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Axiom A Actions

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§1. Introduction

In this paper we generalize Ω-stability theory to actions by Lie groups other than $\mathbb{Z}$ and $\mathbb{R}$. Our results include [15, 17]. They are in answer to the suggestion of Steve Smale in [16] that differentiable dynamical systems be investigated for smooth group actions.

Main Theorem. *An Axiom A group action with no cycles is Ω-stable.*

See §2 for definitions of these terms and for more introductory discussion. We are grateful for the help given us by Moe Hirsch, John Stallings, and Joe Wolf. Except for the end theory in §4, this paper is an application of the invariant manifold techniques developed in our work with Moe Hirsch, which we refer to bibliographically as HPS.

§2. Lie Group Dynamics

In this section we lift some basic ideas of flow theory to action theory. An *action* of a group G on a set M is a homomorphism φ from G into the group of bijections of M. The action is of *class* C^r, $r \geqq 0$, iff G, M, and the evaluation map $(g, x) \mapsto \varphi(g)(x)$ are of class C^r. Since $\varphi(g^{-1}) = \varphi(g)^{-1}$, the bijections $\varphi(g)$ are homeomorphisms if $r=0$ and C^r diffeomorphisms if $r \geqq 1$. *Whenever convenient, we write $\varphi(g, x)$ for $\varphi(g)(x)$.*

The set of C^r actions $r \geqq 0$, of G on a compact M, $A^r(G, M)$, has a natural C^r topology (under which $A^r(G, M)$ is a Baire space) defined as follows. Each C^r action is a certain kind of continuous map $G \to \mathrm{Diff}^r(M)$, so we may consider $A^r(G, M) \subset C^0(G, \mathrm{Diff}^r(M))$. The latter space has the natural compact open topology and thereby endows $A^r(G, M)$ with a natural C^r topology by restriction. See [12]. Convergence $\varphi_n \xrightarrow[n]{} \varphi$ in $A^r(G, M)$ means: for each compact set $S \subset G$, $\varphi_n(g) \xrightarrow[n]{} \varphi(g)$ in the C^r sense uniformly over $g \in S$.

Although this topology on $A^r(G, M)$ is the only natural one, it leads nowhere unless we restrict G somewhat. See §6 for an example where G is too general. *From now on, standing hypotheses are: G is a Lie group with a compact set of generators, M is a compact, smooth, boundaryless, connected manifold, and the actions discussed are at least of class C^0.* When G is discrete, "compact" means "finite" and then we are assuming G is finitely generated.

Let φ be a G-action M. The *φ-orbit* of $p \in M$ is $O(p)$ [or O_p] $= \{gp \colon g \in G\}$. [Following the standard practice, we often think of $g \in G$ as the diffeomorphism

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$\varphi(g) \in \mathrm{Diff}^r(M)$.] A point $x \in M$ is *nonwandering* for φ iff for each neighborhood U of x in M and each compact set $S \subset G$, there exists $g \in G - S$ with $gU \cap U \neq \emptyset$. The set of nonwandering points of φ is denoted by Ω_φ. Clearly, Ω_φ is φ-invariant – i.e. consists of whole φ-orbits. Also Ω_φ is a closed subset of M. If G is compact then Ω_φ is empty and vice versa. The *boundary* of an orbit $O(p)$ is the set of all limit points of sequences $g_n p$ where $\{g_n\}$ is a sequence in G having no cluster point in G. Clearly $\partial O(p) = \partial O(q)$ if $O(p) = O(q)$.

(2.1) **Proposition.** *The boundary of each orbit lies in* Ω_φ.

Proof. Let $x \in \partial O(p)$ and let $\{g_n\}$ be a clusterless sequence in G with $g_n p \to x$. Let S be a given compact subset of G and U a given neighborhood of x in M. Fix k large with $x' = g_k p \in U$. Clearly $g_n g_k^{-1}(x') \underset{n}{\to} x$. Since $\{g_n\}$ has no cluster point in G, neither does $\{g_n g_k^{-1}\}$. Hence, for large n, $g = g_n g_k^{-1} \in G - S$ and $g x' \in U$, i.e. $gU \cap U \neq \emptyset$. Hence $x \in \Omega_\varphi$.

Next we discuss when two actions $\varphi, \psi \colon G \to \mathrm{Diff}(M)$ should be called equivalent. We say φ and ψ are *parametrically conjugate* iff there is a homeomorphism $h \colon M \to M$ such that

$$\begin{array}{ccc} M & \xrightarrow{\varphi(g)} & M \\ \downarrow h & & \downarrow h \\ M & \xrightarrow{\psi(g)} & M \end{array}$$

commutes for all $g \in G$. This says $\psi(g) \equiv h \circ \varphi(g) \circ h^{-1}$. When $G = \mathbb{R}$, such a conjugacy preserves the parametrization of the trajectories, hence the name.

A homeomorphism $h \colon M \to M$ which sends each φ-orbit onto a ψ-orbit is an *orbit conjugacy* between φ and ψ; the orbit pictures (or "phase portraits") of φ and ψ are the same, although the parameterizations of corresponding orbits may be different. The equivalence relation of orbit-conjugacy is well adapted to dynamical systems; parametric-conjugacy implies orbit-conjugacy (clearly) but is too restrictive. We write $\varphi \sim \psi$ to denote orbit conjugacy.

A G-action φ is *structurally stable* iff $\varphi \sim \psi$ for each G-action ψ near φ; φ is *Ω-stable* iff $\varphi|\Omega_\varphi \sim \psi|\Omega_\psi$ for each ψ near φ.

Palais [13] proves:

Theorem. *If G is a compact Lie group then any φ action is parametrically structurally stable.*

That is, any C^1 perturbation of φ is actually parametrically conjugate to φ. For this reason our interest is non-compact G with emphasis on the ultimate behavior of the orbits.

As for $\mathbb{Z}$ and $\mathbb{R}$ actions, hyperbolicity is a crucial idea in studying structural and Ω-stability. One version of this is presented in [6].

Definition. Suppose G is a connected Lie group and φ is a C^1 locally free G-action such that some f in G is normally hyperbolic at the orbit foliation. Then φ is called an *Anosov Action* and f is called an *Anosov element*. See [HPS] and §3 for the definition of "normally hyperbolic."

(2.2) **Theorem.** *If φ is an Anosov Action then φ is structurally stable.*

Proof. Local freeness of φ implies the orbits of φ foliate M. Clearly the same is true for any φ' near φ. Let $\mathscr{F}$ be the φ-orbit foliation. Since φ is C^1 so is $\mathscr{F}$. Let $f_0 \in G$ be an Anosov element and let $f=\varphi(f_0)$. By [HPS, (7.2)], $(f, \mathscr{F})$ is plaque expansive. Let $\varphi'(f_0)=f'$ for φ' near φ. By [HPS (7.1)], $(f, \mathscr{F})$ is structurally stable and so there is a canonical leaf conjugacy $h_{f'}$: $(f, \mathscr{F}) \to (f', \mathscr{L}')$ where $\mathscr{L}'$ is an f' invariant lamination with $T\mathscr{L}'$ near $T\mathscr{F}$. We claim $\mathscr{L}'=$the φ'-orbit foliation. Since Tf leaves both $T\mathscr{F}'$ and $T\mathscr{L}'$ invariant and since both are near $T\mathscr{F}$, [HPS (2.12)] implies $T\mathscr{F}'=T\mathscr{L}'$. Since $\mathscr{F}'$ is C^1 this implies that lamina of $\mathscr{L}'$ are contained in φ'-orbits. Since G is connected and lamina are Riemann-complete, the lamina coincide with the φ'-orbits and (2.2) is proved.

Remark. In [6] a stronger result is given: if φ is C^2 then the φ-orbit foliation is structurally stable "as a foliation." Besides, it is enough to assume the Anosov element lies in connected component of the identity, G_1, or that G/G_1 is finite.

Although (2.2) is elegant, it does not include the case of an Anosov diffeomorphism f, considered as a $\mathbb{Z}$-action, $n \mapsto f^n$. For $\mathbb{Z}$ is too disconnected. Also, the assumption that φ be locally free prohibits singularities—for $\mathbb{R}$-actions no fixed points are allowed. Here are two definitions answering these objections. φ is a G-action on M.

Definition. φ is a *hyperbolic* G-action if the φ-orbits foliate M and some f in the center of G is normally hyperbolic to the φ-orbit foliation of M.

Definition. φ is an *Axiom A* G-action iff the φ-orbits laminate Ω_φ and

(a) Some f in the center of G is normally hyperbolic to the orbit lamination of Ω_φ.

(b) The compact orbits are dense in Ω_φ.

Such an f is called a *hyperbolic element.* Axiom $A(a)$ could be called "Ω-hyperbolicity of φ" and hyperbolicity of φ could consistently be called "M-hyperbolicity of φ". Even for $\mathbb{Z}$-actions it is not known whether $A(a) \Rightarrow A(b)$.

Remark 1. Centralness of f is a big assumption. But hyperbolic G-actions include the $\mathbb{Z}$-action of an Anosov diffeomorphism, the $\mathbb{R}$-action of an Anosov flow, and indeed all Anosov actions by Abelian groups. We need f in the center of G to prove structural stability, see (3.1). If G is not connected and centralness of f is dropped, structural stability fails. See §6.

Remark 2. A lamination is a foliation with less smoothness assumed. See [HPS] and §3.

Remark 3. If φ is a hyperbolic G-action then it satisfies Axiom $A(a)$. If $\Omega=M$ and φ satisfies Axiom $A(a)$ then φ is hyperbolic. (3.10) establishes structural stability for Axiom $A(a)$ G-actions with $\Omega=M$. If G is connected, this was proved already in more generality in (2.2).

Remark 4. As a natural specialization of Axiom A, one might say φ is a Morse-Smale Action iff φ is Axiom A and Ω_φ is finitely many orbits. When $G=\mathbb{Z}$ or $\mathbb{R}$ this agrees with the standard notion. For $G=\mathbb{R}^2$, however, there is another definition of Morse-Smale action which has been investigated largely by Cezar Camacho [1, 2]. He does insist Ω_φ be a finite union of orbits but only requires normal hyperbolicity to the compact orbits. Non compact orbits in Ω are permitted which connect Ω-basic sets of different splitting-types. It remains to be seen what turns out to be the most fruitful definition of Morse-Smale action – or Axiom A action.

Here is a basic result of Smale's Theory [17] generalized to actions.

(2.3) **Ω-Decomposition Theorem.** *Let φ be a C^1 G-action satisfying Axiom A. Then there is a unique decomposition $\Omega_\varphi = \Omega_1 \cup \cdots \cup \Omega_k$ such that the Ω_i are compact, disjoint, φ-invariant, and indecomposable. On Ω_i, φ is topologically transitive.*

The proof of (2.3) occurs after (3.3) in §3. The Ω_i are *basic sets* for φ. "*Indecomposable*" means Ω_i cannot be divided into two disjoint compact nonempty φ-invariant subsets. Since M is connected, Ω is indecomposable if $M=\Omega$. "*Topological transitivity*" of φ on Ω_i means that any two relatively open, nonempty, φ-invariant subsets of Ω_i meet.

To a hyperbolic element f are associated stable manifold structures by [HPS]. Through each orbit $O(x)$ in Ω_φ there pass unique f-invariant manifolds, $W^u(x)$ and $W^s(x)$, transversally intersecting in $O(x)$. The stable manifold $W^s(x)$ is f-invariantly fibered by strong stable manifolds $W^{ss}(x')$, $x' \in O(x)$, consisting of points sharply asymptotic with x' under positive iteration by f. Similarly the unstable manifold. Centralness of f and these characterizations imply that the strong stable and strong unstable fibrations are invariant by the entire G-action, not just by f. For if $g \in G$ and $y \in W^{ss}x$ then $f^n g(y) = g f^n(y)$ is equally sharply asymptotic with $f^n g(x) = g f^n x$ as $f^n y$ is with $f^n x$ when $n \to \infty$. Similarly for W^{uu}. Since $W^s(x)$ consists of the fibers $W^{ss}(x')$ with x' in the invariant set $O(x)$, $W^s(x)$ is φ-invariant. Likewise $W^u(x)$.

There is a partial order on the basic sets of an Axiom A action

$$\Omega_i \prec \Omega_j \quad \text{iff} \quad W^u(x) \cap W^s(y) \not\subset \Omega \qquad \text{for some } x \in \Omega_i,\ y \in \Omega_j.$$

A *cycle* is a sequence $\Omega_{i_1} \prec \cdots \prec \Omega_{i_n} = \Omega_{i_1}$, $n \geqq 2$. A *self cycle* occurs when $n=2$. In (4.14) we show that the existence of cycles is independent of which hyperbolic element f we choose in G.

Our Main Theorem – that Axiom A plus no cycles implies Ω-stability for G-actions – has already been proved when $G = \mathbb{Z}$ [17], $G = \mathbb{R}$ [15], or $\Omega = M$ and G is connected [6] or (2.2). It turns out, to our surprise, that all Axiom A actions are "essentially" one of these types. Precisely, there is an alternative: for Axiom A actions either

$$\Omega_\varphi = M$$

or

$$G \text{ is hyperbolic.}$$

A group G is *hyperbolic* if it has two ends and they are invariant under right-multiplication by all elements of G. See §4 for a discussion of ends and (4.12) for a proof of this alternative.

