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ENTROPY AND STABILITY

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WE PROVE two theorems, the basic one giving an inequality, the second an equality.

THEOREM 1. *Suppose $f: M^n \to M^n$ is a diffeomorphism satisfying Axiom A and the no cycle condition. Then $h(f) \geq \log s(f_*)$.*

Here $h(f)$ is the topological entropy of f and $s(f_*)$ is the spectral radius of $f_*: H_*(M; \mathbb{R}) \to H_*(M; \mathbb{R})$. That is, $s(f_*) = \max |\lambda|$ where the max is taken over all eigenvalues of $f_{*i}: H_i(M; \mathbb{R}) \to H_i(M; \mathbb{R})$ and all dimensions i.

This theorem is already known in the Morse–Smale case[14, 16] and more generally in the Axiom A and no cycle case with the added hypothesis that the non-wandering set is zero dimensional[4]. Also, Manning proves that $h(f) \geq \log s(f_{*1})$, where f_{*1} is the map in 1-dimensional homology and f is a continuous map of quite general spaces. For further discussion of entropy see [10] and [11].

This theorem was conjectured in [15] and [16] and has its most striking application in a significant sharpening of the Lefschetz trace formula when the periodic points of f are counted asymptotically. This follows from the work of Bowen[1] where the following was first proved:

THEOREM [Bowen]. *If $f: M \to M$ is a diffeomorphism satisfying Axiom A then $h(f) =$ $\limsup 1/n \log N_n(f)$.*

Here $N_n(f)$ is the number of periodic points of f of period n, that is, the number of fixed points of f^n. Since

$$\log s(f_*) = \limsup \frac{1}{n} \log \left|\sum \operatorname{trace} f_{*i}^n\right|$$

we have

COROLLARY A. *If $f: M \to M$ is a diffeomorphism satisfying Axiom A and the no cycle condition, then:*

$$(I)\ \limsup \frac{1}{n} \log N_n(f) \geq \limsup \frac{1}{n} \log \left|\sum \operatorname{trace} f_{*i}^n\right|.$$

For a diffeomorphism which has only transversal periodic points the Lefschetz trace formula implies that $N_n(f) \geq |\Sigma(-1)^i \operatorname{trace} f_{*i}^n|$ so that

$$(II)\ \limsup \frac{1}{n} \log N_n(f) \geq \limsup \frac{1}{n} \log \left|\sum (-1)^i \operatorname{trace} f_{*i}^n\right|.$$

The difference between I and II lies in the alternation of signs. Indeed, it is easily seen that the right hand side of II may be identically zero while the right hand side of I is not and hence predicts an infinity of periodic points growing exponentially in number with n. An example of this phenomenon may be found in [15].

The Axiom A and no cycle diffeomorphisms were defined by Smale and proved to be Ω-stable by him[18, 19]. In particular if $f: M^n \to M^n$ satisfies Axiom A and the no cycle condition, then there is a neighborhood of f, $U_f \subset \operatorname{Diff}^1(M)$ with the property that for any $g \in U_f$, $N_n(g) = N_n(f)$ for all n. The Axiom A and no cycle diffeomorphisms are the only known diffeomorphisms with this property.

THEOREM 2a. *For $f_{*i}: H_i(M_j, M_{j-1}; \mathbb{R}) \circlearrowleft$*

$$h(f|\Lambda_j) \begin{cases} > \log s(f_{*i}); & i \neq u \\ \geq \log s(f_{*u}); & \\ = \log s(\tilde{f}_{*u}); & \end{cases}$$

where the coefficients of $\tilde{f}_{*u}$ are perhaps twisted.

As twisted coefficients are a bit obscure, we state the following untwisted theorem:

THEOREM 2b. *With the additional assumption that the bundle* E^u *on* Λ_j *extends semi-invariantly to an oriented bundle and* df *preserves (or reverses) an orientation, then* $h(f|\Lambda_j) = \log s(f_{*u})$.

Here Λ_j is a basic set for f and M_j, M_{j-1} are the two elements of a filtration bracketting Λ_j. See also paragraphs below for discussion of $\tilde{X}$, $\tilde{A}$ and $\tilde{f}$. The first two parts of Theorem 2a, even in the weaker form $h(f|\Lambda_j) \geq \log s(f_*)$ prove Theorem 1 by induction on j via exact sequences. This is a well known process (see e.g. [4] concerning 0-dimensional non-wandering sets).

The strict inequality of the first part of Theorem 2 is quite important as it allows one to neglect all but one dimension in the asymptotic version of the Lefschetz trace formula and thus avoid the cancellation alluded to above. This, in a relative version, together with the computation of $N_n(f)$ as given in [5], [9] proves the third part of Theorem 2.

