FILTRATIONS, DECOMPOSITIONS, AND EXPLOSIONS.

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Our goal in this paper is to establish for flows results analogous to those obtained for discrete dynamical systems in [5].

To formulate the main result, we begin by recalling from [4] the notion of a filtration for a flow. Let ϕ be the flow on compact n-manifold M generated by a C^1 vector field $X = \dot{\phi}$

Definition. A filtration for ϕ *(or* X*) is a finite sequence* $\mathfrak{M} = \{M_0, \ldots, M_k\}$ *of compact submanifolds with boundary such that*

(i) $\varnothing = M_0 \subset M_1 \subset \cdots \subset M_k = M$
(ii) $\dim M_i = n (\forall i \geqslant 1)$
(iii) $\phi_t\,[M_i] \subset \operatorname{int} M_i\ (\forall t > 0)$
(iv) *The flow is transverse to the boundary of each* M_i*—that is, for* $x \in \partial M_i$ $(0 < i < k)$, X_x *is not tangent to* ∂M_i.

Given a filtration $\mathfrak{M}$, the maximal ϕ-invariant subset of $M_i - M_{i-1}$ is denoted

$$K_i(\mathfrak{M}) = \cap \{\phi_t[M_i - \operatorname{int} M_{i-1}] : t \in \mathbf{R}\}$$

and we let $K(\mathfrak{M}) = \cup \{K_i(\mathfrak{M}) : i = 1, \ldots, k\}$. We shall call the filtration *fine* if $K(\mathfrak{M}) = \Omega(\phi)$ (the set of non-wandering points for ϕ). If ϕ does not admit a fine filtration, we will look for a sequence $\mathfrak{M}_i$ of filtrations for which $K(\mathfrak{M}_i)$ tend to $\Omega\ (\phi)$. More precisely,

Definition. A filtration $\tilde{\mathfrak{M}} = \{\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_l\}$ *refines the filtration* $\mathfrak{M} = \{M_0, \ldots, M_k\}$ *if for each* $\alpha = 1, \ldots, l$ *there exists* i_α, $1 \leqslant i_\alpha \leqslant k$, *such that*

$$[\tilde{M}_\alpha - \tilde{M}_{\alpha-1}] \subset [M_{i_\alpha} - M_{i_\alpha - 1}].$$

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Definition. *A sequence* $\mathcal{M}_1, \mathcal{M}_2, \ldots$ *of filtrations for* ϕ *is said to be fine if*

(i) $\mathcal{M}_{i+1}$ *refines* $\mathcal{M}_i$ $(i=1,\ldots)$
(ii) $\cap K(\mathcal{M}_i) = \Omega(\phi)$.

A fine sequence of filtrations will be seen to control the growth of Ω under perturbations.

Definition. *A flow* ϕ *has no* C^0 Ω*-explosions if given any neighborhood* U *of* $\Omega(\phi)$ *in* M, *there is a neighborhood* V *of* $\dot{\phi} = X$ *in* $\mathcal{X}^r(M)$ (*the space of* C^r *vectorfields on* M *with the* C^0 *topology*) *such that a flow* ψ *with* $\dot{\psi} \in V$ *has* $\Omega(\psi) \subset U$.

Our main result, then, is

THEOREM 2. *A necessary and sufficient condition for* ϕ *to have no* C^0 Ω*-explosions is that* ϕ *admit a fine sequence of filtrations.*

The paper has four parts. We begin by recalling some properties of filtrations and demonstrating the sufficiency of the above condition. In the second part, we define the notions of decomposition and open decomposition for an invariant set, and establish that the existence of a fine filtration is equivalent to the existence of either kind of decomposition for Ω, with no cycles. In the third section, we investigate the problem of constructing a fine sequence of filtrations, or equivalently, of finding an open decomposition without cycles inside any specified neighborhood of Ω. We show how such a construction is possible in the absence of C^0 Ω-explosions. Most of our techniques are translations of arguments in [5] to the flow case. We close the paper by showing how the dimension restrictions in a central lemma of [5] can be removed.

1. We begin with some observations concerning filtrations. A filtration for ϕ gives some rough information concerning the dynamic structure of the flow. As usual, given a ϕ-invariant set Λ, we define its stable (unstable) set by

$$W^s(\Lambda) = \{x \in M : \mathrm{dist}(\phi_t(x), \Lambda) \to 0 \text{ as } t \to +\infty\}$$
$$W^u(\Lambda) = \{x \in M : \mathrm{dist}(\phi_t(x), \Lambda) \to 0 \text{ as } t \to -\infty\}.$$

PROPOSITION 1. *If* $\mathcal{M} = \{M_0, \ldots, M_k\}$ *is a filtration for* ϕ, *then:*

(i) *each* $K_i(\mathcal{M})$ *is compact*
(ii) $\Omega(\phi) \subset K(\mathcal{M})$

(iii) *M is the disjoint union of the $W^s(K_i)$ ($W^u(K_i)$). In fact, $M_i \subset \cup \{W^s(K_j): j \leq i\}$.*

(iv) *$K_i = W^s(K_i) \cap W^u(K_i)$. In fact, letting $\Omega_i = \Omega(\phi) \cap K_i(\mathfrak{M})$, we have $K_i = W^s(\Omega_i) \cap W^u(\Omega_i)$ and $W^\sigma(K_i) = W^\sigma(\Omega_i)$ ($\sigma = u, s$).*

The proof of (i) is immediate. To see (ii), note that for any point $x \in [M_i - \text{int } M_{i-1}] - K_i(\mathfrak{M})$, there is some $t \in \mathbf{R}$ such that $\phi_t(x) \not\in M_i - \text{int } M_{i-1}$. If $t > 0$, $\phi_t(x) \in \text{int } M_{i-1}$ and x wanders; if $t < 0$, $\phi_t(x) \in M - M_i$; since this is taken into itself by the reverse flow, x wanders again. We defer proof of (iii) and (iv) to the next section, where they will be established in a more general setting.

The real usefulness of a filtration, however, is that it gives information not only about the flow ϕ, but about (C^0) nearby flows, as well.

PROPOSITION 2. *If $\mathfrak{M}$ is a filtration for ϕ, then:*

(i) *$\mathfrak{M}$ is also a filtration for any flow ψ with $\dot{\psi}$ sufficiently (C^0) near $\dot{\phi}$.*

(ii) *Given a neighborhood U of $K(\mathfrak{M})$, there exists a neighborhood $\mathfrak{U}$ of $\dot{\phi}$ in $\mathfrak{X}^0(M)$ such that for any flow ψ with $\dot{\psi} \in \mathfrak{U}$, the corresponding maximal set for ψ*

$$K_\psi(\mathfrak{M}) = \bigcup_i \bigcap_{t \in \mathbf{R}} \psi_t[M_i - \text{int} M_{i-1}]$$

is contained in U.