If G is two-ended and is connected then G is isomorphic to the direct product of a compact Lie group K and $\mathbb{R}$. This was conjectured by Zippin [20] and proved extemporaneously for us by Joe Wolf.[1] Thus, when G is connected, the new

[1] Afterwards, we find that this result is due to Freudenthal.

part of our Main Theorem amounts to Ω-stability for flows equivariant by a compact group action and is an extension of Mike Field's thesis [4] on equivariant dynamical systems. If G is two-ended, is not connected, and supports the Axiom A action φ then we thought that G was isomorphic to $K \times \mathbb{Z}$, K being a compact group. (When G is discrete and two-ended it does contain a copy of $\mathbb{Z}$ having finite index – see (4.6) and [5, 6.14, 10, Satz V].) John Stallings showed us an example of a group G casting doubt on this conjecture and Moe Hirsch showed us how to make G act on an M^3 obeying Axiom A, $\Omega_\varphi \neq M$, and having no cycles. See §6. But no matter – our proof of Ω-stability is independent of such factorizations. $G = K \times \mathbb{R}$, $G = K \times \mathbb{Z}$.

§3. Hyperbolic Sets for Actions

In this section we apply [HPS] to the hyperbolic element f of our Axiom A action. As in the flow case, much of the stability theory for Ω works equally for a hyperbolic set Λ, so

Definition. A C^1 G-action φ is *hyperbolic at* $\Lambda \subset M$ iff the connected components $\mathscr{L}_x$, $x \in M$, of the φ-orbits laminate Λ and G has some f in its center so that f is normally hyperbolic to the orbit lamination of Λ. Such an f is a *hyperbolic element* for φ at Λ and the lamination $\mathscr{L}$ is called the *φ-orbit lamination.*

Recall from [HPS] that a *lamination* $\mathscr{L}$ of Λ is a continuous foliation of Λ with smooth leaves $\mathscr{L}_x$ and continuous leaf tangent bundle $T\mathscr{L}$. (That is, $x \mapsto T_x(\mathscr{L}_x)$ is a continuous map $\Lambda \to \operatorname{Grass}(TM)$.) The diffeomorphism f is normally hyperbolic at $\mathscr{L}$ iff it permutes the laminas (= leaves of $\mathscr{L}$) and $T_\Lambda M$ splits Tf invariantly

$$T_\Lambda M = N^u \oplus T\mathscr{L} \oplus N^s, \qquad T_\Lambda f = N^u f \oplus \mathscr{L} f \oplus N^s f$$

with

$$\inf_\Lambda m(N^u_x f) > 1 \qquad \sup_\Lambda \|N^s_x f\| < 1$$

$$\inf_\Lambda m(N^u_x f)\, \|\mathscr{L}_x f\|^{-1} > 1 \qquad \sup_\Lambda \|N^s_x f\|\, m(\mathscr{L}_x f)^{-1} < 1.$$

By $m(A)$ we mean the "minimum norm" or "conorm" of the linear transformation A: $m(A) = \inf_{|x|=1} |Ax|$.

The existence of such a hyperbolic element f already puts certain limitations on G. Either all of M is a single orbit (i.e. M is a homogeneous space) or else $\{f^n\}$ has no cluster points in G. For if N^u is nonzero then $m(N^u f^n) \to \infty$ as $n \to \infty$ and $\|N^u f^n\| \to 0$ as $n \to -\infty$, whereas for each $g \in G$, the tangent to $\varphi(g)$ has bounded norm and conorm. Likewise if $N^s \neq 0$ then $\{f^n\}$ has no cluster point in G. On the other hand if $N^u = 0 = N^s$ then $T_x M = T_x O_x$ for all $x \in \Lambda$. Since M is connected, this implies M is a single orbit. Hence: either G contains a center with a copy of $\mathbb{Z}$ embedded in it or else M is a single orbit. The latter sort of action is always structurally stable. (This is easy to check and has nothing to do with Axiom A) and we can ignore it in all that follows.

For an f normally hyperbolic at the lamination $\mathscr{L}$ of Λ there exists a natural stable manifold theory from [HPS] – even if f is not part of an action. For some

$\varepsilon>0$, each $p\in\Lambda$ has a strong stable manifold $W_\varepsilon^{ss}(p)$ characterized by

$$W_\varepsilon^{ss}(p)=\{x\in M\colon d_M(f^n x, f^n p)\leqq\varepsilon \text{ for all } n\geqq 0 \text{ and } d_M(f^n x, f^n p)\to 0, \text{ as } n\to\infty, \text{ faster than is possible along } \mathscr{L}_p\}.$$

Moreover, $W_\varepsilon^{ss}(p)$ is C^1, is tangent to N_p^s, and $fW_\varepsilon^{ss}(p)\subset W_\varepsilon^{ss}(fp)$. Taking the union of $W_\varepsilon^{ss}(x)$ over all $x\in\mathscr{L}_p$ gives the ε-stable manifold of the lamina $\mathscr{L}_p$, $W_\varepsilon^s(\mathscr{L}_p)$. This $W_\varepsilon^s(\mathscr{L}_p)$ is a C^1 immersed manifold, but it has a boundary and may have self intersections or branching. See [HPS §§ 6, 7].

To get rid of the boundary, we can globalize by iteration

$$W^{ss}(x)=\bigcup_{n\geqq 0} f^{-n} W_\varepsilon^{ss}(f^n x)$$

$$W^s(\mathscr{L}_p)=\bigcup_{x\in\mathscr{L}_p} W^{ss}(x).$$

From the characterization of strong stable manifolds it is clear that any $W^{ss}(x_1)$, $W^{ss}(x_2)$ are equal or disjoint.

In general, branching of $W^s(\mathscr{L}_p)$ seems possible. But when φ is hyperbolic at Λ then centralness of f and the characterization of $W_\varepsilon^{ss}(p)$ imply that

$$g\,W^{ss}(p)=W^{ss}(g\,p)$$

for all $g\in G$ and all $p\in\Lambda$. Thus, either $W^s(\mathscr{L}_p)$ equals $W^s(\mathscr{L}_q)$ or they are disjoint. Likewise $W^s(\mathscr{L}_p)$ intersects itself only in relatively open sets. Since $W^s(\mathscr{L}_p)$ has no boundary, this means that $W^s(\mathscr{L}_p)$ is injectively immersed.

To make this clearer, consider an Anosov flow, say ψ_t, on M^3. Let γ be a closed orbit of ψ and $W_\varepsilon^s\gamma$ its local stable manifold, a cylinder with boundary. Consider one of the other orbits, say γ_1, on $W^s\gamma$. Clearly, $W^s\gamma_1=W^s\gamma$. Although $W^{ss}(p_1)=W^{ss}(p_1')$ for many distinct $p_1, p_1'\in\gamma_1$, it is still true that $W^s\gamma$ is injectively immersed. What is *not* true is that the map sending the abstract union $\bigcup_{p_1\in\gamma_1} W^{ss}(p_1)$ onto $W^s(\gamma_1)$ is injective.

Of course, replacing f by f^{-1}, we get the corresponding unstable manifold theory.

Definition. Let f be normally hyperbolic to the lamination $\mathscr{L}$ of Λ. Then (f,Λ) has *local product structure* iff

$$W_\varepsilon^u(\Lambda)\cap W_\varepsilon^s(\Lambda)=\Lambda$$

for some $\varepsilon>0$. If in addition, each lamina $\mathscr{L}_x$ meets each $W^u O_y$ and $W^s O_y$ in relatively open subsets of $\mathscr{L}_x$ then we say $(f,\mathscr{L})$ has *local product structure.*

(3.1) *Local Product Structure Theorem. If f is a hyperbolic element of an Axiom A action φ then $(f,\mathscr{L})$ has local product structure when $\mathscr{L}$ is the φ-orbit lamination of Ω_φ.*

Several preliminaries are required to prove this theorem. First we prove a simple

(3.2) **Intersection Lemma.** *Let φ be hyperbolic at Λ with hyperbolic element f. Let O_1, O_2 be φ orbits in Λ. If x is a point of transverse intersection between $W^u O_1$*

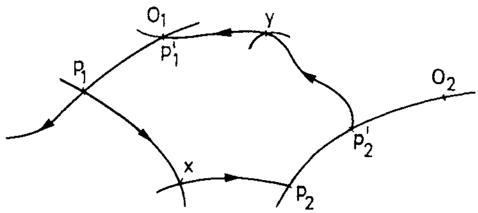

Fig. 1

and $W^s O_2$ then x is also a point of transverse intersection between the strong unstable fiber through x and $W^s O_2$.

Proof. By f-invariance, it suffices to prove this when $x \in W_\varepsilon^u O_1$ and ε is small. Let $x \in W_\varepsilon^{uu}(p)$, $p \in O_1$. Let G_1 be the connected component of 1, the identity, in G. Each $X \in T_1 G$ generates a 1-parameter subgroup in G_1, $\exp X$, where exp is the exponential of G. Each $\exp X$ generates a C^1 flow $(t, z) \mapsto \varphi(\exp(tX), z)$. By van Kampen's Uniqueness Theorem [11],

$$\varphi(\exp(tX), p) \equiv p \Leftrightarrow \frac{d}{dt}\bigg|_{t=0} \varphi(\exp(tX), p) = 0.$$

Thus, the restricted tangent-map $T\varphi$: $T_1 G \times p \to T_p O_1$ is surjective and so the local isotropy subgroup $I_p = \{g \in G_1 : \varphi(g, p) = p\}$ is a C^1 submanifold of G by the Implicit Function Theorem.

Let D be a small smooth disc in G transversally meeting I_p at 1. Then $\varphi | D \times p$ is a diffeomorphism to a neighborhood of p in O_1 and, since O_1 meet $W^{uu}(p)$ transversally at p, $\varphi(D, x)$ meets $W^{uu}(p)$ transversally at x, x near p. Since $\varphi(D, x) \subset O(x)$, $O(x)$ also meets $W^{uu}(p)$ transversally at x. Since $W^s O_2$ meets $W^u O_1$ transversally at x, $T_x(W^s O_2)$ contains a complement of $T_x(W^u O_1)$. Hence, $T_x M$ is spanned by $T_x(W^s O_2)$ plus $T_x O(x)$ plus $T_x(W^{uu}(p))$, proving (3.2).

The next lemma is the key to many problems.

(3.3) **Cloud Lemma.** *Let φ be hyperbolic at Λ with hyperbolic element f. Let O_1, O_2 be compact φ-orbits in Λ. If $W^u O_1$, and $W^s O_2$ have at least one point of transverse intersection then $W^s O_1 \cap W^u O_2 \subset \Omega_\varphi$.*

Proof. See Fig. 1. Let $y \in W^{uu}(p_2') \cap W^{ss}(p_1')$ where $p_1' \in O_1$, $p_2' \in O_2$. Let $x \in W^{uu}(p_1) \cap W^{ss}(p_2)$ when $p_1 \in O_1$, $p_2 \in O_2$, and $W^u O_1$ intersects $W^2 O_2$ transversally at x. By the Intersection Lemma, $W^{uu}(p_1)$ intersects $W^s O_2$ transversally at x. Let U be any neighborhood of y in M and let S be a compact set in G. We must show that $gU \cap U \neq \emptyset$ for some $g \in G - S$.

Since O_1 and O_2 are compact there is a large compact subset $Q \subset G$ such that

$$Q p_1 = O_1 \qquad Q p_2 = O_2$$

for all $p_1 \in O_1$, $p_2 \in O_2$. By Qz we mean $\{qz : q \in Q\}$. Then choose $q_n \in Q$ so that

$$f_n(p_1') = p_1 \qquad \text{where } f_n = q_n f^n.$$

Since $\{\varphi(g)\colon g\in Q\}$ is a compact subset of Diff(M) it is clear that $f_n(y)\to p_1$ as $n\to\infty$. By the usual λ-lemma [14] plus the φ-invariance of $W^u O_1$, $W^s O_1$, plus the commutativity of f^n and g_n it follows that $f_n U$ contains a disc D_n nearly equal to much of $W^{uu} p_1$. In particular, for large n, D_n intersects $W^s O_2$ near x at say $x_n \in f_n U \cap W_a^{ss}(p_{2n})$, for some $x_n\to x$, $p_{2n}\to p_2$, and fixed (large) a.

Again, choose $q'_n\in Q$ such that

$$q'_n f^n(p_{2n}) = p'_2.$$

The λ-lemma applied again produces discs D'_n in $q'_n\circ f^n\circ f_n(U)$ nearly equal to much of $W^{uu}(p'_2)$. Thus,

$$g_n U\cap U \neq \emptyset$$

when $g_n = g'_n f^n q_n f^n = q'_n q_n f^{2n}$. Since $\{f^n\}$ has no cluster point in G and since the q_n, q'_n all lie in a Q which is compact, $\{g_n\}$ has no cluster point in G, so most of the g_n lie outside the given compact set S. This proves that $gU\cap U\neq\emptyset$ for some $g\in G-S$ and completes the proof of the Cloud Lemma.