We were convinced of the equality in Theorem 2 because of Plante's "asymptotic cycles"[12]. We have not pursued this approach for two reasons. On the one hand, the work of Ruelle and Sullivan[13] on "geometric currents" preempted us, and on the other hand, we realized that we need not construct these classes geometrically. The inequalities of Theorem 2 already imply the existence of such classes as well as other algebraic facts.

Let Λ_j, M_j, M_{j-1} be as above and define $X = f^m(M_j)$, $A = X \cap f^{-m}(M_{j-1})$ for m suitably large. Then there is a "relative double cover" $(\tilde{X}, \tilde{A})$ of (X, A) and a lifting $\tilde{f}$ of f to $(\tilde{X}, \tilde{A})$ and we have

THEOREM 2′. *There is a real homology class* $C_u \in H_u(\tilde{X}, \tilde{A})$ *such that* $\tilde{f}_* C_u = \lambda C_u$ *where* $\log \lambda = h(f|\Lambda_j)$.

Remark. Under the hypothesis of Theorem 2b, this class actually lies in $H_n(M_j, M_{j-1}; \mathbb{R})$ and $f_* C_u = \pm\lambda C_u$ depending upon whether df preserves or reverses the orientation of the extended bundle E^u.

For sake of definiteness, we state the conclusion of these results in a special case as the

COROLLARY B. *If* $f: M \to M$ *is an Anosov diffeomorphism such that*

(a) M *is connected*;

(b) $\Omega(f) = M$;

(c) M *and* E^u *are oriented and* f *preserves these orientations, then there are homology classes* $C_u \in H_u(M)$, $C_s \in H_s(M)$ *such that*

(1) $f_* C_u = \lambda C_u$ *where* $\log \lambda = h(f) \neq 0$;

(2) $f_* C_s = (1/\lambda) C_s$ *and* $C_u \cap C_s = 1$;

(3) *if* μ *is an eigenvalue corresponding to neither* C_u *nor* C_s, *then*

$$1/\lambda < |\mu| < \lambda.$$

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THEOREM. *Suppose* $f: M^n \to M^n$ *is a diffeomorphism satisfying Axiom A and the no cycle condition. Then* $h(f) \geq \log s(f_*)$.

Remark. It suffices to prove the theorem with f replaced by any power, f^α of f.

Proof. This is just additivity of h and s together with functorality of f^*, i.e. that $(f^\alpha)_* = (f_*)^\alpha$.

The proof of the theorem is in several steps, the first rather easy, and fairly well known:

LEMMA 1. *We may suppose* M^n *is orientable. That is, let* $\tilde{M}$ *be the orientable double cover of* M *(in case* M *is not) and* $\tilde{f}: \tilde{M} \to \tilde{M}$ *a diffeomorphism covering* f. *Then* $h(\tilde{f}) = h(f)$ *and* $s(\tilde{f}_{*i}) \geq s(f_{*i})$.

Proof. That $h(\tilde{f}) = h(f)$ is basic to entropy theory and well known[3]. The homological statement follows from the diagram

$$\begin{tikzcd} H_i(\tilde{M}) \arrow[r, "\tilde{f}_{*i}"] \arrow[d, shift right, "\pi_*"'] & H_i(\tilde{M}) \arrow[d, shift right, "\pi_*"'] \\ H_i(M) \arrow[r, "f_{*i}"'] \arrow[u, shift right, "T_*"'] & H_i(M) \arrow[u, shift right, "T_*"'] \end{tikzcd}$$

in which $\pi: \tilde{M} \to M$ is the projection and T_* is the transfer map. T_* is induced from the chain map T_* which assigns to each simplex σ in $C_*(M)$ the chain $\sigma_1+\sigma_2$, where σ_1 and σ_2 are the two simplexes which cover σ. Then it follows that $\pi_*T_*=2$ id, so that if $f_{*i}(x)=\lambda x$, then $\tilde{f}_{*i}(T_*x)=\lambda T_*x$, and $\pi_*T_*x=2x$, so that $T_*x\neq 0$ if $x\neq 0$. Thus each eigenvalue of f_{*i} is also one of $\tilde{f}_{*i}$, so that $s(f_{*i})\le s(\tilde{f}_{*i})$, which completes the proof of Lemma 1.

The rest of the proof proceeds as many arguments by induction on a filtration[18],

$$\phi = M_0 \subset M_1 \subset \cdots \subset M_t = M, \text{ where}$$

(a) $fM_j \subset M_j$;

(b) $\Omega[f|(M_j-M_{j-1})]=\Lambda_j$, a basic set[18].

