Our proof seems integrated with striations.

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