(iii) *If $\mathfrak{M}$ is a fine filtration for ϕ, then ϕ has no C^0 Ω-explosions.*

Proof. The first statement follows from the observation that since the flow is transverse to the ∂M_i, we must have $\dot{\phi} \neq 0$ on ∂M_i; hence small perturbations of $\dot{\phi}$ must point in approximately the same direction. The second statement follows from the remark that by compactness of K_i, there is a finite time, T, such that

$$\cap \{\phi_t[M_i - M_{i-1}] : |t| \leq T\} \subset U.$$

The third statement is an immediate consequence of the second, combined with (ii) in the previous proposition.

While fine filtrations exist for a large class of flows (in particular, for those satisfying Axiom A' and the no cycle condition), this class is not generic, as noted in [5] for the discrete case, using the example of Newhouse [2]. (The suspension of this example provides a C^2-open set of vectorfields without fine filtrations on every manifold of dimension 3 or greater.) However, the flows with a fine sequence of filtrations seem to include the Newhouse example, and this condition suffices for prevention of C^0 Ω-explosions:

PROPOSITION 3. *A flow with a fine sequence of filtrations has no C^0 Ω-explosions.*

Proof. We combine statement (ii) of the previous proposition with statement (ii) in Proposition 1, and observe that for a fine sequence of filtrations, $K(\mathfrak{M}_i)\downarrow\Omega(\phi)$, so that for large i, $K(\mathfrak{M}_i)\subset U$, where U is any given neighborhood of $\Omega(\phi)$.

2. We now turn to the decomposition of invariant sets. These are of two kinds: the first will be called an "open decomposition", the second, simply a "decomposition". Let Λ be a compact ϕ-invariant set.

Definition. An open decomposition for Λ is a finite, disjoint family of open sets, $W_1,\ldots,W_k$ in M such that $\Lambda\subset\bigcup_{i=1}^{k}W_i$. We will denote the maximal ϕ-invariant subset of W_i by

$$K_i=\bigcap_{t\in\mathbf{R}}\phi_t[W_i].$$

The open decomposition will be called proper if each K_i is compact. Of course, $\Lambda\subset\bigcup K_i$, because any trajectory of a flow is connected and the W_i are disjoint.

Definition. For an open set $W\subset M$, we define

$$W^s=\bigcup_{t\geqslant 0}\phi_{-t}[W].$$

Definition. If $W_1,\ldots,W_k$ is an open decomposition of Λ, then

(i) *An r-cycle ($r\geqslant 2$) is an ordered r-tuple of distinct indices $(i_1,\ldots,i_r)$ such that*

$$W_{i_j}\cap W^s_{i_{j+1}}\neq\varnothing\qquad(j=1,\ldots,r-1)$$
$$W_{i_r}\cap W^s_{i_1}\neq\varnothing$$

(ii) *A 1-cycle is an index i such that*

$$(\exists x\in M-W_i)(\exists s,t>0)\ \phi_{-s}(x)\in W_i,\ \phi_t(x)\in W_i.$$

Note that if an open decomposition has no r-cycles ($r\geqslant 2$), then we can reindex the open sets so that $i>j\Rightarrow W_j\cap W_i^s=\varnothing$. Following Newhouse [1], we call such a numbering a *filtration ordering*. The terminology is motivated in part by the observation that if $\varnothing=M_0\subset\cdots\subset M_k=M$ is a filtration, then the sets

$W_i = \text{int } M_i - M_{i-1}$ form an open decomposition for $\Omega(\phi)$ with no r-cycles ($r \geq 2$), and in this case the given numbering is a filtration ordering.

We shall be particularly interested in compact invariant sets Λ which contain all limit sets—that is, such that for every $x \in M$, $\alpha(x) \cup \omega(x) \subset \Lambda$. We begin with some data about the maximal invariant sets of an open decomposition for such a Λ.

PROPOSITION 4. *Let Λ be a compact, ϕ-invariant set such that $\alpha(x) \cup \omega(x) \subset \Lambda (\forall x \in M)$. If $W_1, \ldots, W_k$ is an open decomposition for Λ, with no 1-cycles, letting $K_i = \cap_t \phi_t [W_i]$ and $\Omega_i = \Omega(\phi) \cap K_i$, then:*

(i) *The decomposition is proper—i.e. the K_i are compact.*
(ii) *The stable sets $W^s K_i$ partition M—in fact, $(\forall x \in M)(\exists i, j) \alpha(x) \subset K_i$, $\omega(x) \subset K_j$.*
(iii) *$W^s K_i = W^s \Omega_i$, $W^u K_i = W^u \Omega_i$, and*

$$K_i = W^s \Omega_i \cap W^u \Omega_i.$$

Proof.

(i) Suppose $x \in \text{clos } K_i - K_i$. Since $\text{clos } K_i$ is ϕ-invariant, $\alpha(x) \cup \omega(x) \subset \text{clos } K_i$, so that $\alpha(x) \cup \omega(x) \subset \Lambda \cap W_i$. Since W_i is a neighborhood of $\Lambda \cap W_i$, $\phi_t(x) \in W_i$ for $|t|$ large. On the other hand, $x \notin K_i$, so that for some t, $\phi_t(x) \notin W_i$. Thus, we have a 1-cycle; the contradiction implies that $\text{clos } K_i - K_i$ is empty, so K_i is compact.

(ii) We reproduce a well-known argument of Smale ([11], p. 782), to show that $\omega(x)$ is contained in a single K_i; the argument for $\alpha(x)$ follows by time-reversal. Suppose $\omega(x)$ intersects several distinct K_i. Since a single orbit is a connected set and the W_i are disjoint open sets, it follows that if $\phi_{T_0}(x) \in W_i$ and $\phi_{T_1}(x) \in W_j$ with $T_0 < T_1$ and $i \neq j$, then for some time S, $T_0 < S < T_1$, $\phi_S(x) \in M - \cup_{l=1}^k W_l$. Since the orbit of x frequently enters more than one W_i, we can pick a sequence of times $S_n \to +\infty$ when $\phi_{S_n}(x) \in M - \cup W_l$. An accumulation point of this sequence lies outside $\Lambda \subset \cup K_l$, contradicting the assumption that Λ contains all α- and ω-limits.

(iii) To show $K_i \subset W^s \Omega_i \cap W^u \Omega_i$, take $x \in K_i$ and note that $\exists\, j, l$ such that $x \in K_i \cap W^s \Omega_j \cap W^u \Omega_l$. We want to show $j = l = i$. Certainly, $\alpha(x) \subset \Omega_j$, $\omega(x) \subset \Omega_l$. Thus, since K_i is invariant and closed,

$$\varnothing \neq \alpha(x) \subset \Omega_j \cap K_i \subset K_j \cap K_i$$
$$\varnothing \neq \omega(x) \subset \Omega_l \cap K_i \subset K_l \cap K_i.$$

Since the open sets W are disjoint, so are the K, and we have $i = j = l$. On the other hand, suppose $x \in W^s\Omega_i \cap W^u\Omega_i$. Then, since the above set is invariant, $\phi_t(x) \in W^s\Omega_i \cap W^u\Omega_i$ $(\forall t)$. In particular, $\phi_t(x) \in W_i$ for $|t|$ large, and so (since there are no 1-cycles) for all t. Thus, $x \in K_i$. Q.E.D.