Now fix a j and let u and s stand for the dimensions of the fibers of E^u and E^s, where E^u, E^s give the hyperbolic structure on Λ_j. Then the principal new result is the

PROPOSITION A. *$h(f|\Lambda_j)\ge \log s(f_{*i})$, where f_{*i} is the map induced by f on $H_i(M_j, M_{j-1}; \mathbb{R})$. Here the inequality is strict, except in case $i=u$.*

Proof. As Λ_j will be fixed below, we drop the subscript and refer to Λ. Define X, A as in the introduction, by $X=f^\beta(M_j)$, $A=X\cap f^{-\beta}(M_{j-1})$. Then

LEMMA 2. *It suffices to prove our proposition with f_* replaced with $(f|X)_*\colon H_*(X,A)\to H_*(X,A)$.*

Proof. If $m\ge\beta$, then we have the commutative diagram

$$\begin{tikzcd}
(X,A) \arrow[r, "f_1{}^m"] \arrow[d, "i"] & (X,A) \arrow[d, "i"] \\
(M_j,A) \arrow[r, "f_3{}^m"'] \arrow[ur, "f_2{}^m"] & (M_j,A)
\end{tikzcd}$$

in which f_1, f_2, f_3 are all f, but considered with different range and target spaces. Thus in homology we have

$$\begin{tikzcd}
H_*(X,A) \arrow[r, "f^m_{1*}"] \arrow[d, "i_*"] & H_*(X,A) \arrow[d, "i_*"] \\
H_*(M_j,A) \arrow[r, "f^m_{3*}"'] \arrow[ur, "f^m_{2*}"] & H_*(X,A)
\end{tikzcd}$$

We claim f_{1*} and f_{3*} have the same non-zero eigenvalues, which one can easily check directly; this is also proved in [21], as f_{1*} and f_{3*} are "shift equivalent" by our diagrams, above.

In just the same way one uses the diagram

$$\begin{tikzcd}
(M_j,A) \arrow[r, "f_3{}^m"] \arrow[dr, "f_4{}^m"'] & (M_j,A) \\
(M_j,M_{j-1}) \arrow[r, "f_5{}^m"'] \arrow[u, "i"] & (M_j,M_{j-1}) \arrow[u, "i"']
\end{tikzcd}$$

to show that f_{3*} has the same non-zero eigenvalues as f_{5*}. Thus for large values of n,

$$\sum \text{trace}\, f^{\,n}_{*i} = \sum \text{trace}\,(f|X)^n_{*i}$$

which proves Lemma 2.

We return to the proof of Proposition A, except we work with the (X,A), where β is chosen sufficiently large so that our considerations are reduced to a small neighborhood of Λ, which we describe below.

We now consider a Markov partition of Λ[2] and a volume form ω_u for the unstable manifold of a fixed point p which we assume (by taking powers of f if necessary) lies in the interior of some element of the Markov partition. Arrange the notation so that this is the first element of the Markov partition. We also assume that $W^u(p)$ is dense in Λ by taking a power of f if necessary.

Let $B_0\subset X-A^0$ be a ball in the unstable manifold $W^u(p)$, around p, which we assume to be a fixed point, lying in the interior of the first element of a Markov partition. Let T be the intersection matrix of the Markov partition and define

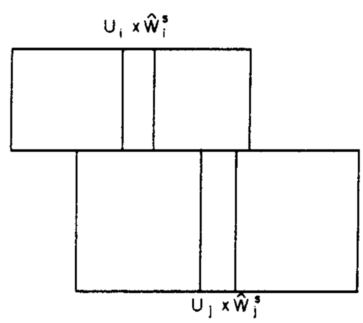
$$B_n = f^n(B_0) \cap (X - A^0).$$

LEMMA 3. *Let λ be the large eigenvalue of T and $\epsilon > 0$. Then there are constants $C_1, C_2 > 0$ such that*

$$C_1(\lambda - \epsilon)^n \leq \text{volume } B_n \leq C_2(\lambda + \epsilon)^n.$$

Proof. We prove this by estimating the number of times B_n passes through each element W_i of the Markov partition. Our problem is that the various W_i intersect, causing a possible over count, and another that the W_i are a neighborhood in Λ and not (necessarily) in $X - A^0$, so that we have trouble deciding which parts of B_m "belong" to which elements of the Markov partition.

The elements of the Markov partition are products, $W_i = \hat{W}_i^u \times \hat{W}_i^s$, where $\hat{W}_i^u \subset W^u(x) \cap \Lambda$ and $\hat{W}_i^s = W^s(x') \cap \Lambda$, x, x' some points in Λ. Choose a closed neighborhood $U_i \subset \hat{W}_i^u$ so that $(U_i \times a) \cap (U_j \times b) = \phi$ for $i \neq j$, where a and b refer to variables in $\hat{W}_i^s$ and $\hat{W}_j^s$. This is possible by staggering.



Then there is an $\epsilon_1 > 0$, such that no disk in $W^u(p)$ of diameter $< \epsilon_1$ contains two plaques $U_i \times a$, $U_j \times b$, where *either* $i \neq j$ or $a \neq b$. We assume also that $d(\Lambda, A) < \epsilon_1$. Thus, there is an integer α such that if $D \subset W^u(p)$ is a disk centered at p and $x \in D$, then $f^\alpha(D)$ contains an ϵ_1 neighborhood of x and this in turn lies in $X - A$.