Proof of the Local Product Structure Theorem (3.1). By assumption φ is an Axiom A action, so Ω_φ is a hyperbolic set for φ. Since the splitting $T_{\Omega_\varphi} M = N^u\oplus T\mathscr{L}\oplus N^s$ is continuous, it is clear that

$$W_\varepsilon^{uu}(p) \pitchfork W_\varepsilon^s(O_q)\neq\emptyset$$
$$W_\varepsilon^{uu}(q) \pitchfork W_\varepsilon^s(O_p)\neq\emptyset$$

for all nearby p, q in Ω_φ. By Axiom A, the compact orbits are dense in Ω_φ and so p, q can be approximated by p', q' in Ω_φ with O_p, O_q compact. By the persistance of transversality, the corresponding intersections $W_{2\varepsilon}^{uu}(p')\cap W_{2\varepsilon}^s(O_{q'})$ and $W_{2\varepsilon}^{uu}(q')\cap W_{2\varepsilon}^s(O_{p'})$ continue to be nonempty and near the ones for p, q. By the Cloud Lemma, the former are in Ω_φ. Since Ω_φ is a closed subset of M, so are the latter. This proves that (f,Ω) has local product structure. As we observed in § 2, stable and unstable manifolds of orbits are φ-invariant. Thus, the orbit lamination of Ω is subordinate to $\mathscr{W}^u$, $\mathscr{W}^s$ and so $(f,\mathscr{L})$ also has local product structure.

As a consequence of local product structure we get Ω-decomposition, just as for flows.

Proof of the Ω-Decomposition Theorem (2.3). Let φ be an Axiom A G-action on M with hyperbolic element f. Let ε be small enough so (f,Ω_φ) has 2ε-local product structure. Let $p\in\Omega_\varphi$. Consider any neighborhoods V, V' of p having diameter $\leqq\varepsilon$. Then

$$\mathrm{Sat}(V\cap\Omega) = \mathrm{Sat}(V'\cap\Omega) \qquad (*)$$

where $\mathrm{Sat}(X) = \mathrm{Closure}\big(\bigcup_{g\in G} gX\big)$ is the *saturate* of the set X. To verify $(*)$ it suffices to prove $V'\cap\Omega\subset\mathrm{Sat}(V\cap\Omega)$. If $z\in V'\cap\Omega$ then Axiom A says p and z can be approximated by p' and z' such that $O(p')$ and $O(z')$ are compact and $p'\in V\cap\Omega$. By 2ε-local product structure, $W_{2\varepsilon}^{uu}(p')\pitchfork W_{2\varepsilon}^s(O(z'))\neq\emptyset\neq W_{2\varepsilon}^{uu}(z')\pitchfork W_{2\varepsilon}^s(O(p'))$ and so $z'\in\mathrm{Sat}(V\cap\Omega)$ by the Cloud Lemma proof. Since z' is arbitrarily near z and $\mathrm{Sat}(V\cap\Omega)$ is closed, z is also in $\mathrm{Sat}(V\cap\Omega)$ completing the proof of $(*)$.

Let $\Omega(p) = \mathrm{Sat}(V\cap\Omega)$ for any neighborhood V of p having diameter $\leqq\varepsilon$. Then $\Omega(p)$ is compact, non-empty, and φ-invariant. From $(*)$ we see that either $\Omega(p) =$

$\Omega(p')$ or $\Omega(p) \cap \Omega(p') = \emptyset$, $p, p' \in \Omega$. In this sense the family $\{\Omega(p)\}_{p \in \Omega}$ is a nonoverlapping covering of Ω by neighborhoods. Ω being compact, finitely many of the $\Omega(p)$'s cover Ω and they form the Ω-decomposition $\Omega_\varphi = \Omega_1 \cup \cdots \cup \Omega_m$.

Let V, V' be open neighborhoods of p, $p' \in \Omega_i$. Then $\Omega(p) = \Omega(p') = \Omega_i$ proves that $g(V \cap \Omega_i) \cap (V' \cap \Omega_i) \neq \emptyset$ for some $g \in G$, i.e. $\varphi | \Omega_i$ is topologically transitive. Clearly, topological transitivity implies indecomposability.

Uniqueness of $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$, assuming the Ω_i are φ-invariant, compact, disjoint, and indecomposable, is immediate: if $\Omega'_1 \cup \cdots \cup \Omega'_n$ is another Ω-decomposition then so is $\Omega = \bigcup_{i,j} \Omega_i \cap \Omega'_j$, a contradiction to indecomposability unless the Ω_i are the Ω'_j.

Here is another qualitative result about Ω for an Axiom A action.

(3.4) **Theorem.** *An Axiom A action has no self cycles.*

Proof. Suppose Ω_i has a self cycle, i.e. suppose $z \in W^u \Omega_i \cap W^s \Omega_i$ for some point z of $M - \Omega_i$. Then $z \in W^{uu}(p) \cap W^{ss}(p')$ for some $p, p' \in \Omega_i$. Let U be a given neighborhood of z and S a given compact subset of G. By topological transitivity on Ω_i, any small neighborhood of p meets a φ-orbit in Ω_i passing arbitrarily near p'. By Axiom A it can be approximated by a compact orbit O passing arbitrarily near p and p'. In particular, $U \cap W^u O$ and $U \cap W^s O$ will be nonempty. Fix such an O and choose a compact set $Q \subset G$ such that $Qx = O$ for all $x \in O$. By the λ-lemma as in the proof of the Cloud Lemma, there is an element f_n in G of the form

$$g_n = f^n \circ q_n \circ f^n = f^{2n} \circ q_n$$

such that $g_n U \cap U \neq \emptyset$, $q_n \in Q$, and n is arbitrarily large. Since $\{f^n\}$ has no cluster point in G, neigher does $\{g_n\}$. Thus most g_n lie outside S and z is proved to be nonwandering, i.e. $z \in \Omega_j$ for some j. If $j \neq i$ then $-$ Ω_j being compact, invariant, and disjoint from Ω_i $-$ $f^n z$ could not tend to Ω_i. Thus $z \in \Omega_i$, a contradiction. This proves (3.4).

The next theorem is the intent of the remark after (7.4) of [8]. According to (3.1), it applies to $\Omega_\varphi = \Lambda$ when φ is an Axiom A action.

(3.5) **Theorem.** *Let the G-action φ be hyperbolic at Λ with hyperbolic element f. Suppose $(f, \mathscr{L})$ has local product structure where $\mathscr{L}$ is the φ-orbit lamination of Λ. Then there exists a neighborhood U of Λ such that any point x whose forward f-iterates remain in V lies on $W^{ss}_\varepsilon(x')$ for some $x' \in \Lambda$. Similarly for backward f-iterates and W^{uu}.*

Proof. This is a special case of [HPS, (7A.1)].

(3.6) **Corollary.** *If φ is an Axiom A G-action with hyperbolic element f and Ω decomposition $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ then ∂O_x meets at least two Ω_i's, $x \in M - \Omega$.*

Proof. $\partial O_x \subset \Omega$ by (2.1). If $\partial O_x \subset \Omega_i$ for some single i then $f^n x \to \Omega_i$ as $n \to \pm\infty$. By (3.5), $x \in W^{uu}(x') \cap W^{ss}(x'')$ for some $x', x'' \in \Omega_i$, i.e. Ω_i has a self cycle, contradicting (3.4).

Finally, we discuss perturbations of φ when φ is a C^1 G-action with hyperbolic set Λ and $\mathscr{L}$ is the φ-orbit lamination of Λ. The perturbation theory of [HPS] requires that $(f, \mathscr{L})$ be "plaque expansive". A lamination $\mathscr{L}$ of Λ is called C^1-*smoothable* iff $\mathscr{L}$ extends to a C^1 local foliation near each $p \in \Lambda$. The local foliations need not be coherent.

(3.7) **Proposition.** *If φ is a G-action and the φ-orbits laminate a compact set Λ then this lamination of Λ is C^1-smoothable.*

Proof. The construction in the proof (3.2) gives these C^1 local foliations.

(3.8) **Corollary.** *If φ is an Axiom AG-action with hyperbolic element f and if $\mathscr{L}$ is the φ-orbit lamination of Ω_φ then $(f, \mathscr{L})$ is plaque expansive.*

Proof. By (3.7), $\mathscr{L}$ is C^1 smoothable. By (7.4(i)) of [HPS], $(f, \mathscr{L})$ is plaque expansive.

The next result says that the canonical perturbation theory of [HPS, § 7] is natural respecting G-actions.

(3.9) **Persistence Theorem.** *Suppose φ is a G-action with hyperbolic set $\Lambda \subset M$. Then φ has a neighborhood U in $A^1(G, M)$ such that to each $\varphi' \in U$ there corresponds an orbit conjugacy*

$$h_{\varphi'}: (\varphi, \Lambda) \to (\varphi', \Lambda')$$

where Λ' is a canonically determined φ'-invariant set near Λ. Also, $h_{\varphi'} \to$ inclusion as $\varphi' \to \varphi$.

Proof. Let $\mathscr{L}$ be the φ-orbit lamination of Λ. Let $f_0 \in G$ be a hyperbolic element for φ and let $f = \varphi(f_0)$, $f' = \varphi'(f_0)$. By (3.8), $(f, \mathscr{L})$ is plaque expansive and clearly f' is a C^1 perturbation of f. By [HPS, (7.4(ii))] there is a canonical f'-invariant lamination $\mathscr{L}'$ near $\mathscr{L}$ and a canonical leaf conjugacy $h_{f'}: (f, \mathscr{L}) \to (f', \mathscr{L}')$ near the inclusion $\Lambda \hookrightarrow M$. We must show $\mathscr{L}'$ is the φ'-orbit lamination of $\Lambda' = h_{f'}\Lambda$ and that $h_{f'}$ carries φ-orbits to φ'-orbits.

$h_{f'}$ is characterized as follows, using a fixed smooth bundle η in $T_\Lambda M$, complementary to $T\mathscr{L}$, and a fixed small plaquation $\mathscr{P}$ of $\mathscr{L}$. Given $x \in \Lambda$, $h_{f'}(x)$ is the unique point of $\exp_x \eta(\varepsilon)$ whose f' orbit can be closely shadowed by an f-pseudo-orbit which respects $\mathscr{P}$.

Let W be a seed of G, i.e. a compact set of generators with $W^{-1} = W$. Let $\{x_m\}$ be an f-pseudo-orbit through x which respects $\mathscr{P}$ and closely shadows $\{f'^n(x')\}$, $x' = h_{f'}x$. Thus $x_{n+1} = \varphi(g_n, f x_n)$, g_n is near 1, and $x_0 = x$. Let $g \in W$. Then g', g'' can be found near g such that

$$z' = \varphi'(g', x') \in \exp_{\varphi(g, x)} \eta(\varepsilon) \qquad x' = h_{f'} x$$
$$z'' = \varphi'(g, x') \in \exp_{\varphi(g'', x)} \eta(\varepsilon).$$

To find g', g'' reconsider the proof of (3.2): let D be a smooth disc in G transverse at 1 to the isotropy subgroup of x and regard the map

$$D \times \exp_x \eta(\varepsilon) \to M$$
$$(d, z) \to \varphi(gd, z)$$

which is a local diffeomorphism to a neighborhood of $\varphi(g, x)$ in M. Since it sends $D \times 0$ to a neighborhood of $\varphi(g, x)$ in $\mathscr{L}_{\varphi(g, x)}$, since $\varphi' \doteq \varphi$, and since $h_{f'} \doteq$ inclusion we can find $d, d' \in D$ such that $g' = gd$, $g'' = gd''$ work.

We claim that $\{\varphi(g, x_n)\}$ and $\{\varphi(g'', x_n)\}$ are f-pseudo-orbits which respect $\mathscr{P}$.

By centralness of f_0 we can write

$$\begin{aligned}\varphi(g, x_{n+1}) &= \varphi(g, \varphi(g_n, f x_n)) \\ &= \varphi(g g_n g^{-1} f_0 g, x_n) \\ &= \varphi(g g_n g^{-1}, f \varphi(g, x_n)).\end{aligned}$$

When the g_n are very near 1, the $g g_n g^{-1}$ are near 1 and so $\{\varphi(g, x_n)\}$ is an f-pseudo-orbit which respects $\mathscr{P}$. Since g'' is near g, and is thus also confined to a compact set, the same is true of $\{\varphi(g'', x_n)\}$.