We note that by (i) in the above, an open decomposition of Λ allows us to write Λ as a disjoint union of the closed, invariant sets $\Lambda_i = K_i \cap \Lambda = W_i \cap \Lambda$. In general, we can define a decomposition of a compact ϕ-invariant set Λ.

Definition. *Let Λ be compact and ϕ-invariant. A decomposition of Λ is a finite, disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that*

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

We recall from [4] *that in case $\Lambda = \Omega(\phi)$, a spectral decomposition is a decomposition such that each Ω_i contains a dense orbit.*

We would like to define the notion of cycles in this situation. Recalling the defintion of $W^s\Lambda_i$ and $W^u\Lambda_i$ from Sec. I, we define a relation on the Λ_i by: $\Lambda_i \prec \Lambda_j$ if $W^s\Lambda_i \cap W^u\Lambda_j \neq \varnothing$.

Definition

(i) *An r-cycle $(r \geqslant 2)$ for the decomposition $\Lambda_1, \ldots, \Lambda_K$ is an ordered r-tuple of distinct indices $(i_1, \ldots, i_r)$ such that $\Lambda_{i_1} \prec \Lambda_{i_2} \prec \cdots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.*

(ii) *A 1-cycle is an index i such that*

$$(W^s\Lambda_i \cap W^u\Lambda_i) - \Lambda_i \neq \varnothing.$$

(iii) *As before, a filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leqslant j$.*

It should be pointed out that there is an important *a priori* difference between the two kinds of decompositions. While for an open decomposition of Λ "no cycles" is a condition affecting all orbits that come near Λ, the corresponding condition for a decomposition of Λ takes into account only those orbits that are asymptotic to Λ. On the other hand, if all orbits are asymptotic to Λ and Λ is the maximal invariant set for the open decomposition, the two conditions are the same, as shown in the following proposition. A corollary of this is the proposition on p. 455 of [4] for flows; similar arguments yield the result in the discrete case.

PROPOSITION 5. *Let Λ be a compact ϕ-invariant set containing all limit sets $\alpha(x) \cup \omega(x)$, $x \in M$. Then a necessary and sufficient condition for Λ to be* *the maximal invariant set of an open decomposition with no cycles is that Λ have a decomposition with no cycles.*

Proof. The necessity follows from the observation (i) in Proposition 4. We will prove sufficiency in a sequence of lemmas motivated by arguments in [10], [9] and [6]. Fix $\Lambda = \cup_{i=1}^{K} \Lambda_i$ a decomposition of Λ with no cycles and a filtration ordering.

LEMMA 1. $[\text{clos } W^u\Lambda_i] \cap W^u\Lambda_j \neq \varnothing \Rightarrow [\text{clos } W^u\Lambda_i] \cap \Lambda_j \neq \varnothing$.

Proof. Since clos $[W^u\Lambda_i]$ is closed and invariant, the α-limit set of any of its points is contained in it. But the α-limit set of any point in $W^u\Lambda_j$ is contained in Λ_j.

LEMMA 2. *If $i \neq j$, then* $\text{clos } [W^u\Lambda_i] \cap \Lambda_j \neq \varnothing \Rightarrow \text{clos } [W^u\Lambda_i] \cap \{W^s\Lambda_j - \Lambda_j\} \neq \varnothing$.

Proof. Since the Λ_l are compact and disjoint, we can pick open sets $U_l \supset \Lambda_l$ with $\bar{U}_l$ disjoint and such that for $l \neq l'$ and $-1 \leqslant t, s \leqslant 1$,

$$\phi_t(\bar{U}_l) \cap \phi_s(\bar{U}_{l'}) = \varnothing.$$

For each $j = 1, \ldots, k$ consider the compact set

$$L_j = \phi_{-1}[\bar{U}_j] - U_j.$$

By assumption, there exists a sequence of points $x_n \in W^u\Lambda_i$ with $x_n \to \Lambda_j$. Assuming $x_n \in U_j$, the fact that $x_n \in W^u\Lambda_i$ implies the existence of $T_n < 0$ such that $\phi_{T_n}(x_n) \not\in U_j$. Since $x_n \to \Lambda_j$, we can pick the $T_n \to -\infty$ so that for $0 \geqslant p > T_n$, $\phi_p(x_n) \in U_j$. Thus,

$$(\forall n) \quad \phi_{T_n}(x_n) \in L_j$$

and these points have an accumulation point $x \in L_j$. By choice of the T_n, however, $\phi_t(x) \in \bar{U}_j$ $(\forall t > 0)$, because, if $x = \lim \phi_{T_n}(x_n)$, then for any given $t > 0$, $\phi_t(x) = \lim \phi_{T_n + t}(x_n)$. Hence, $\omega(x) \subset \bar{U}_j$, and so by Proposition 4, $\omega(x) \subset \Lambda_j$, that is, $x \in W^s\Lambda_j$.

On the other hand, since x is a limit of points $\phi_{T_n}(x_n)$ in the closed set $M - U_j$, we can conclude that

$$x \in W^s\Lambda_j - U_j \subset W^s\Lambda_j - \Lambda_j.$$

LEMMA 3. *If $i \neq j$,* $\text{clos } [W^u\Lambda_i] \cap \Lambda_j \neq \varnothing \Rightarrow i > j$.

Proof. Suppose $i<j$. The point x chosen in the previous lemma is an element of $W^u\Lambda_l$ for some l. Since $x \in W^s\Lambda_j \cap W^u\Lambda_l$, we must have $l>j$ ($l=j$ would give a 1-cycle). Thus, $x \in$ clos $[W^u\Lambda_i] \cap W^u\Lambda_l$, with $l>j$. Since $i<j$, $i \neq l$ and we can apply Lemma 1 to see that j in the hypothesis of Lemma 3 can be replaced by some $l>j$. Repeating this process, we get a strictly increasing sequence $l_1<l_2<\cdots$ of indices l for which clos $[W^u\Lambda_i] \cap \Lambda_l \neq \varnothing$. But this is impossible, since there are only finitely many indices altogether.

LEMMA 4. $\operatorname{clos}[W^u\Lambda_i] \subset \cup \{W^u\Lambda_j : j \leqslant i\}$.

Proof. Every $x \in \operatorname{clos}[W^u\Lambda_i]$ has its α-limit inside some Λ_j, so $x \in W^u\Lambda_j$; but then by Lemma 1, $j \leqslant i$.

LEMMA 5. *For each* $i=1,\ldots,k$, $\cup_{j\leqslant i} W^s\Lambda_j$ *is an open set containing* $\cup_{j\leqslant i} W^u\Lambda_j$.

Proof. Applying Lemma 4 to the reverse flow, we see that $\cup_{j\geqslant i+1} W^s\Lambda_j$ is closed, so $\cup_{j\leqslant i} W^s\Lambda_j = M - \cup_{j\geqslant i+1} W^s\Lambda_j$ is open. Suppose $x \in W^u\Lambda_j$, $j \leqslant i$. Then $x \in W^s\Lambda_l$ for some l. But since we have a filtration ordering, $l \leqslant j$, and so the lemma follows.