Thus we have the estimates

$$C_1 T^n \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot (1, 1, \ldots, 1) \leq \text{vol } B_n \leq C_2 T^{n+\alpha} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot (1, 1, \ldots, 1).$$

As $\dfrac{T^n \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}{\lambda^n} \to v$, the positive eigenvector associated with λ, one has that, as $n \to \infty$,

$$\frac{T^{n+1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot (1, 1, \ldots, 1)}{T^n \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot (1, 1, \ldots, 1)} \to \frac{\lambda^{n+1} v \cdot (1, \ldots, 1)}{\lambda^n v \cdot (1, \ldots, 1)} = \lambda.$$

Hence

$$C_1(\lambda - \epsilon)^n \leq \operatorname{vol} B_n \leq C_2'(\lambda + \epsilon)^n,$$

where $C_2' = C_2(\lambda + \epsilon)^\alpha$. This proves Lemma 3.

Next, there are semi-invariant extensions of the bundles E^u, E^s [6; p. 131] to a neighborhood of Λ, which are expanded and contracted in some adapted metric. We know that $W^u(p)$ is dense in Λ. Thus the τ-tubular neighborhood $V_\tau(W^u(p))$ of $W^u(p)$, for τ fixed and small, contains a neighborhood of Λ, and hence $X - A^0$, for a suitable choice of β, taken so that the extended bundles E^u, E^s are also defined on $X - A^0$. Here and below we take the fibers of normal bundles to be close to E^s. See [8; §§6, 7] for this type of discussion.

Now let η be a differential form of dimension u in $C^\infty(X, A; \mathbf{R})$, and D_0 a disk transverse to E^s with a bounded angle, lying in $X - A^0$, and

$$D_n = f^n(D_0) \cap (X - A^0).$$

Then,

LEMMA 4. *Given $\epsilon > 0$, there is a constant $C > 0$ such that $|\int_{D_h} \eta| \leq C(\lambda + \epsilon)^n$, all $h > 0$.*

The proof uses the

LEMMA 4.1. *We can define a "projection" $\pi_h : D_h \to B_{h+\beta}$ in the tubular neighborhood $\nu_\tau(B_{h+\beta})$ for some fixed β and all large h. Furthermore, $f^h(D) \cap (X - A^0) \to \pi_h(D_h)$ in the C^r sense.*

Proof. This is essentially contained in [7], [8], but we indicate how part of the proof goes.

Let $\nu'_\tau(W^u(p))$ be the τ-tubular neighborhood of $W^u(p)$; this contains in turn

$$\nu'_\tau(W^u(p)) \supset \bigcup_{x \in \Lambda \cap W^u(p)} \nu_\tau(W_d^u(x)) \equiv \nu_\tau,$$

where $\nu_\tau(W_d^u(x))$ is the τ-tubular neighborhood of the d-ball about x in $W^u(x)$. Also ν_τ contains a neighborhood of Λ. To see this, note that $\Lambda \cap W^u(p)$ is dense in Λ so that the uniformly large $\tau \times d$ boxes contain an ϵ-neighborhood of Λ for some ϵ. See [8; §§6, 7] for this type of discussion.

There is the "semi-invariant disk family" $\tilde{W}_\rho^s(x)$, for each $x \in W_\delta^u(\Lambda) = \bigcup_{y \in \Lambda} W_\delta^u(y)$. Let $F = \overline{f(W_\delta^u(\Lambda)) - W_\delta^u(\Lambda)}$. ($F$ is a "proper fundamental domain.") If ρ is small enough, then

$$\bigcup_{x \in W_\delta^u(\Lambda)} \tilde{W}_\rho^s(x) - \bigcup_{x \in F} \tilde{W}_\rho^s(x)$$

is a neighborhood of Λ. But in fact, if we let

$$C = \bigcup_{x \in \Lambda \cap W^u(p)} W_\delta^u(x),$$

then

$$U(\delta, \rho) = \bigcup_{w \in C} \tilde{W}_\rho^s(w) - \bigcup_{w \in F \cap C} \tilde{W}_\rho^s(w)$$

is also a neighborhood of Λ.

To see this last, note that the proof of [6; (4.1)] gives a uniform estimate and $W^u(p) \cap \Lambda$ is dense in Λ by our hypotheses. Furthermore we choose ρ and δ much smaller than τ and d, so that in particular, $U(\delta, \rho) \subset \nu_\tau$, defined above. In fact, for any $x \in \Lambda \cap W^u(p)$,

$$\tilde{B}(x) \equiv \bigcup_{w \in W_\delta^u(x)} \tilde{W}_\rho^s(w) \subset \nu_\tau(W_d^u(x)) \subset \nu_\tau.$$

Thus, to define the "projection" on $U(\delta, \rho)$ along the fibers of ν_τ, it suffices to do this for each box $\tilde{B}(x)$, if only finitely many boxes are used. So we define a partial projection $\pi_x : \tilde{B}(x) \to W_\delta^u(x)$ just to send $z \in \nu_\tau$ fiber of $\nu_\tau(W_\delta^u(x))$ to its base point.