Also we claim that $\{\varphi(g, x_n)\}$ closely shadows $\{f'^n(z')\}$ while $\{\varphi(g'', x_n)\}$ closely shadows $\{f'^n(z'')\}$. Again by centralness of f_0

$$f'^n(z') = \varphi'(f_0^n, \varphi'(g', x')) = \varphi'(g', f'^n(x')).$$

Since $\varphi' \doteq \varphi$, x_n is close to $f'^n(x')$, g' is near g, and g is confined to compact set of G, this $\varphi'(g', f'^n(x'))$ is near $\varphi(g, x_n)$. Similarly

$$f'^n(z'') = \varphi'(f_0^n, \varphi'(g, x')) = \varphi'(g, f'^n x')$$

is near $\varphi(g'', x_n)$. Hence the pseudo-orbits do closely shadow $\{f'^n(z')\}$ and $\{f'^n(z'')\}$. By the characterization of $h_{f'}$ we conclude $h_{f'}(\varphi(g, x)) = z'$, $h_{f'}(\varphi(g'', x)) = z''$. That is, for any $g \in W$ and all $x \in \Lambda$

$$\begin{aligned}&(1) \qquad h_{f'}(\varphi(g, x)) = \varphi'(g', h_{f'} x) \\ &(2) \qquad h_{f'}(\varphi(g'', x)) = \varphi'(g, h_{f'} x)\end{aligned} \qquad \text{for some } g', g'' \text{ near } g.$$

Next we extend these equations to all $g \in G$. We claim that for any $g \in G$ and all $x \in \Lambda$

$$\begin{aligned}&(3) \quad h_{f'}(\varphi(g, x)) = \varphi'(g', h_{f'} x) \\ &(4) \quad h_{f'}(\varphi(g'', x)) = \varphi'(g, h_{f'} x)\end{aligned} \qquad \begin{aligned}&\text{for some } g', g'' \text{ in the same connected} \\ &\text{component of } G \text{ as } g.\end{aligned}$$

By (1), (2) it suffices to prove (3), (4) for g of the form $g_1 g_2$ when (3), (4) are known for g_1 and g_2. Then

$$\begin{aligned}h_{f'}(\varphi(g_1 g_2, x)) &= h_{f'}(\varphi(g_1, \varphi(g_2, x))) \\ &= \varphi'(g_1', h_{f'} \varphi(g_2, x)) \\ &= \varphi'(g_1' g_2', h_{f'} x)\end{aligned}$$

for some g_1', g_2' in the same components of G as g_1, g_2. The product $g_1' g_2'$ lies in the same component as $g_1 g_2$, since G_1, the component of 1, is a normal subgroup of G. This proves (3) for $g = g_1 g_2$. The proof of (4) is similar:

$$\begin{aligned}\varphi'(g_1 g_2, h_{f'} x) &= \varphi'(g_1, \varphi'(g_2, h_{f'} x)) \\ &= \varphi'(g_1, h_{f'} \varphi(g_2'', x)) \\ &= h_{f'}(\varphi(g_1'', \varphi(g_2'', x))) \\ &= h_{f'}(\varphi(g_1'' g_2'', x))\end{aligned}$$

for some g_1'', g_2'' in the same component of G as g_1, g_2. The product $g_1'' g_2''$ lies in the same component as $g_1 g_2$, proving (4) for $g = g_1 g_2$.

From (4) we deduce that each $\mathscr{L}'_x$ is invariant by $\varphi'(g, \ )$, $g \in G_1$. For

$$\varphi'(g, x') = \varphi'(g, h_{f'}(x)) = h_{f'}(\varphi(g'', x)) \subset h_{f'}(\mathscr{L}_x) = \mathscr{L}'_{x'}$$

since $g'' \in G_1$ when $g \in G_1$. By (3) and the fact that $\mathscr{L}_x = \varphi(G_1, x)$, we also have

$$\begin{aligned} \mathscr{L}'_{x'} &= h_{f'}(\mathscr{L}_x) \\ &= h_{f'}\Big(\bigcup_{g \in G_1} \varphi(g, x)\Big) \subset \bigcup_{g' \in G_1} \varphi'(g', h_{f'} x) \\ &= \varphi'(G_1, x'). \end{aligned}$$

Thus, the connected component of the φ'-orbit through $x' \in \Lambda'$ is exactly $\mathscr{L}'_{x'}$, i.e. the φ'-orbits laminate Λ' and $\mathscr{L}' = h_{f'} \mathscr{L}$ is that lamination. Also, from (3), (4) we deduce

$$h_{f'}(\mathscr{L}_{\varphi(g, x)}) = \mathscr{L}'_{h_{f'}(\varphi(g, x))} = \mathscr{L}'_{\varphi'(g, h_{f'} x)}$$

which shows that not only is $h_{f'}$ an orbit conjugacy but it is also a G/G_1 parameter preserving conjugacy. This completes the proof of (3.9).

(3.10) **Corollary.** *If φ is a hyperbolic G-action then it is structurally stable. In particular, if φ satisfies Axiom A and if $\Omega = M$ then φ is Ω-stable.*

Proof. Apply (3.9) to $\Lambda = M$ and φ' near φ. Since $h_{\varphi'}$ is near the identity and is continuous, $h_{\varphi'}(M) = M$. Thus $\varphi' \sim \varphi$ and φ is structurally stable. We observed in § 2 that if φ satisfies Axiom $A(a)$ and $\Omega_\varphi = M$ then it is hyperbolic. Structural stability always implies Ω-stability.

§ 4. Ends of a Group

In [5] Hans Freudenthal defines the concept of "end" for a topological group or space. See also [3]. It is a notion concerning "directions that lead to ∞". The ends maximally compactify the group. Intuition says that in $\mathbb{R}$ there are *two* directions leading to ∞, $t \to +\infty$ and $t \to -\infty$, i.e. $\mathbb{R}$ has two ends. In $\mathbb{R}^2$ there is only one direction leading to ∞, i.e. all directions are equivalent and $\mathbb{R}^2$ has only one end. Likewise, $\mathbb{Z} \times \mathbb{Z}$. The cylinder $\mathbb{R} \times S^1$ has two ends. A compact space has no ends, because there is no way to go toward ∞.

It turns out that groups have exactly 0, 1, 2, or c ends [5]. Examples are: compact group have 0 ends, $\mathbb{R}^2$ has 1 end, $\mathbb{R}$ has 2 ends, and the free group on two generators has c ends. We shall show that if φ is an Axiom A G-action then either $\Omega_\varphi = M$ or else G has exactly two ends. It is reasonable that G has *at least* two ends when $\Omega_\varphi \neq M$ because there are G orbits in $M - \Omega$ connecting different basic sets of Ω_φ. See he figure below and use (2.1). The hard part is to eliminate the possibility of c ends. The presence in G's center of a copy of $\mathbb{Z}$ turns the trick.

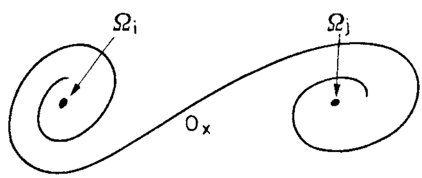

Fig. 2. Ends of an orbit in different basic sets

In [5], Freudenthal mainly develops end theory for finitely generated discrete groups whereas we are interested in general Lie groups. In [10] Hopf develops end theory for his groups but usually assumes they are connected. In this section, we present a self-contained treatment of end theory for compactly generated topological groups, not only because we need an end theory in exactly this generality, but also because we want to stress the existence of an end theory working simultaneously in the discrete and continuous categories.

CONVENTION: "LEFT" VERSUS "RIGHT". In everything done below, our order of group multiplication is the reverse of Freudenthal's. Our end theory is a left end theory, his a right end theory. This is not mere perversity, but the unhappy accident that left end theory is adapted to left group actions (homomorphisms $\varphi: G \to \mathrm{Diff}(M)$) while right end theory is adapted to right group actions (antihomomorphisms $\psi: G \to \mathrm{Diff}(M)$, $\psi(g_1 g_2) = \psi(g_2)\psi(g_1)$). Not being British, we prefer left group actions and accordingly develop a left end theory. Of course, right and left end theories are equivalent, but we apologize to the reader if extra reflection is needed when comparing our ends to Freudenthal's.

It is first necessary to do some topology in G, even though G may be discrete. We say that U is a *seed* of the topological group G iff U is a compact neighborhood of the identity, $U^{-1} = U$, and $\mathrm{Int}(U)$ generates G. One may imagine the group G as growing from U: $U^n \uparrow G$ as $n \to \infty$, where U^n = all products of $\leqq n$ elements of U. Any compactly generated group has a seed: let V be a compact neighborhood of the identity generating G, then put W = a compact neighborhood of V and $U = W \cup W^{-1}$.

Definitions. The *U-hull* of a set $S \subset G$ is

$$\mathscr{H}_U(S) = \{x \in G: x = us \text{ for some } u \in U, \ s \in S\}.$$

Thus, $\mathscr{H}_U S = US$ is a "thickening" of S. The U-frontier of S is

$$\partial_U(S) = \mathscr{H}_U(S) \cap \mathscr{H}_U(S^c)$$

when $S^c = G - S$. (Freudenthal calls $\partial_U S$ the "Franse" which literally means "fringe".) The set S is *U-bounded* iff $S \subset U^n$ for some n.

(4.1) **Proposition.** *Let G be compactly generated with seeds U, U'. A set S is U-bounded iff U'-bounded; $\partial_U(S)$ is bounded iff $\partial_{U'}(S)$ is bounded. If S is closed then $\mathscr{H}_U(S)$ is closed. A set is closed and bounded iff compact.*

Proof. The sets $(\mathrm{Int}\, U)^n$ are open, increase monotonically with n, and cover G. Since U' is compact, $U' \subset (\mathrm{Int}\, U)^n \subset U^n$ for some n. Symmetrically, $U \subset U'^{n'}$ for some n'. This proves the first two assertions of (4.1).

The U-hull of S is just the product $U \cdot S$. Since U is compact, $U \cdot S$ is closed whenever S is closed (a general fact).

Let S be a closed bounded set in G. Being bounded, S is contained in some U^n. Since U^n is compact and S is a closed subset of U^n, S is compact. Let S be a compact subset of G. Since $\{(\mathrm{Int}\, U)^n\}$ is an ascending open cover of G, $S \subset (\mathrm{Int}\, U)^n \subset U^n$ for some n. Hence S is bounded. Any compact set is closed. This completes the proof of (4.1).

Although these notions of "hull" and "frontier" are reminiscent of corresponding topological notions, they do not arise from some topology on G. The structure they define more closely resembles a Zeeman tolerance-space [19].

Definition. A set S is U-connected iff S cannot be divided, $S=S_1\cup S_2$, where S_1, S_2 are nonempty and $S_1\cap\mathscr{H}_U(S_2)=\emptyset=S_2\cap\mathscr{H}_U(S_1)$.

It is easy to see that S is U-connected iff each pair of its points g, g' can be joined by a chain in S, $g, u_1 g, u_1 u_2 g, \ldots, u_1 \ldots u_k g=g'$ where $u_1, \ldots, u_k\in U$. In particular, the whole group G is U-connected.

Definition. A *neighborhood of infinity* in G is an unbounded set $Q\subset G$ having $\partial_U Q$ bounded.

Definition. If G is a compactly generated group then an *end* of G is a class $\mathfrak{e}$ of subsets $Q\subset G$ such that

(a) each $Q\in\mathfrak{e}$ is a neighborhood of infinity.

(b) if $Q, Q'\in\mathfrak{e}$ then $Q\cap Q'\in\mathfrak{e}$.

(c) $\mathfrak{e}$ is maximal respecting (a), (b).

The set of ends of G is denoted $\mathscr{E}_G$. By (4.1), it does not matter which seed U is used in the definition.

If G is compact then (a), (b) are incompatible and G has no end. Conversely, any noncompact G has a least one end: let $\mathfrak{e}_0$ consist of all the complements of bounded sets. By Zorn's lemma, enlarge $\mathfrak{e}_0$ as much as possible without contradicting (a), (b). The result is an end $\mathfrak{e}$ of G. Note that $\mathfrak{e}_0$ is contained in every end by (c) but if $\mathfrak{e}_0$ *is* an end of G then $\mathfrak{e}_0$ is the only end of G.

If $\mathfrak{e}\in\mathscr{E}_g$ and if $P\subset G$ has $P\cap Q$ a neighborhood of infinity for all $Q\in\mathfrak{e}$ then (c) implies $P\in\mathfrak{e}$. For $\tilde{\mathfrak{e}}=\mathfrak{e}\cup\{P\cap Q\}_{Q\in\mathfrak{e}}$ satisfies (a), (b), and $\tilde{\mathfrak{e}}\supset\mathfrak{e}$. Similarly, if $P\Delta Q_0$ is bounded for some $Q_0\in\mathfrak{e}$ then $P\in\mathfrak{e}$. (By $A\Delta B$ we mean the symmetric difference $(A-B)\cup(B-A)$.) For if $Q\in\mathfrak{e}$ then

$$\partial_U(P\cap Q)\subset\mathscr{H}_U(Q\Delta Q_0)\cup\partial_U(Q\cap Q_0)$$

which is bounded, and so $\tilde{\mathfrak{e}}=\{P\cap Q\}_{Q\in\mathfrak{e}}$ satisfies (a), (b), and $\tilde{\mathfrak{e}}\supset\mathfrak{e}$. In particular the U-hull of any $Q\in\mathfrak{e}$ has $\mathscr{H}_U Q-Q=\partial_U Q$ and so $\mathscr{H}_U Q\in\mathfrak{e}$.

In a natural way, the ends of G can be adjoined to G, forming a compact space $\bar{G}$. In fact, the set of ends of any space X is the maximal, totally disconnected set compactifying X [5]. A sequence a_n in G converges to an end $\mathfrak{e}$ iff for each $Q\in\mathfrak{e}$, $a_n\in Q$ for all large n. A sequence $\mathfrak{e}_n$ of ends of G converges to an end $\mathfrak{e}$ iff for each $Q\in\mathfrak{e}$ and all large n there exist $Q_n\in\mathfrak{e}_n$ with $Q_n\subset Q$. (This definition of sequential convergence in $\bar{G}$ is equivalent to taking as neighborhood basis at $\mathfrak{e}$, $\{Q\}_{Q\in\mathfrak{e}}$.)