LEMMA 6. *Any compact neighborhood* Q *of* $P_i = \cup_{j\leqslant i} W^u\Lambda_i$ *contains a compact neighborhood* V_i *of* P_i *such that* $\forall t \geqslant 0$ $\phi_t[V_i] \subset V_i$.

Proof. We can assume without loss of generality that $Q \subset \cup_{j\leqslant i} W^s\Lambda_i$, so that in particular all points of $Q-P_i$ have their α-limit outside Q. It then follows that $P_i = \cap \{\phi_t[q] : t \geqslant 0\}$, and we adapt to this case an argument given by Smale in [6, Lemma 4.2]:

Given a real number $r \geqslant 0$, we let

$$A_r = \cap \{\phi_t[Q] : 0 \leqslant t \leqslant r\}$$

and note that each A_r is a compact neighborhood of P_i. We also note that the sets A_r are nested ($A_r \subset A_s$ if $r>s$) and $\cap A_r = P_i$.

We claim that for sufficiently large r, $\phi_t[A_r] \subset$ int A_1 for $0 \leqslant t \leqslant 1$. To see this, consider the sets $U_r = \cup_{0\leqslant t\leqslant 1} \{\phi_t[A_r]\}$-int A_1, which is a nested family of compact sets with empty intersection, hence $U_{r_1} \cap \cdots \cap U_{r_k} = \varnothing$ for some finite set of indices.

But then, consider any r satisfying the above, and let $0 \leqslant T \leqslant 1$. Then

$$\begin{aligned}
\phi_T[A_r] &= \phi_T[\cap \{\phi_t Q : 0 \leqslant t \leqslant r\}] \cap \operatorname{int} A_1 \\
&= \cap \{\phi_t Q : T \leqslant t \leqslant r+T\} \cap \cap \{\phi_t Q : 0 \leqslant t \leqslant 1\} \\
&\subset \cap \{\phi_t Q : 0 \leqslant t \leqslant r+T\} \subset A_r.
\end{aligned}$$

Furthermore, for any $t>0$, pick an integer $n>t$, and note that

$$\phi_t A_r = \phi_{t/n}\left[\phi_{t/n}\left[\cdots\phi_{t/n}A_r\right]\cdots\right] \subset A_r.$$

Thus, $V_i = A_r$ works.

LEMMA 7. *There is an open decomposition $W_1, \ldots, W_K$ for Λ with no cycles, with $\Lambda_i = K_i = \cap \phi_t[W_i]$.*

Proof. Noting that $W^u\Lambda_1 = \Lambda_1$, we take $W_1 = \operatorname{int} V_1$, V_1 as in the previous lemma, and such that $V_1 \cap \Lambda_i = \varnothing$ for $i>1$. To construct W_2, we start with a neighborhood U of Λ_2 whose closure is disjoint from $V_1 \cup_{i>2}\Lambda_i$. We then note that $V_1 \cup U^u$ is a neighborhood of $W^u\Lambda_1 \cup W^u\Lambda_2$ so by Lemma 6 contains a compact neighborhood V_2 of $W^u\Lambda_1 \cup W^u\Lambda_2$ with $\phi_t[V_2] \subset V_2 (\forall t \geqslant 0)$. We let $\tilde{U} = V_2 \cap U$, and avoid 1-cycles by taking $W_2 = \tilde{U}^s \cap \tilde{U}^u$. We note that by the forward-invariance of V_1 and V_2,

$$W_2 \cap W_1 = \tilde{U}^s \cap \tilde{U}^u \cap V_1 = \tilde{U}^u \cap \varnothing = \varnothing$$

and $W_2 \subset V_2$, so that $W_2^u \subset V_2$.

We can now proceed to construct W_3 by a similar process starting with a neighborhood of Λ_3 disjoint from $V_1 \cup V_2 \cup \cup_{i>3}\Lambda_i$, and so on. The construction of W_i insures no 1-cycles immediately, and the fact that $W_i \subset V_i$, together with forward-invariance of V_i, insure that $W_i^u \subset V_i$, so that $W_i^u \cap W_j^s = \varnothing$ for $j>i$.

To see that $\Lambda_i = K_i$, we recall from Proposition 4 (iii) that K_i is the set of points $\{x : \omega(x) \cup \alpha(x) \subset \Lambda_i\}$, so that if $\Lambda_i \neq K_i$, we would have a 1-cycle for the decomposition of Λ.

The last lemma is the sufficiency statement of Proposition 5, and thus concludes its proof. The reader may have noticed that the proofs of the last two lemmas have amounted to the construction of something very much like a filtration for ϕ. We are lacking certain differentiability statements—our V_i are not necessarily submanifolds-with-boundary, and we have no transversality at the boundary.

We obtain these refinements by means of Lyapunov functions.

Definition. Let K be a closed ϕ-invariant set. A Lyapunov function for (ϕ, K) is a differentiable function

$$L : M \to \mathbf{R}$$

such that

(1) *K is the set of critical points of L.*

(2) *$\dot{\phi}(L) < 0$ off K.*

PROPOSITION 6. *Suppose Λ is a compact ϕ-invariant set containing all α- and ω-limit sets, and let $W_1, \ldots, W_k$ be an open decomposition for Λ, with no cycles. Then there exists a C^∞ Lyapunov function L for $K = \cup K_i$, with each K_i contained in a different critical level of L.*

Proof. We present in detail an argument from [3]. Let

$$A_i = \bigcup_{j \leqslant i} W^u \Lambda_j \quad \text{and} \quad B_i = \bigcup_{l > i} W^s \Lambda_l.$$

Then note that by Lemma 6, A_i is a uniformly asymptotically stable set for ϕ with domain of attraction $\cup_{j \leqslant i} W^s \Lambda_j = M - B_i$; similarly, B_i is uniformly asymptotically stable for the backward flow ϕ_{-t}, with domain of attraction $M - A_i$. Thus by [7] there are C^∞ non-negative functions f_i, g_i defined respectively on $M - B_i$ and $M - A_i$, with (i) $f_i^{-1}[0] = A_i$, $g_i^{-1}\,[0] = B_i$, (ii) $Df_i = 0$ on A_i, $Dg_i = 0$ on B_i, (iii) $(d/dt)\, f_i(\phi_t(x)) < 0$, $x \not\in A_i$, $(d/dt)\, g_i(\phi_t(x)) > 0$, $x \not\in B_i$, and (iv) $\lim_{p \to B_i} f_i = \mathrm{Lim}_{p \to A_i} g_i(p) = \infty$.