Now given $z \in U(\delta, \rho)$, say $z \in \tilde{W}_\rho^s(w)$ and $w \in W_\delta^u(x)$, $x \in \Lambda \cap W^u(\rho)$. Define $\pi_0(z) = \pi_x(z)$. If $f^h(z) \in U(\delta, \rho)$ for $0 \le h \le k$, then $f^h(z) \in \tilde{W}_\rho^s(f^k w)$ and $f^k(w) \in W_\delta^u(x_h)$, where $x_h \in \Lambda \cap W^u(\rho)$. Moreover, the distance between $f^h(z)$ and $f^h(w)$, measured along $\tilde{W}_\rho^s(f^k w)$ is less than $\rho\theta^h$, where $0 < \theta < 1$ is the contraction along the $\tilde{W}_\rho^s$'s.

If, ρ, δ are chosen small enough, then

(1) $f^h(z) \in \nu_\tau W_d^u(x_h)$;
(2) $q_h = \pi(f^h z) \in W_{2\delta}^u(x_h)$;
(3) $f(q_{h-1}) \in W_{2\delta}^u(x_h)$;
(4) $f(W_d^u(x_h)) \supset W_d^u(x_h)_{+1}$;

and by induction

(5) $f^h(W_d^u(x_0)) \supset W_d^u(x_h)$.

Here (1) is obvious and implies (2). Then (3) and (4) follow easily, geometrically.

We return to the proof of (4.1) and note first that there is a finite set of the $\tilde{B}_x$'s which cover D_0. Thus we write D_0 as a finite union

$$D_0 = E_1 \cup \cdots \cup E_t,\ E_i \cap E_j \text{ a thin set,}$$

where each $E_l \subset \tilde{B}(x_l)$, $l = 1, \ldots, t$.

Now take β sufficiently large that $B_\beta \supset W_\delta^u(x_l)$, for $l = 1, \ldots, t$. We define

$$\pi_h : D_h \to B_{h+\beta}, \text{ as follows:}$$

$$\text{if } z \in D_0, \text{ say } z \in E_l, \text{ then } f^h(z) \in \nu_\tau W_d^u(x_{l,h}),$$

where $x_{l,h} = (x_l)_h$. But

$$B_{h+\beta} \supset f^h(W_d^u(x_l)) \supset W_d^u(x_{l,h})$$

so that it is OK to set

$$\pi_h(f^h z) = \text{proj of } f^h z \text{ in } \nu_\tau W_d^u(x_{l,h})$$

as then $\pi_h(f^h z) \subset B_{h+\beta}$. It follows that

$$\pi_h | f^h(E_l) \cap (X - A^0) \text{ is } C',$$

just because the bundle ν_τ is.

That $f^n(D) \cap (X - A^0)$ converges to $\pi_n(D_n)$ is a familiar argument, used in the proof of the stable manifold theorem[7] and elsewhere[8]. The crucial fact is that we can use the semi-invariant disk family to show that π_h moves points by no more than $\tau\theta^h$ where $0 < \theta < 1$ is the contraction along the $\tilde{W}_\rho^s$'s. Since the tangent planes to $W^u(\rho) \cap U(\rho, \delta)$ lie in a continuous bundle E^u defined on a compact neighborhood, the result follows.

For large h, $f^h(E_l)$ is transverse to the smooth bundle ν_τ, which gives convergence in the C^1-sense, as in [7] and [8].

We continue with the proof of Lemma 4. Using Lemma 4.1, there are $\epsilon_h > 0$, $\epsilon_h \to 0$ as $h \to \infty$, such that (recall $\int_K$ means $\int_{K \cap (X - A^0)}$)

$$\left| \int_{f^h(D)} \eta \right| \le \left| \int_{\pi_h(D)} \eta \right| + \epsilon_h \operatorname{vol} \pi_h(D),$$

so that

$$\left|\int_{f^h(D)} \eta\right| \le (C+\epsilon_h)\operatorname{vol}(\pi_h(D))$$
$$\le (C+\epsilon_h)\operatorname{vol}(B_{h+\beta})$$
$$\le (C+\epsilon_h)(\lambda+\epsilon)^\beta(\lambda+\epsilon)^h$$
$$\le C'(\lambda+\epsilon)^h.$$

This completes the proof of Lemma 4.