Products between ends are not defined but the products $a\mathfrak{e}$ and $\mathfrak{e}a$ for $a\in G$ make sense:

$$a\mathfrak{e}=\{aQ\}_{Q\in\mathfrak{e}} \qquad \mathfrak{e}a=\{Qa\}_{Q\in\mathfrak{e}}.$$

First note that aQ is unbounded and if $a\in U^m$ then $\partial_U(aQ)\subset\partial_{U^{m+1}}(Q)$ which is bounded. Also Qa is unbounded and $\partial_U(Qa)=(\partial_U Q)a$. Clearly $\{aQ\}_{Q\in\mathfrak{e}}$ and $\{Qa\}_{Q\in\mathfrak{e}}$ satisfy (b), (c) so $a\mathfrak{e}$, $\mathfrak{e}a$ are ends.

Let us observe that $a\mathfrak{e}\equiv\mathfrak{e}$. It suffices to show that $(aQ)\Delta Q$ is bounded, $Q\in\mathfrak{e}$. Let $a\varepsilon U^m$. If $x\in Q-aQ$ then $q=x=aq'$ for some $q\in Q$, $q'\in Q^c$ and so $a^{-1}x\in\mathscr{H}_{U^m}(Q^c)$, $1x\in\mathscr{H}_{U^m}(Q)$ imply $x\in\partial_{U^m}(Q)$. Thus $(aQ)\Delta Q\subset\partial_{U^m}(Q)$ which is bounded. Similarly, left multiplication by a fixed element $a \in G$ does not affect convergence $x_n \to \mathfrak{e} \in \mathscr{E}_G$, i.e. $(a x_n)$ and (x_n) have the same limit points in $\mathscr{E}_G$.

Right multiplication is another matter. Each $a \in G$ defines a homeomorphism $r_a \colon \bar{G} \to \bar{G}$ whose inverse is $r_{a^{-1}}$. Some ends may move, others remain fixed.

It would seem natural to define

$$\mathfrak{e}^{-1} = \{Q^{-1}\}_{Q \in \mathfrak{e}} \qquad Q^{-1} = \{q^{-1}\}_{q \in Q}.$$

Unless G is Abelian, $\mathfrak{e}^{-1}$ is not a left end, it is a right end. For if $Q \in \mathfrak{e}$ then,

$$\begin{aligned}(\partial_U Q)^{-1} &= (\mathscr{H}_U Q)^{-1} \cap (\mathscr{H}_U (Q^c)^{-1} = (UQ)^{-1} \cap (U(Q^c)^{-1}) \\ &= (Q^{-1} U^{-1}) \cap (Q^{-1})^c U^{-1} = \partial^r_U (Q^{-1})\end{aligned}$$

where ∂^r_U denotes the right frontier (Freudenthal's convention). Thus

inversion induces a natural bijection $\mathscr{E}^l_G \leftrightarrow \mathscr{E}^r_G$.

The following proposition says that U-connectedness is no serious restriction, especially for ends.

(4.2) **Proposition.** (i) *Any* $S \subset G$ *with* $\partial_U S$ *bounded has only finitely many* U-*components.* (ii) *Any* $Q \in \mathfrak{e} \in \mathscr{E}_G$ *contains a unique maximal,* U-*connected* $Q' \in \mathfrak{e}$.

Proof. (i) The set $\partial_U S$ has only finitely many components, say $K_1, \ldots, K_m$, because $\partial_U S$ is contained in a compact set and points in distinct U-components of a set clearly cannot accumulate. Let $S_1, \ldots, S_m$ be the largest U-connected subsets of S containing $K_1, \ldots, K_m$. If $\partial_U S = \emptyset$ then (4.2i) is true because G is U-connected. Thus, we may assume $\partial_U S \neq \emptyset$. Choose any $s \in S$ and $s' \in \partial_U S$. Since G is U-connected consider a U-chain from s to s'. By definition of $\partial_U S$, it does not leave S until $\partial_U S$. Hence, every $s \in S$ belongs to $S_1 \cup \cdots \cup S_m$, i.e. (4.2i) is proved.

(ii) Let $Q \in \mathfrak{e}$ and let $Q = Q_1 \cup \cdots \cup Q_n$ be the distinct U-connected components of Q arising from (i). By maximality, $\mathscr{H}_U(Q_i) \cap Q_j = \emptyset$ for each $i \neq j$. Thus $\partial_U Q_i \subset \partial_U Q$ and is bounded. If all the Q_i fail to lie in $\mathfrak{e}$ then each Q_i misses some $S_i \in \mathfrak{e}$ and so $Q = Q_1 \cup \cdots \cup Q_u$ misses $S_1 \cap \cdots \cap S_n \in \mathfrak{e}$, contradicting (b) in the definition of ends. Also by (b), only one Q_i can lie in $\mathfrak{e}$ since the Q_i are disjoint. This completes the proof of (4.2ii).

The following four theorems are what we require from end theory. They answer the questions: How many ends does a group have? How are the ends of groups, subgroups, and factor groups related? How can we recognize a one-ended group? How nearly does a two ended group resemble $\mathbb{Z}$?

(4.3) **Theorem.** *If* G *is a compactly generated locally compact group then* G *has* 0, 1, 2, *or* c *ends.*

(4.4) **Theorem.** *Let* G *be a compactly generated locally compact group and* H *be a normal compactly generated closed subgroup of* G. (i) *If* H *is compact then there is a natural bijection between* $\mathscr{E}_G$ *and* $\mathscr{E}_{G/H}$. (ii) *If* G/H *is bounded then there is a natural bijection between* $\mathscr{E}_G$ *and* $\mathscr{E}_H$.

(4.5) **Theorem.** *Let G be a compactly generated locally compact group and H be a normal, compactly generated closed subgroup of G. If H and G/H are unbounded then G is one-ended.*

(4.6) **Theorem.** *Any two-ended group G with ends $\mathfrak{e}_-, \mathfrak{e}_+$ contains a closed, infinite cyclic subgroup $H=\{h^n\}_{n\in\mathbb{Z}}$ such that $h^n \to \mathfrak{e}_\pm$ as $n\to\pm\infty$ and G/H, $H\backslash G$ are bounded.*

Remark 1. If G is discrete then (4.3) is Satz 3.3 of [5], (4.4ii) is [10, Satz 4], and (4.6) is [5, 6.14]; (4.5) is a unification of Satz 6 and Zusatz zu Satz 6 of [5].

Remark 2. When H is not normal G/H is not a group, but by G/H being bounded we mean that each coset gH be contained in $\pi_l U^m$ for some fixed m. The map π_l is the projection $G\to G/H$. Similarly for $H\backslash G$. Note that G/H is bounded iff $H\backslash G$ is bounded. For inversion in G induces a homeomorphism $H\backslash G \leftrightarrow G/H$ which exchanges $\pi_r(U^m)$ and $\pi_l(U^m)$. Using ideas like this, versions of (4.4, 5) can be proved when H is not normal. For instance (4.4i) becomes $\mathscr{E}_G \leftrightarrow \mathscr{E}_{H\backslash G}$.

Here is the key lemma.

(4.7) **Lemma.** *Let $\mathfrak{e}_1$, $\mathfrak{e}_2$ be ends of G and $a_n\to\mathfrak{e}_1$ with $a_n\in G$. Then either a_n^{-1} accumulates at $\mathfrak{e}_2$ or else $\mathfrak{e}_2 a_n \to \mathfrak{e}_1$.*

Proof [5, 6.6]. Suppose that a_n^{-1} does not accumulate at $\mathfrak{e}_2$. Then $\mathfrak{e}_2$ has a compact neighborhood $\bar{Q}_2$ in $\bar{G}$ such that a_n^{-1} does not accumulate at any point of $\bar{Q}_2$. Let $\bar{Q}_1$ be any neighborhood of $\mathfrak{e}_1$ and let $Q_i=\bar{Q}_i\cap G$, $i=1,2$. Since $\partial_U(Q_2)$ is bounded,

$$\partial_U(Q_2)\,a_n=\partial_U(Q_2 a_n)\subset Q_1 \qquad n \text{ large}.$$

Indeed it is easy to see that any bounded set S is sent inside Q_1 by $S\mapsto S a_n$, n large.

Since $\partial_U(Q_1)=\partial_U(Q_1^c)$ is bounded, Q_1^c has only finitely many U-connected components by (4.2). Suppose $Q_2 a_n \not\subset Q_1$ for infinitely many values of n. For each such n, $Q_2 a_n$ contains whole U-connected components of Q_1^c because U-chains in Q_1^c do not meet $\partial_U(Q_2 a_n)$. Since Q_1^c has only finitely many of them, one gets contained in $Q_2 a_n$ infinitely often, and we can choose a fixed x

$$x\in Q_1^c\cap Q_2 a_n \qquad \text{infinitely often.}$$

In other words, $x a_n^{-1}\in Q_2$, and so some accumulation points of $x a_n^{-1}$ lie in $\bar{Q}_2$. But $x a_n^{-1}$ and a_n^{-1} have the same accumulation points. This contradicts the choice of $\bar{Q}_2$. Hence $Q_2 a_n\subset Q_1$ for all large n, i.e. $\mathfrak{e}_2 a_n\to\mathfrak{e}_1$. This completes the proof of (4.7).

Proof of (4.3). Let G have at least three ends. We must show it has c ends, so it suffices to prove that the set $\mathscr{E}$ of ends is perfect.

Choose any end $\mathfrak{e}$ and a sequence $a_n\to\mathfrak{e}$, $a_n\in G$. The sequence a_n^{-1} is unbounded and so we may assume $a_n^{-1}\to\mathfrak{e}'$ some end of G. Choose two ends of G distinct from $\mathfrak{e}'$ and each other, say $\mathfrak{e}''$, $\mathfrak{e}'''$. By Lemma 4.7, $\mathfrak{e}'' a_n\to\mathfrak{e}$ and $\mathfrak{e}''' a_n\to\mathfrak{e}$, because a_n^{-1} does not accumulate at $\mathfrak{e}''$ or $\mathfrak{e}'''$. From this it follows that $\mathfrak{e}$ is an accumulation point of other ends – either of $\mathfrak{e}'' a_n$, $\mathfrak{e}''' a_n$, or of both. (Note that right multiplication by an element $a\in G$ gives a bijection of $\mathscr{E}$ to itself, so $\mathfrak{e}'' a_n$ and $\mathfrak{e}''' a_n$ cannot both equal $\mathfrak{e}$. Since $\mathscr{E}$ is compact by construction, this shows that it is perfect and (4.3) is proved.

Proof of (4.4i). H is a compact, normal subgroup of G and $\pi\colon G\to G/H$ is the projection. Define

$$\pi\mathfrak{e}=\{\pi Q\}_{Q\in\mathfrak{e}}\qquad \mathfrak{e}\in\mathscr{E}_G$$
$$\pi^{-1}\bar{\mathfrak{e}}=\{\pi^{-1}T\}_{T\in\bar{\mathfrak{e}}}\qquad \bar{\mathfrak{e}}\in\mathscr{E}_{G/H}.$$

We claim that $\pi\mathfrak{e}$, $\pi^{-1}\bar{\mathfrak{e}}$ lie in unique ends of G/H, G, say $\pi_{\#}\,\mathfrak{e}$, $\pi_{\#}^{-1}\,\bar{\mathfrak{e}}$, and the maps $\pi_{\#}\colon\mathscr{E}_G\to\mathscr{E}_{G/H}$, $\pi_{\#}^{-1}\colon\mathscr{E}_{G/H}\to\mathscr{E}_G$ are inverse to each other. This is hardly surprising since the effect of H is to thicken things up by a bounded amount and this does not affect ends.

Since π is continuous, the π-image of each compact set is compact. Since H is compact the π^{-1}-image of each compact set is compact. Thus, π and π^{-1} preserve boundedness.

Given $\mathfrak{e}\in\mathscr{E}_G$, we want to show $\pi\mathfrak{e}$ lies in a unique end of G/H. It suffices to prove that $\pi\mathfrak{e}=\{\pi Q\}_{Q\in\mathfrak{e}}$ satisfies properties (a), (b) in the definition of ends since Zorn's Lemma lets us extend $\pi\mathfrak{e}$ to satisfy (c).

(a) Let $Q\in\mathfrak{e}$, let V be a seed of $H\backslash G=G/H$ and let $U=\pi^{-1}(V)$. Then U is a seed of G and $U\supset H$. We claim

$$\partial_V(\pi Q)\subset\pi\,\partial_{U^3}(Q).$$

Let $Hg\in\partial_V(\pi Q)$. Then $Hg=v_1Hg_1$ for some $Hg_1\in\pi Q$ and some $v_1\in V$. Also $Hg=v_2Hg_2$ for some Hg_2 in $(\pi Q)^c$ and some $v_2\in V$. Thus

$$v_2h_2g_2=g=v_1h_1g_1\qquad \begin{aligned}x_1&=h_1'g_1\in Q\\ x_2&=h_2'g_2\in Q^c\end{aligned}$$

for some $h_1,h_1',h_2,h_2'\in H$, $v_1,v_2\in V$. Thus

$$x_i=h_i'g_i=h_i'h_i^{-1}v_i^{-1}g\in\mathscr{H}_{U^3}(g)$$

$i=1,2$. This proves that $g\in\partial_{U^3}(Q)$. Since $\partial_{U^3}(Q)$ is bounded so is $\partial_V(\pi Q)$. Clearly πQ is unbounded since Q is. This proves (a) for $\pi\mathfrak{e}$.