Pick a number c, $0 < c < \frac{1}{2}$, such that $f_i^{-1}[0,c) \cap g^{-1}{}_i[0,c) = \varnothing$. Then the set

$$f_i^{-1}[c, \infty) \cap g_i^{-1}[c, \infty)$$

is diffeomorphic to

$$f_i^{-1}[c] \times [c, 1-c] \approx g_i^{-1}[c] \times [c, 1-c]$$

(see, for example, [8]) and we can thus find a C^∞ function h_i defined on this set, taking values in $[c, 1-c]$, which agrees with $1 - g_i$ on a neighborhood of $g_i^{-1}[c]$ and with f_i on a neighborhood of $f_i^{-1}[c]$, and which decreases along trajectories of ϕ. Then the function L_i defined by

$$L_i(p) = \begin{cases} 1 - g_i(p) & \text{if } p \in g_i^{-1}[0,c) \\ h_i(p) & \text{if } p \in f_i^{-1}[c, \infty) \cap g_i^{-1}[c, \infty) \\ f_i(p) & \text{if } p \in f_i^{-1}[0,c) \end{cases}$$

is a C^∞ function defined on M such that:

(i) $L_i(p) \in [0,1] \quad \forall p \in M$

(ii) $L_i^{-1}[0] = A_i, \quad L_i^{-1}[1] = B_i, \quad DL_i = 0$ on $A_i \cup B_i$

(iii) $\dot{\phi}_p(L_i) = d/dt\, L_i(\phi_t(p)) < 0$ for $p \not\in A_i \cup B_i$.

Taking $L=\sum_{i=1}^{k-1} L_i$, we get a C^∞ function $L:M\to[0,k-1]$ such that

(1) $L^{-1}[i]\supset K_{i+1} \qquad i=0,\dots,k-1$

(2) $DL=0$ on $K_i=\cup_{j\leqslant i} W^u\Lambda_j \cap \bigcup_{l\geqslant i} W^s\Lambda_l$

(3) $\dot{\phi}(L)<0$ off K.

Thus, L is the required Lyapunov function. Q.E.D.

We show how this yields a filtration for ϕ.

PROPOSITION 7. *If L is a C^n Lyapunov function for K, with finitely many critical values, then there is a filtration $\varnothing=M_0\subset\dots\subset M_k=M$ for ϕ, with $K=\cup K_i$ (where $K_i=\cap_t\,\phi_t[M_i-\mathit{int}\ M_{i-1}]$).*

Proof. Let $c_1<c_2<\cdots<c_k$ be the (finite number of) critical values of L. We pick $d_i\in(c_i,c_{i+1})$, a regular value, and let $M_i=L^{-1}(-\infty,d_i]$. Then clearly the maximal invariant subset of $M_i-\text{int}\ M_{i-1}$ is $K_i=L^{-1}(c_i)\cap\{\text{critical points of } L\}$, and the M_i are the desired filtration.

We can summarize the results of this section in a

THEOREM 1. *Let Λ be a compact ϕ-invariant set containing all α- and ω-limit sets. Then the following conditions are equivalent:*

(i) *There is a decomposition $\Lambda=\Lambda_1\cup\cdots\cup\Lambda_k$ of Λ with no cycles.*
(ii) *There is an open decomposition $W_1,\dots,W_k$ for Λ with no cycles and $\Lambda=K$.*
(iii) *There is a filtration $\varnothing=M_0\subset\cdots\subset M_k=M$ for ϕ, with $\Lambda=\cup_{i=1}^k K_i$.*

We note that in the case when $\Lambda=\Omega(\phi)$, Theorem 1 gives necessary and sufficient conditions for the existence of a fine filtration.

3. We turn now to the necessity statement in Theorem 2. Given a flow ϕ with no C^0 Ω-explosions, our problem is to construct a fine sequence of filtrations for ϕ. By the results of Sec. 2, it suffices to construct a sequence of open decompositions of $\Omega(\phi)$ with no cycles, whose intersection is $\Omega(\phi)$. This last condition amounts to constructing an open decomposition $W_1,\dots W_K$ of $\Omega(\phi)$ with no cycles such that $\cup W_i\subset U$, where U is some arbitrarily specified neighborhood of $\Omega(\phi)$ in M.

This construction is in two steps. In Lemma 8, we show how any open cover of $\Omega(\phi)$ naturally generates a no-cycle open decomposition. Then in Proposition 8, we show how a properly chosen cover generates an open decomposition inside U, provided there are no C^0 Ω-explosions.

The first of these steps can be formulated, as before, for any ϕ-invariant set containing all limit sets.

LEMMA 8. *Let Λ be a compact ϕ-invariant set containing all limit sets, and let $V_1,\ldots,V_l$ be (not necessarily disjoint) open subsets of M with $\Lambda \subset \cup_{i=1}^{l} V_i$. Then there exists an open decomposition $W_1,\ldots,W_K$ for Λ with no cycles, such that each V_i (possibly shrunk slightly) is contained in one of the W_j.*

Proof. Given the V_i, we let $U_1,\ldots,U_m$ be the connected components of $\cup_{i=1}^{l} V_i$. By shrinking the V_i a little, we can assume the U_i have disjoint closures. We eliminate r-cycles ($r \geqslant 2$) in this open decomposition by calling U_i and U_j equivalent if they are contained in a common cycle, and letting $\tilde{W}_1,\ldots,\tilde{W}_K$ be the unions of the equivalence classes. We then eliminate 1-cycles by enlarging the $\tilde{W}_i$, letting

$$W_i = \tilde{W}_i^s \cap \tilde{W}_i^u \supset \tilde{W}_i.$$

The resulting open decomposition clearly has no cycles and contains the (slightly shrunk) V_i's. It is in some sense the "minimum" no-cycle open decomposition containing the V_i.

We will use an argument by contradiction to prove that the construction of Lemma 8 can be carried out inside a specified neighborhood of $\Omega(\phi)$. The heart of the argument is the following proposition, a kind of specialized C^0 closing lemma. We fix a Riemann metric on M.

PROPOSITION 8. *Given a flow ϕ, an open neighborhood $\mathcal{U}$ of $\Omega(\phi)$, and $\epsilon > 0$, there exists an open cover $V_1,\ldots,V_l$ of $U(\phi)$ by subsets of $\mathcal{U}$ such that, if $W_1,\ldots,W_K$ is the corresponding no-cycle open decomposition given by Lemma 8 and $x \in W_j$, then there exists a (C^∞) flow ψ satisfying*

(i) $\|\dot{\psi} - \dot{\phi}\| < \epsilon$ *pointwise on M*
(ii) *x is periodic under ψ.*

We defer the proof of Proposition 8 for a moment to show how it yields the main result.

COROLLARY . *Given a flow ϕ with no C^0 Ω-explosions and an open neighborhood $\mathcal{U}$ of $\Omega(\phi)$, there exists an open decomposition $W_1,\ldots,W_K$ with no cycles and $\cup W_i \subset \mathcal{U}$.*

Proof of Corollary. Since ϕ has no C^0 Ω-explosions, we can pick $\epsilon > 0$ such that any flow ψ with $\|\dot{\psi} - \dot{\phi}\| < \epsilon$ pointwise on M must have $\Omega(\psi) \subset \mathcal{U}$. Thus, the open decomposition in Proposition 8 must be contained in $\mathcal{U}$, for if

$x \in W_j - \mathfrak{U}$, then there exists a flow ψ with $\|\dot{\psi}-\dot{\phi}\| < \epsilon$ but with $x \in \Omega(\psi) - \mathfrak{U}$, a contradiction. Q.E.D. (Corollary)

In the course of proving Proposition 8, we will first produce a curve which is to serve as the desired periodic orbit, and then modify the vectorfield so that the curve becomes an integral curve. We begin with a consideration of which curves allow such a modification.