Now suppose we are given an eigenvalue $\sigma \to \gamma\sigma$ in $H_u(X, A; \mathbf{R})$ and suppose first that γ is real. Then there is a singular cycle $\Sigma\, \gamma_i\sigma_i$ representing σ such that the σ_i are all transverse to E^s with bounded angle. Let η be a differential form dual to σ, i.e., a closed form such that $\int_{\Sigma \gamma_i\sigma_i} \eta = 1$ and $\int_\tau \eta = 0$ for all other singular τ's filling out a basis of $H_u(M)$. Note that these sums are all finite. Here, we are integrating over a chain having its support in $X - A_0$. But below this will not be the case; $\int_K \eta$ means $\int_{K\cap(X-A^0)} \eta$.

Computing,

$$\left|\int_{f^n(\Sigma\gamma_i\sigma_i)} \eta\right| = \left|\sum \gamma_i \int_{f^n(\sigma_i)} \eta\right| \le C_1(\lambda+\epsilon)^n, \text{ by Lemma 4.}$$

But,

$$\left|\int_{f^n(\Sigma\gamma_i\sigma_i)} \eta\right| = \left|\int_{\Sigma\gamma_i\sigma_i} f^{*n}\gamma\right| = |\gamma|^n.$$

Thus $|\gamma|^n \le C_1(\lambda+\epsilon)^n$; that is, for each $\epsilon > 0$, there is a $C_1 = C_1(\epsilon)$ such that $|\gamma|^n \le C_1(\lambda+\epsilon)^n$. Thus $|\gamma| \le \lambda + \epsilon$ for each ϵ so that $|\gamma| \le \lambda$. This proves the following in case γ is real:

LEMMA 5. *If γ is an eigenvalue of f^*: $H_u(X, A) \to H_u(X, A)$, then $|\gamma| \le \lambda$.*

The complex case. Let γ be a complex eigenvalue, and $[\sigma]$ be a corresponding eigenvector. We think of $[\sigma]$ as being a real vector in a two dimensional vector space V and write $f_*[\sigma] = |\gamma|\cdot R([\sigma])$, where R is a rotation of V. Let η be a corresponding real form, dual to $[\sigma]$, and let σ be a singular cycle representing $[\sigma]$.

Choose a sequence $n_i \to \infty$ such that $R^{n_i} \to$ the identity on V. Then

$$\left|\int_{f^{n_i}\sigma} \eta\right| = \left|\int_{f_*^{n_i}[\sigma]} \eta\right| = \left|\int_{|\gamma|^{n_i}R^{n_i}[\sigma]} \eta\right|$$
$$= |\gamma|^{n_i}\left|\int_{R^{n_i}[\sigma]} \eta\right| \ge C|\gamma|^{n_i} \text{ for large } i.$$

Thus $|\gamma| \le \lambda$ as in the real case.

Next, suppose we are given a real eigenvalue γ of f_{*l}: $H_l(X, A; \mathbf{R}) \to H_l(X, A; \mathbf{R})$ where $l < u$. As before, we let ξ be an l-dimensional cycle representing the homology class of the γ-eigenvector. Let η be a closed differential form dual to ξ. We wish to estimate $\int_{f_*^n\xi} \eta$, for n large.

We think of ξ as a geometric complex K of dimension l, with a real coefficient a_i associated with each l-simplex σ_i. Consider the cartesian product, $K \times D$, D a $(u-l)$-disk.

For any map $H: K \times D \to M$, so that $H|K \times 0$ represents ξ, we have that

$$|\gamma|^n = \left|\int_D \left(\int_{\xi_t} H^*f^{*n}\eta\right) dt\right|$$

where $\xi_t = H|K \times t$, $t \in D$, weighted with the coefficients a_i on $\sigma_i \times t$. Here D has $(u-l)$-volume $= 1$. Then

$$|\gamma|^n \le C_1 \int_{K\times D} \|H^*f^{*n}\eta\|.$$

Note that we have absorbed the coefficients a_i into the constant C_1.

Now H is chosen to be transverse to E^s with bounded angle, and to be a diffeomorphism on each closed u-simplex D_i^u for some simplicial subdivision of $K \times D$. Then

$$|\gamma|^n \leq C_1 \sum \int_{D_i^u} \|H^* f^{*n} \eta\| \leq C_2 \sum \int_{D_i^u} \|f^{*n}\eta\| \leq C_3 \sum \int_{D_i^u} \alpha^n \|f^{*n}\omega\|,$$

where ω is the volume form for the E^u bundle, and $\alpha < 1$. Here ω is only C^0 as a form, but we do not use any differentiation. Further, ω is defined up to sign only, so that all further integration is taken in a positive sense. α can be taken less than 1 because in the adapted metric, f uniformly stretches every vector in E^u. Consequently the volume of any u-dimensional parallelopiped in the bundle E^u grows faster than any l-dimensional parallelopiped by a factor of δ^{u-l}, where $\delta > 1$ is the minimum expansion on E^u.