(b) If $Q,Q'\in\mathfrak{e}$ and U is as above then by (c) for $\mathfrak{e}$, $\pi^{-1}\pi Q$, $\pi^{-1}\pi Q'\in\mathfrak{e}$. By (b) for $\mathfrak{e}$, $\pi^{-1}\pi Q\cap\pi^{-1}\pi Q'\in\mathfrak{e}$ and so $\pi Q\cap\pi Q'=\pi(\pi^{-1}\pi Q\cap\pi^{-1}\pi Q')\in\pi\mathfrak{e}$ proving (b) for $\pi\mathfrak{e}$.

Hence $\pi_{\#}\colon\mathscr{E}_G\to\mathscr{E}_{H\backslash G}$ is well defined.

The proof that $\pi_{\#}^{-1}\colon\mathscr{E}_{G/H}\to\mathscr{E}_G$ is well defined is slightly easier. Given $\bar{\mathfrak{e}}\in\mathscr{E}_{G/H}$ we want to show that $\{\pi^{-1}T\}_{T\in\bar{\mathfrak{e}}}=\pi^{-1}\bar{\mathfrak{e}}$ satisfies (a), (b) in the definition of ends. Let U, V be as above. We observe that

$$\partial_U(\pi^{-1}T)\subset\pi^{-1}\partial_V(T)$$

which is bounded. $\pi^{-1}T$ is unbounded since T is. This proves (a) for $\pi^{-1}\bar{\mathfrak{e}}$; (b) is clear since $\pi^{-1}(T\cap T')=\pi^{-1}T\cap\pi^{-1}T'$. Hence $\pi_{\#}^{-1}\colon\mathscr{E}_{G/H}\to\mathscr{E}_G$ is well defined.

Since $\pi^{-1}\pi(Q)\in\mathfrak{e}$ for all $Q\in\mathfrak{e}\in\mathscr{E}_G$, we have $\pi_{\#}^{-1}\circ\pi_{\#}=$ identity on E_G. Since $\pi\circ\pi^{-1}=$ identity on G/H, we have $\pi_{\#}\circ\pi_{\#}^{-1}=$ identity on $\mathscr{E}_{G/H}$. Hence, $\pi_{\#}{}^{-1}=\pi_{\#}^{-1}$, completing the proof of (4.4i).

Proof of (4.4ii). H is a normal, compactly generated closed subgroup of G and G/H is bounded. Let U be a seed of G and H so large that $G/H\subset\pi U$ where

$\pi: G \to G/H$ is the projection. For $\mathfrak{e} \in \mathscr{E}_G$, $\bar{\mathfrak{e}} \in \mathscr{E}_H$ we define

$$i\mathfrak{e} = \{Q \cap H\}_{Q \in \mathfrak{e}} \qquad j\bar{\mathfrak{e}} = \{\mathscr{H}_U T\}_{T \in \bar{\mathfrak{e}}}$$

and claim that i, j induce inverse bijections between $\mathscr{E}_G$, $\mathscr{E}_H$.

Since $\pi U = G/H$, all points of G are in the U-hull of H, $\mathscr{H}_U H = G$. Let $Q \in \mathfrak{e} \in \mathscr{E}_G$. Clearly $Q \cap H$ has bounded frontier, $\partial_{U \cap H}(Q \cap H)$. Also $Q \cap H$ is unbounded since Q is unbounded and points of Q far from $\partial_{U \cap H}(Q \cap H)$ have their whole U-hulls (including thus some points of H) in the set Q. This proves (a) for $i\mathfrak{e}$; (b) is clear since $(Q \cap H) \cap (Q' \cap H) = (Q \cap Q') \cap H$. Thus $i_\# : \mathscr{E}_G \to \mathscr{E}_H$ is well defined by $i\mathfrak{e} \subset i_\# \mathfrak{e}$.

Let $T \in \bar{\mathfrak{e}} \in \mathscr{E}_H$. We claim that

$$\partial_U(\mathscr{H}_U T) \subset \mathscr{H}_U(\partial_{U^3 \cap H}(T)).$$

Let $g \in \partial_U(\mathscr{H}_U T)$. Then

$$g = u_1 g_1 \qquad g_1 = u_1' t_1$$

$$g = u_2 g_2$$

for some $g_1 \in \mathscr{H}_U T$, $g_2 \in (\mathscr{H}_U T)^c$, $u_1, u_2, u_1' \in U$, $t_1 \in T$. But $G = \mathscr{H}_U H$ so $g = uh$ and $g_2 = u_2' h_2$ for some $h, h_2 \in H$, $u, u_2' \in U$. Since $g_2 \notin \mathscr{H}_U T$, $h_2 \in H - T$. Thus

$$u^{-1} u_2 u_2' h_2 = u^{-1} u_2 g_2 = h = u^{-1} g = u^{-1} u_1 u_1' t_1$$

for $h_2 \in H - T$, $t_1 \in T$. This proves $h \in \partial_{U^3 \cap H}(T)$ and $g \in \mathscr{H}_U(\partial_{U^3 \cap H}(T))$ as claimed. Since the latter is bounded, so is the former. Clearly $\mathscr{H}_U T$ is unbounded since T is. This proves (a) for $j\bar{\mathfrak{e}} = \{\mathscr{H}_U T\}_{T \in \bar{\mathfrak{e}}}$; (b) is clear since $\mathscr{H}_U(T \cap T') = \mathscr{H}_U(T) \cap \mathscr{H}_U(T')$. Hence $j_\# : \mathscr{E}_H \to \mathscr{E}_G$ is well defined by $j\bar{\mathfrak{e}} \subset j_\# \bar{\mathfrak{e}}$.

The composition $i_\# j_\#$ is the identity on $\mathscr{E}_H$ because $ij(T)$ is just the $U \cap H$-hull of $T \in \bar{\mathfrak{e}}$, and the hull of any $T \in \bar{\mathfrak{e}}$ is in $\bar{\mathfrak{e}}$. The composition $j_\# i_\#$ is the identity on $\mathscr{E}_G$ since the difference between $ji(Q)$ and Q is bounded, $Q \in \mathfrak{e}$, and so $ji(Q) \in \mathfrak{e}$. Thus, $j_\# = i_\#^{-1}$ and (4.4ii) is proved.

Proof of (4.5). H is a normal, compactly generated closed subgroup of G and H, G/H are unbounded. We must prove G is one-ended. Since G contains the unbounded subset H, it is unbounded, noncompact, and thus has $\geqq 1$ ends.

Let U be a seed of G and H. Let $Q \in \mathfrak{e} \in \mathscr{E}_G$. We shall show Q^c is bounded, which implies G is one-ended.

Let $U^n \supset \partial_U Q$. Since $G/H = H \backslash G$ is unbounded there are many cosets Hg not meeting U^n. Since H is unbounded, so is coset Hg. Thus, many points of every coset do not lie in U^n and many cosets miss U^n altogether.

As we saw before, H is U-connected and therefore so is each coset Hg. Thus, if Hg is a coset which misses U^n then $(Q \cap Hg) \cup (Q^c \cap Hg)$ would be a division of Hg having the U-hull of one piece disjoint from the U-hull of the other. Hence, if Hg misses U^n then either $Hg \subset Q$ or else $Hg \subset Q^c$.

Consider any coset Hg and write $g = u_1 \ldots u_k$, $u_i \in U$. Choose any $h \in H \cap (U^{n+k})^c$, i.e. choose an element of H far from $\partial_U Q$. Since H is not bounded, this is possible. Then

$$h, u_k h, \ldots, u_1 \ldots u_k h = gh$$

is a U-chain from h to gh avoiding $\partial_U Q$. Since H is normal, $gh \in gH = Hg$. Thus, h and some element of Hg are both in Q or both in Q^c.

Suppose $H \cap Q$ is bounded, say $H \cap Q \subset U^m$. Clearly $m = n-1$. Every coset Hg contains an element of Q^c by the preceding chain-construction. Thus, the cosets Hg missing U^n are all in Q^c. On the other hand, if $g = u_1 \dots u_k$, $k \leqq n$, then each point of $Hg \cap (U^{2n})^c$ can be joined to some $h \in H \cap (U^n)^c$ by a U-chain avoiding $\partial_U Q$, and since such an h lies in Q^c, we get

$$Hg \cap (U^{2n})^c \subset Q^c$$

which shows that Q is bounded, in fact $Q \subset U^{2n}$. This contradicts (a) in the definition of ends, so $H \cap Q$ cannot be bounded.

Since $H \cap Q$ is unbounded, the chain construction shows that every coset Hg contains an element of Q. Thus, all cosets missing U^n are wholly contained in Q. Since $H \backslash G$ is unbounded, there is some coset, say Hg_1 with $g_1 = u_{11} \dots u_{1k}$, $k \geqq n+1$, which misses U^n; $Hg_1 \subset Q$. Let $g = u_1 \dots u_l$, $l \leqq n$. Using the chain construction twice we can find a U-chain from any $x \in Hg \cap (U^{2n+k})^c$ to $g_1 H = Hg_1$, avoiding $\partial_U Q$:

$$\begin{aligned} Hg &= gH \ni x \\ &= gh \\ &= u_1 \dots u_l h,\ u_2 \dots u_l h,\ \dots,\ h,\ u_{1k} h,\ \dots,\ u_{11} \dots u_{1k} h \\ &= g_1 H \in g_1 H \\ &= Hg_1. \end{aligned}$$

Thus, $Hg \cap (U^{2n+k})^c \subset Q$ and so

$$Q^c \subset U^{2n+k}.$$

This proves that every $Q \in \mathfrak{e} \in \mathscr{E}_G$ is the complement of a bounded set, i.e. $\mathfrak{e} \equiv \mathfrak{e}_0$ and G is one-ended.

To prove (4.6) we use three lemmas. Let r_a denote right multiplication on $\bar{G}$ by a, $r_a: x \mapsto xa$.

(4.8) **Lemma.** *Each $r_a \in \mathrm{Homeo}(\bar{G})$ and $a \mapsto r_a$ is a continuous monomorphism $G \to \mathrm{Homeo}(\bar{G})$.*

Proof. Continuity of r_a at points of G is a consequence of G being a topological group. If $\mathfrak{e}$ is an end of G then the definition $\mathfrak{e}a = \{Qa\}_{Q \in \mathfrak{e}}$ makes it clear that r_a is continuous at $\mathfrak{e}$. Since $r_{a^{-1}} = r_a^{-1}$, $r_a \in \mathrm{Homeo}(\bar{G})$. Continuity of $a \mapsto r_a$ need only be checked at $a=1$, the identity of G, and at some end $\mathfrak{e}$ of G. Let $a_n \to 1$. (If G is discrete then $a_n \equiv 1$.) Let $\bar{Q}$ be a neighborhood of $\mathfrak{e}$ in $\bar{G}$, $Q = \bar{Q} \cap G$. Then $r_{a_n}(\mathfrak{e}) = \{Qa_n\}_{Q \in \mathfrak{e}}$. Clearly $Q' a_n \subset Q$ for n large, $Q' = Q - \partial_U Q$, and any fixed seed U of G. Hence $r_{a_n}(\mathfrak{e}) \to \mathfrak{e}$ and (4.8) is proved.

(4.9) **Lemma.** *Let S be the isotropy subgroup of the ends of G,*

$$S = \{s \in G \colon \mathfrak{e}s = \mathfrak{e} \textit{ for all ends } \mathfrak{e} \textit{ of } G\}.$$

Then S is a closed, normal subgroup of G. If G is two-ended then either $G = S$ or $G/S \approx \mathbb{Z}_2$.

Proof. S is clearly a normal subgroup since gsg^{-1} fixes the ends of G, $g\in G$. By (4.8) it is closed. If G is two-ended then each $g, g'\in G-S$ switch the ends of G. Thus, so does g^{-1} and $g^{-1}g'\in S$ so $g'\in gS$. This shows G/S has only two cosets, so $G/S\approx\mathbb{Z}_2$.

Definition [5]. Let G be a compactly generated group. G is *elliptic, parabolic*, or *hyperbolic* iff G has exactly 0, 1, or 2 ends $\mathfrak{e}$ which are fixed under multiplication by all elements of G, $g\mathfrak{e}=\mathfrak{e}=\mathfrak{e}g$, $g\in G$.