Definition.

(i) *Let* $\mathfrak{Z}(\phi) = \{x : \dot{\phi}(x) = 0\}$

(ii) *Given a nonzero vector* $Y \in M_x$, *where* $x \notin \mathfrak{Z}(\phi)$, *we define the inclination of* Y *(relative to* ϕ*) to be the length of the normalized difference*

$$\sigma(Y) = \left\| \frac{1}{\|Y\|} Y - \frac{1}{\|\dot{\phi}_{(x)}\|} \dot{\phi}_{(x)} \right\|.$$

If the angle between Y *and* $\dot{\phi}_{(x)}$ *is* θ $(0 \leqslant \theta \leqslant \pi)$, *then* $\sigma(Y) = 2\,\sin(\theta/2)$. *Note that* $0 \leqslant \sigma \leqslant 2$, $\sigma = 0$ *if and only if* Y *points in the same direction as* $\dot{\phi}$, *and that* $\tilde{Y} = (\|\dot{\phi}\|/\|Y\|)\, Y$ *is a vector parallel to* Y *with* $\|\tilde{Y} - \dot{\phi}\| = \sigma(Y)\|\dot{\phi}\|$. *We define the inclination of a curve* γ *to be the inclination of its velocity vector* $\dot{\gamma}$.

LEMMA 9. *Given* $\epsilon > 0$ *and a flow* ϕ. *Suppose* γ *is a* C^1 *curve in* M *(an embedded closed interval or circle) such that at each point* x *in the image of* γ *one of the following conditions holds*:

(i) $\|\dot{\phi}_{(x)}\| < \epsilon/2$, *or*

(ii) $x \notin \mathfrak{Z}(\phi)$, *and* γ *has inclination* $\sigma < \epsilon/\|\dot{\phi}\|$ *at* x.

Then, given any neighborhood U *of the image of* γ, *there exists a flow* ψ *on* M *satisfying*:

(*a*) $\dot{\psi} = \dot{\phi}$ *off* U

(*b*) $\|\dot{\psi} - \dot{\phi}\| < \epsilon$ *on* M

(*c*) γ *is a (segment of an) integral curve of* ψ.

Proof. By reparametrizing γ, we can insure that at each point for which (i) holds, $\|\dot{\gamma}\| < \epsilon/2$, and at each point for which (ii) holds, the difference vector $\dot{\gamma} - \dot{\phi}$ has length arbitrarily close to $\sigma\|\dot{\phi}\|$, say $\|\dot{\gamma} - \dot{\phi}\| < \sigma\|\dot{\phi}\| + \delta$. Thus, for every point of γ, the length of the difference vector is less than ϵ, either because by (i) $\|\dot{\gamma} - \dot{\phi}\| \leqslant \|\dot{\gamma}\| + \|\dot{\phi}\| < \epsilon$ or because by (ii) $\|\dot{\gamma} - \dot{\phi}\| < \sigma\|\dot{\phi}\| + \delta < \epsilon$ (for δ small). Taking a tubular neighborhood N of γ inside U, we can extend the vectorfield $\dot{\gamma}-\dot{\phi}$ to N by making it constant along fibers; if γ is a closed curve, then N is a neighborhood of γ, and we multiply $\dot{\gamma}-\dot{\phi}$ by a "bump" function φ which is 1 on γ and 0 off N, to obtain a vectorfield ξ on M with $\xi=\dot{\gamma}-\dot{\phi}$ on γ. If γ is not a closed curve, we extend it beyond its endpoints, extend N to be a tubular neighborhood of the extended curve, and take φ again so that $\varphi=1$ on the original curve γ, and $\varphi=0$ off of the extended neighborhood N. Again, we end up with a vectorfield ξ on M such that $\xi=\dot{\gamma}-\dot{\phi}$ at points of γ and $\|\xi\|<\epsilon$. Then the vectorfield $\dot{\psi}=\xi+\dot{\phi}$ satisfies:

(a) $\|\dot{\psi}-\dot{\phi}\|<\epsilon$
(b) $\dot{\psi}=\xi+\dot{\phi}=\dot{\gamma}$ for points of γ.

Proof of Proposition 8. We will produce a closed curve through x which satisfies conditions (i) or (ii) of Lemma 9 at each point; the conditions will be insured by restricting our construction piecewise to well-chosen subsets of M.

Recall that a coordinate chart $\varphi: U \to \mathbf{R}^n$ is called a *flow box* for ϕ if $\varphi[U]$ is the product $\mathfrak{D} \times I$ of an $(n-1)$-dimensional disc and an interval, with $\dot{\phi}$ a constant multiple of $\partial/\partial t$—that is, ϕ flows along the "I" factor at constant speed. For definiteness, we will always take $I=(-1,1)$. A flow box will be called δ-*narrow* if any point on its left edge, $\mathfrak{D} \times \{-1\}$, can be joined to any point on its right edge, $\mathfrak{D} \times \{1\}$ by an arc with inclination everywhere $<\delta$ (relative to $\dot{\phi}$). A box is *extra* δ-*narrow* if the boxes obtained by division into thirds, $\mathfrak{D} \times (-1, -\frac{1}{3})$, $\mathfrak{D} \times (-\frac{1}{3}, \frac{1}{3})$, and $\mathfrak{D} \times (\frac{1}{3}, 1)$ are all δ-narrow. Narrowness is a bound on the ratio between the diameter of $\mathfrak{D}$ and the time of transit across U, so that any flow box can be made as narrow as desired by shrinking $\mathfrak{D}$ without changing I (i.e. the transit time).

For technical reasons, we will take a pair of coverings of $\Omega(\phi)$ by open sets. For each point $x \in \mathfrak{Z}(\phi)$, we pick a ball U_x about x on which $\|\dot{\phi}\|<\epsilon/2$; for each $x \in \Omega(\phi)-\mathfrak{Z}(\phi)$, we pick U_x an extra-δ-narrow flow box with x at its center, where δ is chosen so that $\delta\|\dot{\phi}\|<\epsilon$ everywhere on U_x. (We can simply pick $\delta<\epsilon/\sup_M\|\dot{\phi}\|$.)

By bounding the diameters of the U_x away from 0, we can find for each x a second open set, $V_x \subset U_x$, such that if $x \notin \mathfrak{Z}(\phi)$, then V_x is a δ-narrow flow box with x at its center and contained in the middle third $(\mathfrak{D} \times (-\frac{1}{3}, \frac{1}{3}))$ of U_x, and a finite collection $x_1, \ldots, x_l \in \Omega(\phi)$ such that the $V_{x_i}=V_i$ cover $\Omega(\phi)$ and so that if $V_{x_i} \cap V_{x_j} \neq \varnothing$ and $x_i, x_j \notin \mathfrak{Z}(\phi)$, then $V_{x_i} \cup V_{x_j} \subset U_{x_i} \cap U_{x_j}$.