Hence

$$|\gamma|^n \leq C_4 \alpha^n \sum \int_{D_i^u} \|H^* f^{*n} \omega\|,$$

as H^* is a diffeomorphism on each D_i^u. Thus

$$|\gamma|^n \leq C_4 \alpha^n \sum \int_{|D_i^u|} H^* f^{*n} \omega.$$

Now we are back in the case of estimating the growth of u-forms over disks, so that

$$|\gamma|^n \leq C_5 \alpha^n \operatorname{vol} B_{n+j} \leq C_6 \alpha'^n \lambda^n,$$

where $\alpha < \alpha' < 1$. Hence $|\gamma| < \lambda$.

The case where γ is complex is treated just as before. Thus

LEMMA 6. *If γ is an eigenvalue of $f_{*l}: H_l(X, A; \mathbf{R}) \circlearrowleft$ for $l < u$, then $|\gamma| < \lambda$.*

Proof of Proposition A. If we consider the dual filtration of f^{-1} given by

$$\phi = \overline{M - M_t} \subset \overline{M - M_{t-1}} \subset \cdots \subset \overline{M - M_0} = M$$

our proof up to now applies, and shows that for any eigenvalue γ of

$$f_{*i}^{-1}: H_i(\overline{M - M_{j-1}}, \overline{M - M_j}) \circlearrowleft$$

$$|\gamma| \leq h(f^{-1}|\Lambda) = h(f|\Lambda) = \lambda,\ i \leq s$$

with strict inequality holding except (possibly) when $i = s$. But as the coefficients are real,

$$f_i^{-1*}: H^i(\overline{M - M_{j-1}}, \overline{M - M_j}) \circlearrowleft$$

has the same property for $i \leq s$. But now by duality (see e.g. [4; p. 64]) it follows that

$$f_{n-i*}: H_{n-i}(M_j, M_{j-1}) \circlearrowleft$$

has this property for $i \leq s$, and therefore so does f_{*i}, for $i \geq u$ because $u + s = n$. This completes the proof of Proposition A.

Hence, using the homology sequence of the pair (M_j, M_{j-1}), it follows by induction on j and Proposition A, that $h(f|M_j) \geq \log s(f^*|M_j)$ for each j; see [4, p. 67]. For $M_j = M$, the theorem follows.

§2. ENTROPY AS AN EIGENVALUE

We begin with a proof of Theorem 2b. Here, the Lefschetz trace formula determines N_n [18] as

$$N_n = (-1)^n \Delta^n \sum (-1)^i \operatorname{trace} f_{*i}^n$$

where we use f_{*i} to mean $f_{*i}: H_i(M_j, M_{j-1}; \mathrm{R})\circlearrowright$. Therefore, $h(f|\Lambda_j) = \limsup (1/n) \log (-1)^n \Delta^n \Sigma (-1)^i \operatorname{trace} f_{*i}^n$ and the only map on the right which can possibly contribute an eigenvalue which is large enough is f_{*u}. This proves Theorem 2b.

To prove Theorem 2a we use the auxiliary space introduced by Guckenheimer[5]. For the sake of completeness and to include a few more details, we include a description of this space in the appendix.

LEMMA (Guckenheimer). *There is a relative manifold $(\tilde{X}, \tilde{A})$ and a relative double cover $\pi: (\tilde{X}, \tilde{A}) \to (X, A)$ such that*

(1) *(X, A) is as in the proof of Proposition A.*

(2) *$(\tilde{X}, \tilde{A})$ has the same (relative) structure as a manifold with basic set satisfying Axiom A.*

(3) *There is a map $\tilde{f}: (\tilde{X}, \tilde{A}) \to (\tilde{X}, \tilde{A})$ covering f.*

(4) $N_n(\tilde{f}) = (-1)^u \Sigma (-1)^i \operatorname{trace} \tilde{f}_{*i}^n$.

Note that the local structure of (X, A) (e.g. smoothness, stable manifolds, etc.) pulls back to $(\tilde{X}, \tilde{A})$. Now Proposition A applies to $\tilde{f}$ so that

$$h(f) = h(\tilde{f}) \geq \log s(\tilde{f}_{*i})$$

where strict inequality holds except (possibly) when $i = u$. Thus

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} N_n(\tilde{f}_*) = \log s(\tilde{f}_{*u}).$$

But $\tilde{f}_*$ is the same as f_* with coefficients in the orientation sheaf of $\tilde{X}$. This proves Theorem 2a.