It is not hard to see that *a group with infinitely many ends cannot be hyperbolic.* For if $\mathfrak{e}_1, \mathfrak{e}_2$ are the right invariant ends and $\mathfrak{e}_3$ is a third end then we can find a sequence in G, $a_n\to\mathfrak{e}_3$ such that $a_n^{-1}\to\mathfrak{e}_4$, use (4.7) to conclude (by right-invariance of $\mathfrak{e}_1, \mathfrak{e}_2$) that a_n^{-1} accumulates at $\mathfrak{e}_1$ and at $\mathfrak{e}_2$, so $\mathfrak{e}_1=\mathfrak{e}_4=\mathfrak{e}_2$, a contradiction to the distinctness of $\mathfrak{e}_1, \mathfrak{e}_2$. Hence, by (4.3), every compactly generated group is hyperbolic, parabolic, or elliptic.

Question. A problem which Freudenthal poses, but which remains open we think, is whether a group with infinitely many ends can be parabolic – i.e. have exactly one invariant end.

(4.10) **Lemma.** *Let H be a hyperbolic group and U be a seed of H. If a_n tends to one end of H then $a_n^{-1}U$ tends to the other.*

Proof. Let the ends be $\mathfrak{e}_\pm$ and suppose $a_n\to\mathfrak{e}_+$ but $a_n^{-1}u_n$ does not accumulate at $\mathfrak{e}_-$ for some sequence $u_n\in U$. By (4.7) $\mathfrak{e}_-(a_n^{-1}u_n)\to\mathfrak{e}_+$ since

$$(a_n^{-1}u_n)^{-1}=u_n^{-1}a_n\in\mathscr{H}_U(a_n)\to\mathfrak{e}_+.$$

Since H is two-ended, convergence means equality: $\mathfrak{e}_-(a_n^{-1}u_n)=\mathfrak{e}_+$, n large, contradicting hyperbolicity.

Remark. If hyperbolicity is weakened to two-endedness than (4.10) become false. For example, let $G=\mathbb{Z}_2\cdot\mathbb{Z}$ where $\cdot$ means semi-direct-product relative to the $\mathbb{Z}_2$-action on $\mathbb{Z}$, $m\mapsto -m$. (Thus, writing $\mathbb{Z}_2$ multiplicatively, $(a,n)\cdot(b,m)=(ab, n+am)$.) Then G is two ended but $(-1,n)=(-1,n)^{-1}$ both tend to the same end as $n\to\infty$.

Question. If G is two-ended and H is the isotropy subgroup of the ends as in (4.9) then is $G\approx\mathbb{Z}_2\cdot H$?

Proof of (4.6). We may assume the ends of G are right invariant. Otherwise replace G by the subgroup S of index two constructed in (4.9). Let U be a seed of G. Let $Q_\pm\in\mathfrak{e}_\pm$ be such that

$$\mathscr{H}_U Q_-\cap\mathscr{H}_U Q_+=\emptyset=U\cap\mathscr{H}_U Q_\pm.$$

Let V be a large bounded set so $Q_-\cup V\cup Q_+=G$, $V\supset U$.

There is a $Q'_+\in\mathfrak{e}_+$, $Q'_+\subset Q_+$, such that any $a\in Q'_+$ has $a^{-1}\in Q_-$. This is implied by (4.10). There is a $Q''_+\in\mathfrak{e}_+$, $Q''_+\subset Q'_+\subset Q_+$, such that $Q_+a\subset Q_+$ for all $a\in Q_+$. Otherwise there exists a sequence $a_n\to\mathfrak{e}_+$ such that $Q_+a_n\not\subset Q_+$. Since $\partial_U Q_+$ is bounded, $(\partial_U Q_+)a_n\subset Q_+$, n large. As in the proof of (4.7) this implies that Q_+a_n contains one of the finitely many U-connected components of Q^c_+ infinitely often, i.e. $x\in Q_+a_n$ for some constant $x\in Q^c_+$. This says $xa_n^{-1}\in Q_+$, for some fixed x. But by (4.10), $a_n^{-1}\to\mathfrak{e}_-$ and left multiplication by x does not affect such convergence.

Hence Q''_+ exists as asserted. Symmetrically, there exist $Q''_- \subset Q'_- \subset Q_-$, $Q'_-, Q''_- \in \mathfrak{e}_-$, such that any $a \in Q'_-$ has $a^{-1} \in Q_+$ and any $a \in Q''_-$ has $Q_- a \subset Q_-$.

Choose any $h \in Q''_+$ with $h^{-1} \in Q''_-$. Since $h \in Q''_+$, $Q_+ h \subset Q_+$. In particular, h^2, $h^3, \ldots$ all belong to Q_+. We claim $h^n \to \mathfrak{e}_+$. Otherwise there is an infinite sequence of powers h^k occurring in some bounded subset of G. By local compactness, (4.1), a subsequence of these converge to some $x \in G$. But h^n, h^m being near x means $h^n(h^m)^{-1}$ and $h^m(h^n)^{-1}$ are near 1, in particular they are in U. We may assume $n<m$. Then

$$h^m(h^n)^{-1} = h^{m-n} = h^k h = u \in U \Rightarrow h^{-1} = u^{-1} h^k$$

with $k = m-n-1 \geqq 0$. We already saw that $h, h^2, \ldots, \in Q_+$. Thus,

$$h^{-1} = u^{-1} h^k \in U \cup \mathscr{H}_U Q_+$$

which is disjoint from Q_- by our original choice of $Q_\pm$. This contradicts $h^{-1} \in Q_-$. Therefore $h^n \to \mathfrak{e}_+$ as $n \to \infty$. By (4.10), $h^n \to \mathfrak{e}_-$ as $n \to -\infty$. Thus, $H = \{h^n\}$ is a closed infinite cyclic subgroup of G. It remains to show G/H and $H \backslash G$ are bounded.

If H happens to be normal then by (4.5), $H \backslash G = G/H$ is bounded, for otherwise G would be one-ended. But there is no reason to think H is normal. Instead, consider again the sets $Q_\pm \in \mathfrak{e}_\pm$ used above. Since $Q_+ h \subset Q_+$ we get a decreasing sequence $Q_+ \supset Q_+ h \supset Q_+ h^2 \supset \ldots$. Since $h^n \to \mathfrak{e}_+$, $(\partial_U Q_+) h^n \to \mathfrak{e}_+$, $\partial_U Q_+$ being bounded. Hence $\bigcap_{n \geqq 0} Q_+ h^n = \emptyset$. Also, since $Q_+, Q_+ h \in \mathfrak{e}_+$, the difference $Q_+ - Q_+ h$ is a bounded set. Let W be a seed of G containing V and $Q_+ - Q_+ h$. Then

$$Wh \supset (Q_+ - Q_+ h) h = Q_+ h - Q_+ h^2$$

and in general $Wh^k \supset Q_+ h^k - Q_+ h^{k+1}$. Hence $W \cup Wh \cup Wh^2 \cup \cdots \supset Q_+ \cup V$. This says that $\mathscr{H}_W(H) \supset Q_+ \cup V$.

We chose h so that $h^{-1} \in Q''_-$. Thus everything true for h relative to Q_+ is true for h^{-1} relative to Q_-. That means (enlarging W to also include the bounded set $Q_- - Q_- h^{-1}$) $\mathscr{H}_W(H) = G$ and so each $g \in G$ can be expressed $g = wh$ for some $h \in H$, $w \in W$. Thus, G/H is bounded. For each $g \in G$, $g^{-1} = w' h'$ for some $h' \in H$, $w' \in W$. Thus each $g = (h')^{-1}(w')^{-1}$ and so $H \setminus G$ is bounded too.

From (4.4, 5) we deduce

(4.11) **Corollary.** *If φ is an Axiom A G-action on M with more than one orbit then G has either one or two ends.*

Proof. Let H be the subgroup of G generated by a hyperbolic element f. Since f^n does not cluster in G, H is a closed non-compact subgroup of G isomorphic to $\mathbb{Z}$. Since f is central, H is normal in G. If G/H is bounded then (4.4ii) says G is two-ended because H is. If G/H is unbounded then (4.5) says G is one-ended.

We have now come to the main goal of §4.

(4.12) **Theorem.** *If φ is an Axiom A G-action on M then either $\Omega_\varphi = M$ or else G is hyperbolic.*

Proof. Assume $\Omega_\varphi \neq M$. By (4.11) G has either one or two ends; we must show it does not have just one end and that the ends are right-invariant.

The Ω-Decomposition Theorem, (2.3), says that $\Omega_\varphi = \Omega_1 \cup \cdots \cup \Omega_m$ disjointly with φ topologically transitive on each Ω_i. Let $x \in M - \Omega_\varphi$ and consider the orbit of x, O_x. By (3.6), ∂O_x cannot be entirely in one basic set, it must meet at least two, say

$$\partial O_x \subset \Lambda_- \cup \Lambda_+ \qquad \Lambda_+ = \Omega_i \qquad \Lambda_- = \bigcup_{j \neq i} \Omega_j$$

and $f^{n_k} x \to x_\pm \in \Lambda_\pm$ for some sequence $n_k \to \pm\infty$ as $k \to \pm\infty$.

Let U be a fixed seed of G. Let $N_\pm$ be small disjoint neighborhoods of $\Lambda_\pm$. Then $G = L_- \cup S \cup L_+$ where

$$S = \{g \in G : gx \in M - (N_- \cup N_+)\}$$

$$L_\pm = \{g \in G : gx \in N_\pm\}.$$

Clearly S is compact. Since $\Lambda_\pm$ are φ-invariant $\varphi(U, \Lambda_\pm) = \Lambda_\pm$. Hence

$$\varphi(U, N_-) \cap \varphi(U, N_+) = \emptyset$$

for small $N_\pm$ by continuity of φ.

For each $ug_\pm \in \mathscr{H}_U(L_\pm)$, $g_\pm \in L_\pm$,

$$\varphi(ug_\pm, x) = \varphi(u, \varphi(g_\pm, x)) \in \varphi(U, N_\pm)$$

and so $\mathscr{H}_U(L_-) \cap \mathscr{H}_U(L_+) = \emptyset$. [Here we *must* use a left-end theory.] Hence $\partial_U(L_\pm) \subset \mathscr{H}_U S$ so $\partial_U(L_\pm)$ is bounded. Since $L_\pm$ contains infinitely many powers of f and since $\mathbb{Z} \approx \{f^n\}$ is a closed noncompact subgroup of G, $L_\pm$ is unbounded. By Zorn's Lemma there are ends $\mathfrak{e}_\pm$ containing $L_\pm$. Since $L_- \cap L_+ = \emptyset$, $\mathfrak{e}_- \neq \mathfrak{e}_+$, i.e. G has $\geqq 2$ ends. By (4.11), it has exactly two, $\mathfrak{e}_-$ and $\mathfrak{e}_+$.

It remains to show $\mathfrak{e}_\pm$ are right-invariant. Suppose G is two-ended but not hyperbolic. Consider the isotropy subgroup H of the ends as in (4.9). Choose any $g \in U \cap (G - H)$. Then g switches the ends: $\mathfrak{e}_+ g = \mathfrak{e}_-$, $\mathfrak{e}_- g = \mathfrak{e}_+$. Hence, if $f^{n_k} \in L_+$ and k is very large then $f^{n_k} g \in L_-$. (We have *not* yet established that $f^n \to \mathfrak{e}_\pm$ as $n \to \pm\infty$, merely that this holds for a subsequence.) Since f is central,

$$\varphi(f^{n_k} g, x) = \varphi(g f^{n_k}, x) = \varphi(g, \varphi(f^{n_k}, x)) \in \varphi(U, N_+).$$

But $\varphi(U, N_+) \cap N_- = \emptyset$, so $f^{n_k} g$ *cannot* belong to L_-. Thus, no such g can exist and the proof of (4.12) is complete.

(4.13) **Corollary.** *If φ is an Axiom A G-action with hyperbolic element f and if $\Omega_\varphi \neq M$ then f^n converges to one end of G as $n \to \infty$ and to the other as $n \to -\infty$.*

Proof. Let $\Lambda_\pm$, $N_\pm$, $L_\pm$, S, n_k be as in (4.12). We may assume $f \in U$, the seed of G. Since S is compact and $\{f^n\}$ is closed noncompact, there is an integer N such that $n \geqq N$ implies $f^n \notin S$. Since $\varphi(U, N_+) \cap N_- = \emptyset$, $f^{n+1} \in L_+$ for any $n \geqq N$ such that $f^n \in L_+$. Since $n_k \to \infty$ as $k \to \infty$ we can start at any $f^{n_k} \in L_+$ with $n_k \geqq N$ and be assured of staying in L_+ for all f^n, $n \geqq n_k$. Hence $f^n \to \mathfrak{e}_+$ as $n \to \infty$. By (4.10), $f^{-n} \to \mathfrak{e}_-$ as $n \to -\infty$.

Remark. (4.12) could be paraphrased as: Ω-hyperbolicity of a G-action implies hyperbolicity of G. This pleasant coincidence of terminology is not unique. The geodesic flow on the hyperbolic plane is hyperbolic and the hyperbolic trigonometric functions appear in its tangent flow. It remains to involve hyperbolic PDE's.