We wish to show that the collection $\{V_1, \ldots, V_l\}$ satisfies the conclusion of Proposition 8. Letting $\{W_j\}$ be the no-cycle open decomposition of $\Omega(\phi)$ generated by the open cover $\{V_1 \cdots V_l\}$, we fix $x \in W_j$ and wish to produce a closed curve through x which will eventually become a closed orbit.

By definition, $x \in \tilde{W}_j^s \cap \tilde{W}_j^u$; tracing this back, we see that it means that x belongs to a cycle of the open cover $\{V_i\}$, in the sense that we can find a sequence of points y_α $(\alpha = 1, \dots, m)$ and times $t_\alpha > 0$ $(\alpha = 0, \dots, m)$ such that:

(i) Each y_α belongs to one of the V_i, which we call V_α,
(ii) $\phi_{t_0}(x) \in V_1$
(iii) $\phi_{t_\alpha}(y_\alpha) \in V_{\alpha+1}$ $(\alpha = 1, \dots, m-1)$
(iv) $\phi_{t_m}(y_m) = x$.

Note that if $x \in \tilde{W}_j$ for some j, then $x \in V_i$ for some i, and we can take $m = 2$, $V_\alpha = V_i$ $(\alpha = 1, 2)$ and t_1 small, so that i)–iv) are trivial to satisfy.

The curve Γ will be formed of orbit segments and curves γ_α joining them which satisfy one of the hypotheses of Lemma 9. If V_α is one of the open sets V_i $(i \leqslant r)$ on which $\|\dot{\phi}\| < \epsilon/2$, any curve γ_α joining $\phi_{t_{\alpha-1}}(y_{\alpha-1})$ to y_α and lying in V_α satisfies the first condition of Lemma 9. If V_α is a flow-box, we would like to join the first entry of the orbit of $y_{\alpha-1}$ into the left edge of V_α to the last exist of y_α from the right edge of V_α by a curve γ_α of small inclination. When the orbit of y_α enters $V_{\alpha+1}$ after leaving V_α, this is fine, but it is possible that $y_\alpha \in V_\alpha \cap V_{\alpha+1}$ and this argument could break down. However, since V_α contains points of $\Omega(\phi)$, we can find a point $w \in V_\alpha$ which leaves V_α and even U_α by the right edge (let $\phi_+(w)$ be the latter exit point) and thereafter re-enters U_α by its left edge, at $\phi_{++}(w)$, (see Fig. 1). Since $\phi_{t_{\alpha-1}}(y_{\alpha-1}) \in V_\alpha$, its forward orbit in U_α crosses the transverse disc $\mathfrak{D} \times \{\frac{1}{3}\}$ at a point $\phi_+(y_{\alpha-1})$, and similarly the backward orbit of y_α in U_α crosses the transverse disc $\mathfrak{D} \times \{-\frac{1}{3}\}$ at a point $\phi_-(y_\alpha)$. The extra narrowness allows us to join $\phi_+(y_{\alpha-1}) \in \mathfrak{D} \times \{\frac{1}{3}\}$ to $\phi_+(w) \in \mathfrak{D} \times \{1\}$, by an arc of inclination $\sigma < \delta$, then follow the orbit of $\phi_+(w)$ to $\phi_{++}(w) \in \mathfrak{D} \times \{-1\}$, then travel to $\phi_-(y_\alpha) \in \mathfrak{D} \times \{-\frac{1}{3}\}$ via an arc of low inclination, and thence to y_α itself. The (broken) arc γ_α thus described, starting from $\phi t_{\alpha-1}(y_{\alpha-1})$ and ending at y_α, satisfies condition (ii) in Lemma 9.

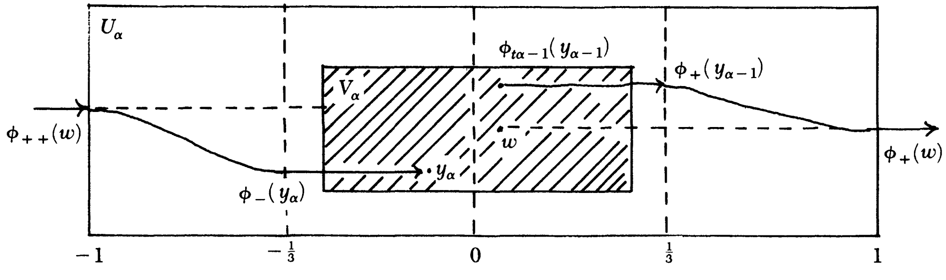

Figure 1. Constructing γ_α.

When we are all through, we have constructed a closed, C^1 curve Γ, pieced together from the arcs γ_α ($\alpha=2,\ldots,m$) and the orbit segments $[x,\phi_{t_1}(x)]$, $[y_\alpha,\phi_{t_\alpha}(y_\alpha)]$, and $[y_m,x]$. We note that Γ satisfies condition (i) or (ii) of Lemma 9. Since inclination depends continuously on a vector, any small C^1-perturbation of Γ will also satisfy the hypotheses of Lemma 9, provided it is simple. If dim $M\geqslant 3$ transversality arguments allow us to C^1-perturb Γ to a simple, closed curve satisfying (i) or (ii) at each point. If dim $M=2$, transversality does not let us eliminate intersections via a C^1-perturbation, but only allows us to conclude that there are finitely many intersections, all transversal. However, we can think of Γ as a graph, with vertices at the intersection points, and then find a simple, closed sub-graph through x. This can be done by going along Γ until we hit a point of intersection, regarding this as a crossroads and turning onto the path corresponding to the later of the two times of crossing. This yields a simple, closed, continuous curve Γ through x with nonzero tangent vector defined continuously everywhere except for possible jump discontinuities at the old crossing points. The curve can, however, be modified on a small interval about each of these points to become a C^1 simple, closed curve; since the right and left limits of $\dot{\Gamma}$ at such a point both satisfy the conditions of Lemma 9, it is possible to make sure that the modification also satisfies Lemma 9.

Thus, we end up with a closed, simple C^1 curve through x which at every point satisfies the hypotheses of Lemma 9. The conclusion of that lemma then yields a C^0 perturbation ψ of ϕ with Γ as one of its integral curves. Q.E.D. (Proposition 8)

We close this section with a few remarks concerning the space in which our perturbations are operating. In Sec. 1 we showed that if ϕ is a flow with a fine sequence of filtrations, then any flow ψ whose velocity vectorfield $\dot{\psi}$ is uniformly near $\dot{\phi}$ (in the C^0 sense) has $\Omega(\psi)$ not much bigger than $\Omega(\phi)$. The above proposition, on the other hand, when incorporated into the corollary, is a strengthened version of the opposite implication: we have shown that a flow ϕ without a fine sequence of filtrations has a C^∞ flow ψ whose velocity vectorfield $\dot{\psi}$ is C^0-near ϕ and which exhibits an Ω-explosion. If we distinguish several interpretations of the statement "ϕ has no C^0 Ω-explosions" by adding a phrase of the form "among flows of C^r vectorfields," then no C^0 Ω-explosions among flows of C^∞ vectorfields is the weaker hypothesis (*a priori*, we allow Ω-explosions for C^0-nearby flows of low differentiability) while no C^0 Ω-explosions among flows of C^0 vectorfields is the stronger conclusion. Our theorem, of course, shows these conditions to be equivalent; but as M. Hirsch has pointed out to us, the weaker hypothesis might be easier to check in practice. We summarize our results, then, in