Theorem 2′ is proved by comparing two computations of the zeta function of $\tilde{f}: (\tilde{X}, \tilde{A}) \to (\tilde{X}, \tilde{A})$, that due to Guckenheimer (above) and that due to Manning. Manning proved[9] that

$$(M)\begin{cases} \zeta(f|\Lambda_j) = \dfrac{p(t)}{q(t)\det(I - tT)}; \\[2ex] |\mu| > \dfrac{1}{\lambda}, \ \mu \text{ any zero } p \text{ or } q \end{cases}$$

where T is a positive integral matrix and $\log \lambda$ is the entropy of $f|\Lambda_j$. Guckenheimer's computation together with our Proposition A, gives

$$(G \,\&\, A)\begin{cases} \zeta(f|\Lambda_j) = \dfrac{\tilde{p}(t)}{\tilde{q}(t)\det(I - t\tilde{f}_{*u})}; \\[2ex] |\nu| > \dfrac{1}{\lambda}, \ \nu \text{ any zero of } \tilde{p} \text{ or } \tilde{q}. \end{cases}$$

The second half of this last follows from Proposition A because each of $\tilde{p}$ and $\tilde{q}$ is a product of terms of the form $\det(I - t\tilde{f}_{*j}) j \neq u$. The Perron–Frobenius theorem says that the zero's of $\det(I - tT)$ include $1/\lambda\omega$ where ω varies over a complete set of roots of unity. It follows that the roots of $\det(I - t\tilde{f}_{*u})$ include $\{1/\lambda\omega\}$ and thus that the roots of $\det(\tilde{f}_{*u} - tI)$ include $\{\lambda\omega\}$, ω as above. Taking $\omega = 1$, proves Theorem 2′.

To prove Corollary B, we note first that $f_{*u}: H_u(M: \mathrm{R})\circlearrowright$ has eigenvalues of the form $\{\lambda\omega\}$, $\lambda > 1$ and ω as above. But as we are on the C-dense case, the matrix T has the property that T^k

has all entries strictly positive, for $k \geq$ (some integer). But then, again by Perron–Frobenius, T has an eigenvalue $\lambda > 1$, and all others of modulus $< \lambda$.

Thus let C_u be a homology class associated with λ and $C_s \in H_s(M:\mathbb{R})$ the dual class. This proves (1) and (2) of the corollary, and (3) follows by duality and the strict inequality above.

APPENDIX (AFTER GUCKENHEIMER[5])

Recall that we choose the (X, A) in the proof of Proposition A small enough that
(a) the bundle E^u extends to $X - f(A^0)$;
(b) the derivative df extends to ϕ

$$\begin{tikzcd} \bar{E}_1^u \arrow[r, "\phi"] \arrow[d, "\pi"] & \bar{E}_2 \arrow[d, "\pi"] \\ X - A^0 \arrow[r] & X - f(A^0) \end{tikzcd}$$

where $\bar{E}_1^u$ and $\bar{E}_2^u$ are restrictions of $\bar{E}^u$. (Note: using the later work of [6] as above, ϕ can be taken $= df$). Then let $(\tilde{X}, \tilde{A}) \xrightarrow{\pi} (X, A)$ be the relative double cover of X, A which orients the bundle $\bar{E}^u$. In detail (as this seems to be a novel notion), let

$$\pi_0: \tilde{X}_0 \to X - f(A^0) = X_0$$

be the double cover of X_0 orienting $\bar{E}^u$ to the bundle $\tilde{E}_0^u$ over X_0. Let $\tilde{X} = \tilde{X}_0 \cup A/\sim$ where $x \sim y$ if
(1) $\pi_0 x = \pi_0 y$ and $\pi_0 x \in A$ or
(2) $\pi_0 x = y$ and $y \in A$.
Define $\tilde{E}^u = \tilde{E}_0^u | \tilde{X} - \tilde{A}$, $\tilde{A} =$ image of A in $\tilde{X}$. Note we still have a map $\pi: (\tilde{X}, \tilde{A}) \to (X, A)$, that the deck transformation $T: \tilde{X} \to \tilde{X}$ is fixed point free off of $\tilde{A}$ so that π is a relative double cover and $(\tilde{X}, \tilde{A})$ a relative smooth manifold. In general, now, there is an orientation of $\tilde{E}^u$ so that T reverses it at each point. It is sometimes, but not always, unique up to signs (e.g. it is not unique for the "horse shoe").

Next, there is a lifting of f to $\tilde{f}$

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \tilde{X} \arrow[d, "\pi"] \\ X \arrow[r] & X \end{tikzcd}$$

This was defined in [5] by
(i) $\tilde{f}|\tilde{A} =$ unique covering of f
(ii) $\tilde{f}|\tilde{X} - \tilde{A}$ is the unique covering of f preserving the orientation of $\tilde{E}^u$.
Such an $\tilde{f}$ is clearly unique if it exists. To see that it exists, proceed through the connected components of $\tilde{X} - \tilde{A}^0$ one at a time, as follows. Take $x \in C$. $\tilde{f}x$ is determined as one of y, Ty, where $\pi y = \pi Ty = \pi f x$. But for one and only one of y, Ty will this preserve the orientation as T reverses it. And this choice prevails uniformly over C by connectedness. Thus $\tilde{f}$ is continuous on C. Finally, $\tilde{X} - \tilde{A}^0$ will generically have only finitely many components. In particular, finitely many components of $\tilde{X} - \tilde{A}_0$ cover $\tilde{\Lambda} = \pi^{-1}(\Lambda)$. Use them.

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