THEOREM 2. *For ϕ a flow on a compact manifold M with velocity vectorfield $\dot{\phi}$, the following are equivalent:*

(i) *ϕ has no C^0 Ω-explosions among flows of C^0 vectorfields.*
(ii) *Given a neighborhood $\mathfrak{U}$ of $\Omega(\phi)$, there is an open decomposition for $\Omega(\phi)$ by subsets of $\mathfrak{U}$, with no cycles.*
(iii) *ϕ has a fine sequence of filtrations.*
(iv) *ϕ has no C^0 Ω-explosions among flows of C^r vectorfields for any r, $0 \leqslant r \leqslant \infty$.*

4. We close this paper with some considerations of the dimension restrictions in [5]. Because of a technicality, the main theorem is stated for dim $M \geqslant 3$. This restriction can be removed in two ways. One is to note that given a diffeomorphism $f \in$ Diff (M) on a manifold of dimension r, we can "suspend" it as the Poincaré map of a flow ϕ_f on a manifold of dimension $r+1$.

Another way of removing the restriction is to prove directly the following topological lemma, whose proof in [5] required dim $M \geqslant 3$. Since the lemma itself might be of some independent interest, we give a proof below.

LEMMA 13. *Let M be a manifold of dimension $\geqslant 2$ with distance d coming from a Riemann metric. Suppose a finite collection $\{(p_i, q_i) \in M \times M$: $i = 1, \dots k\}$ of pairs of points of M is specified, together with a small positive constant $\delta > 0$ such that:*

(i) *For each i, $d(p_i, q_i) < \delta$*
(ii) *If $i \neq j$, then $p_i \neq p_j$ and $q_i \neq q_j$.*

Then there exists $f \in$Diff (M) such that

(a) *$d(f(x), x) < 2\pi\delta$ for every $x \in M$*
(b) *$f(p_i) = q_i$ for $i = 1, \dots, k$.*

We remark that the above is well known to be false when $M = S^1$, and that S. J. Blank has an independent proof when dim $M \geqslant 2$, using a fine triangulation of the manifold.

Proof of Lemma. We consider the "suspension" of the identity, by taking $M \times S^1$ with the vectorfield $X(p, \theta) = 2\pi\partial/\partial\theta$. The induced flow is, of course, $\phi_t(p, \theta) = (p, \theta + 2\pi t)$. Its time-one map, ϕ_1, takes $M \times \{0\}$ to itself and induces the identity map there.

Given the points p_i, $q_i \in M$, we consider the points $(p_i, \pi/2)$ and $(q_i, 3\pi/2)$ in $M \times S^1$. We take, for each i, a curve $\gamma_i(t)$ in M, $(0 \leqslant t \leqslant 1)$ of constant speed, joining p_i to q_i, of length $< \delta$. Then the curve g_i in $M \times S^1$ given by

$$g_i(t) = \left(\gamma_i(t), \pi t + \frac{\pi}{2}\right) \qquad 0 \leqslant t \leqslant 1$$

has velocity vector $\dot{\gamma}_i + \frac{1}{2}X$, whose inclination is $<2\pi\delta$. We can change g_i slightly so that $\dot{g}_i(t) = \frac{1}{2}X$ at $t=0$, 1, and then use Lemma 9 to find a vectorfield Y with $\|Y-X\| < 2\pi\delta$ for which the curves g_i are segments of integral curves. But then the flow of Y takes $(p_i, \pi/2)$ to $(q_i, 3\pi/2)$, so that $(p_i, 0)$ and $(q_i, 2\pi) = (q_i, 0)$ are joined by an integral curve of Y. Hence, the time-one map of the flow of Y is a diffeomorphism of $M \times \{0\}$, (a) within distance

$$\int_0^1 \|Y-X\|\, dt \; < 2\pi\delta \text{ of the identity,}$$

and (b) taking p_i to q_i. Q.E.D.

Since the rest of [5] uses no dimension assumptions, this establishes the main result when dim $M \geqslant 2$. The case of diffeomorphisms of the circle is easy to take care of directly. There are four cases: (i) If there is at least one topologically transversal periodic point, there must be at least one other periodic point and an open decomposition separating them has no cycles; there are no $C^0\Omega$-explosions, (ii) A transitive homeomorphism (irrational rotation) has $\Omega = S^1$, so explosions are impossible and only one open decomposition is possible; (iii) an intransitive homeomorphism (Denjoy example) always has 1-cycles and always has Ω-explosions (by the Denjoy theorem); (iv) a homeomorphism for which the rotation number is rational but all periodic points are topologically degenerate, always has cycles, and can always be perturbed to irrational rotation number.

Thus, the main theorem of [5] is true for both discrete and continuous dynamical systems on compact manifolds of all dimensions.

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REFERENCES.

[1] S. Newhouse, "Hyperbolic limit sets," *Trans. Amer. Math. Sci.*, **167** (1972) pp. 125–150.
[2] ———, "Diffeomorphisms with infinitely many sinks," *Topology* **12** (1974) pp. 9–18.
[3] C. C. Pugh., and M. Shub, "The Ω-Stability theorem for flows," *Inventiones Math.*, **11** (1970), p. 150–158.

[4] M. Shub, "Stability and genericity for diffeomorphisms," *Dynamical Systems* (ed., M. Peixoto) (Proceedings of Symposium on Dynamical Systems, Salvador, Bahia, Brazil, 1971) Academic Press, pp. 493–514.

[5] ———, and S. Smale, "Beyond hyperbolicity," *Annals of Math.*, **96** (1972), pp. 587–591.

[6] S. Smale, "The Ω-stability theorem," *Global Analysis, Proc. Symp. Pure Math.*, **14** (Providence: A.M.S., 1970), pp. 289–299.

[7] F. W. Wilson, "Smoothing derivatives of functions and applications," *Trans. Amer. Math. Soc.*, **139** (1969), pp. 413–428.

[8] ———, "The structure of the level surfaces of a Lyapunov function," *J. Diff. Eq.*, **3** (1967), pp. 323–329.

[9] J. Palis, "On Morse-Smale dynamical systems," *Topology* **4**, (1969), pp. 385–404.

[10] S. Smale, "Morse inequalities for a dynamical system," *Bull.* Amer. Math. Soc., **66** (1960), pp. 43–49.

[11] ———, "Differentiable dynamical systems," Bull. Amer. Math. Soc., vol. 73, (1967), pp. 747–817.

LINKED CITATIONS

References

[7] **Smoothing Derivatives of Functions and Applications**
F. Wesley Wilson, Jr.
Transactions of the American Mathematical Society, Vol. 139. (May, 1969), pp. 413-428.
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http://links.jstor.org/sici?sici=0002-9947%28196905%29139%3C413%3ASDOFAA%3E2.0.CO%3B2-C