Let us return to the problem of intrinsically defining when an action has cycles. Let φ be an Axiom A G-action with Ω decomposition $\Omega_1 \cup \cdots \cup \Omega_m = \Omega_\varphi + M$. By (4.12) G is a hyperbolic group, say its ends are $\mathfrak{e}_\pm$. Let U be a seed of G and let $N_1, \ldots, N_m$ be small open neighborhoods of $\Omega_1, \ldots, \Omega_m$. For any $x \in M - \Omega$ let

$$L_i = \{g \in G : gx \in N_i\} \qquad i = 1, \ldots, m.$$

Since Ω_i is φ-invariant, the $\mathscr{H}_U L_i$ are disjoint, just as in the proof of (4.12), at least if the N_i are small enough. Also as in (4.12), each L_i belongs to an end $\mathfrak{e}_i$ of G. Since G is two-ended we conclude that ∂O_x meets at most two Ω_i's. By (3.6), ∂O_x meets at least two Ω_i's, say $\Omega_{i_1}, \Omega_{i_2}$. Define

$$\partial_\pm O_x = \bigcap_{Q \in \mathfrak{e}_\pm} \overline{\varphi(Q, x)} = \text{limit points of } \underset{g \to \mathfrak{e}_\pm}{gx}.$$

Then $\partial O_x = \partial_- O_x \cup \partial_+ O_x \subset \Omega_{i_1} \cup \Omega_{i_2}$. We observe that $\partial_+ O_x$ lies in one basic set and $\partial_- O_x$ in the other. For L_{i_1}, L_{i_2} are disjoint, nonempty, one belonging to $\mathfrak{e}_-$, the other to $\mathfrak{e}_+$, say $L_{i_1} \in \mathfrak{e}_-$, $L_{i_2} \in \mathfrak{e}_+$. Then

$$\partial_- O_x = \bigcap_{\substack{Q \in \mathfrak{e}_- \\ Q \subset L_{i_1}}} \overline{\varphi(Q, x)} \qquad \partial_+ O_x = \bigcap_{\substack{Q \in \mathfrak{e}_+ \\ Q \subset L_{i_2}}} \overline{\varphi(Q, x)}$$

shows that $\partial_- O_x \subset \Omega_{i_1}, \partial_+ O_x \subset \Omega_{i_2}$.

Without reference to a particular hyperbolic element of φ define

$$\Omega_i \prec \Omega_j \quad \text{iff } \partial_- O_x \subset \Omega_i,\ \partial_+ O_x \subset \Omega_j \text{ for some } O_x \subset M - \Omega.$$

This gives a partial order on $\Omega_1, \ldots, \Omega_m$, unique up to the reversal caused by interchanging $\mathfrak{e}_-$ and $\mathfrak{e}_+$. If f is a hyperbolic element of φ, let $\underset{f}{\prec}$ denote the partial order on the Ω_i defined in §2 by

$$\Omega_i \underset{f}{\prec} \Omega_j \qquad \text{iff } W^u \Omega_i \cap W^s \Omega_j \not\subset \Omega$$

where the stable and unstable manifolds were constructed using f. By (4.13), either $f^n \to \mathfrak{e}_+$ or else $f^n \to \mathfrak{e}_-$ as $n \to \infty$. If $f^n \xrightarrow[n]{} \mathfrak{e}_+$ then $\prec = \underset{f}{\prec}$. If $f^n \xrightarrow[n]{} \mathfrak{e}_-$ then $\prec = \underset{f^{-1}}{\prec}$. Summing this up, we state

(4.14) **Proposition.** *If φ is an Axiom A action then the cycles of any two hyperbolic elements are equal up to reversal. In particular, the no cycle assumption is independent of which hyperbolic element is chosen.*

The next result has the consequence that there is an order on a hyperbolic group which makes it like $\mathbb{Z}$ modulo bounded sets.

(4.15) **Proposition.** *Let G be a hyperbolic group with ends $\mathfrak{e}_\pm$. Then G has a seed U and there is a map $\tau: G \to \mathbb{Z}$ such that $\tau(1) = 0$ and*

(1) *There are positive constants c_k, $C_k \to \infty$ such that if $k \geq 3$ then*

$$a' a^{-1} \in U^k \Rightarrow |\tau a - \tau a'| \leq C_k$$

and

$$|\tau a-\tau a'|\leqq c_k \Rightarrow a' a^{-1}\in U^k.$$

(2) *There is a positive constant K such that for all* $a\in G$

$$\tau(a')\geqq K \quad \Rightarrow \tau(a' a)>\tau(a)$$
$$\tau(a')\leqq -K \Rightarrow \tau(a' a)<\tau(a).$$

(3) $$\mathscr{H}_U(x)\cap\mathscr{H}_U(\tau^{-1}(n+1))\neq\emptyset \quad \textit{for each } x\in\tau^{-1}(n),\ n\in\mathbb{Z}$$

(4) $$a_n\to e_\pm \quad \textit{iff } \tau(a_n)\to\pm\infty.$$

Remarks. (1) asserts bi-continuity of τ modulo U, (2) is a weak sort of translation invariance, (3) is an Archimedean Law modulo U. By 1 we mean the identity element of G. Recall that hyperbolicity of G means G is two ended with right-invariant ends.

Proof of (4.15). By (4.6), G has a subgroup $H=\{h^n\}$ such that $h^n\to e_\pm$ as $n\to\pm\infty$ and G/H, $H\backslash G$ are bounded. Let U be a seed for G with $h\in U$ and $\mathscr{H}_U H=G=\bigcup_{k\in\mathbb{Z}}\mathscr{H}_U(h^k)$. Let

$$H_n=\{h^k,\ -\infty<k\leqq n\} \qquad T_n=\mathscr{H}_U(h^n)-\mathscr{H}_U(H_{n-1}).$$

Clearly $G=\bigcup_{n\in\mathbb{Z}} T_n$ disjointly and the T_n are bounded sets. We claim

$$(*)\ h^{N+n}\in T_n \qquad n\in\mathbb{Z}$$

when h^N is the largest positive power of h lying in U. We observe

$$h^N\in U \Rightarrow h^{N+n}\in U h^n=\mathscr{H}_U(h^n) \qquad n\in\mathbb{Z}.$$

But if $h^{N+n}\in\mathscr{H}_U(h^m)$, $m<n$, then $h^{N+n}=u h^m$ for some $u\in U$, and so $h^{N+n-m}\in U$, a contradiction to h^N being the last power of h in U. Hence $h^{N+n}\notin\mathscr{H}_U(H_{n-1})$ proving $(*)$.

Define $\tau\colon G\to\mathbb{Z}$ by

$$\tau(g)=n+N \qquad \text{iff } g\in T_n.$$

By $(*)$ $h^0=1\in T_{-N}$ so $\tau(1)=0$.

(1) Fix any $k\geqq 3$. Let c_k+1 be the first positive power of h in $G-U^{k-2}$. Let C_k be the last positive power of h in U^{k+2}. If $a\in T_n$ and $a'\in T_m$ then $n-m=\tau a-\tau a'$ and $a=u h^n$, $a'=u' h^m$ for some $u, u'\in U$, so

$$a' a^{-1}=u' h^{m-n} u^{-1} \qquad \text{i.e. } (u')^{-1}(a' a^{-1})\, u=h^{m-n}.$$

If $|m-n|\leqq c_k$ then $h^{m-n}\in U^{k-2}$ so $a' a^{-1}\in U^k$. If $a' a^{-1}\in U^k$ then $h^{m-n}\in U^{k+2}$ so $|m-n|\leqq C_k$. This proves (1).

(2) Since $h^k\to e_\pm$ as $k\to\pm\infty$ there is a constant K such that

$$\{h^k\colon k\geqq K\}\cap(U^2 H^- U)=\emptyset$$
$$\{h^k\colon k\leqq -K\}\cap(U^2 H^+ U)=\emptyset$$

where $H^\pm=\{h^{\pm k}\colon k\geqq 0\}$. If $a\in T_m$, $a'\in T_n$, $a' a\in T_s$ then $a=u h^m$, $a'=u' h^n$, $a' a=u'' h^s$, so $a'a = u'h^n u h^m$ implies

$$h^n = (u')^{-1} u'' h^{s-m} u^{-1} \in U^2 h^{s-m} U.$$

But $s-m = \tau(a'a) - \tau(a)$. If $n \leqq -K$ then $s-m \leqq -1$, i.e. $\tau(a'a) < \tau(a)$ as claimed. If $n \geqq K$ then $s-m \geqq 1$, i.e. $\tau(a'a) > \tau(a)$ as claimed, proving (2).

(3) Let $x \in \tau^{-1}(n)$. Then $x \in T_{n-N}$ and so $x = u h^{n-N}$ for some $u \in U$. Thus

$$\mathscr{H}_U(x) \ni u^{-1} x = h^{n-N} = h^{-1} h^{n+1-N} \in \mathscr{H}_U(h^{n+1-N}) \subset \mathscr{H}_U(\tau^{-1}(n+1))$$

by (*) proving (3).

(4) Since $T_n \subset \mathscr{H}_U(h^n) = U h^n$ and $\tau^{-1}(n+N) = T_n$, (4) follows from $h^n \to \mathfrak{e}_\pm$ as $n \to \pm\infty$.

5. Ω-Stability

Here we prove our Main Theorem: Axiom A plus no cycles implies Ω-stability for a G-action φ. There are two cases $\Omega = M$ and $\Omega \subsetneqq M$. (3.10) gives the result when $\Omega = M$.

Suppose $\Omega \subsetneqq M$. Let $f_0 \in G$ be a hyperbolic element for φ. Let $f = \varphi(f_0)$ and let $\mathscr{L}$ be the φ-orbit lamination of Ω. By (3.1), (f, Ω) has local product structure. Since φ obeys Axiom A, it leaves $W^u O_x$, $W^s O_x$ invariant. (This was observed in § 2 and follows at once from centralness of f and the asymptotic characterizations of the strong stable and unstable laminations.) Hence $\mathscr{L}$ is subordinate to $\mathscr{W}_\varepsilon^u$, $\mathscr{W}_\varepsilon^s$ and so local product structure for (f, Ω) implies local product structure for $(f, \mathscr{L})$. By (7 A.1) of [HPS] f has a neighborhood U in $\mathrm{Diff}^1(M)$ and Ω_φ has a neighborhood U in M such that if $f' \in U$, then $h_{f'}(\Omega)$ is the largest f'-invariant subset of U when $h_{f'}$ is the canonical candidate for a leaf conjugacy from (7.4) of [HPS].

Let φ' be a G-action near φ in $A^1(G, M)$ and let $f' = \varphi'(f_0)$. By (3.9) the φ'-orbits laminate $\Omega' = h_{f'}(\Omega_\varphi)$ and $h_{f'}$ is an orbit conjugacy $(\varphi, \Omega_\varphi) \to (\varphi', \Omega')$. Since the compact φ-orbits are dense in Ω, the compact φ'-orbits are dense in Ω'. Hence $\Omega' \subset \Omega_{\varphi'}$. Since Ω' is the largest f'-invariant set near Ω, the φ' orbit of any point in $\Omega_{\varphi'} - \Omega'$ is never wholly near Ω'. It remains to rule out such a "global Ω-explosion".

By (2.3), Ω decomposes into basic sets $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$. By (4.12) G is hyperbolic and there is a natural partial order $\prec$ on the Ω. We are assuming, from now on, that in this order there are no cycles.

As in [15] global Ω-stability modulo local Ω-stability is true in more generality than Axiom A actions. Namely, let $\Lambda_0, \ldots, \Lambda_m$ be compact, disjoint, φ-invariant subsets of M such that

$$\Omega_\varphi \subset \Lambda_1 \cup \cdots \cup \Lambda_m$$

where φ is a continuous G-action on the compact metric space M and G is hyperbolic. Let the ends of G be called $\mathfrak{e}_-$, $\mathfrak{e}_+$ and define the ends of an orbit $O = O_x$ as

$$\partial_\pm O = \bigcap_{Q \in \mathfrak{e}_\pm} Cl\{gx : g \in Q\}.$$

Since $\mathfrak{e}_\pm$ are fixed by right multiplication, $\partial_\pm O$ is independent of $x \in O$ and $\partial_\pm O$ is φ-invariant. Clearly $\partial_\pm O$ is compact, nonempty. Equivalenty, $\partial_\pm(O_x)$ is the set of limit points of gx as $g \to \mathfrak{e}_\pm$.

Let $W^u \Lambda_i$ denote the unstable set of Λ_i, $\{x \in M: \partial_-(O_x) \subset \Lambda_i\}$ and let $W^s \Lambda_i$ denote the stable set $W^s \Lambda_i = \{x \in M: \partial_+(O_x) \subset \Lambda_i\}$. We claim that if $\partial_\pm(O_x)$ meets Λ_i then it is contained in Λ_i. For suppose $\partial_+(O_x)$ meets Λ_1 and Λ_2. Take open disjoint neighborhoods N_1, N_2 of Λ_1, Λ_2 as in the proof of (4.12). The sets $\{g \in G: gx \in N_1\}$, $\{g \in G: gx \in N_2\}$ are elements of different ends. But each is contained in $\mathfrak{e}_+$, contradicting the fact that G has exactly two ends. Similarly ∂_-. Thus, there are disjoint decompositions

$$\bigcup_{i=0}^{m} W^u \Lambda_i = M = \bigcup_{i=0}^{m} W^s \Lambda_i.$$

We say that $\Lambda_i \prec \Lambda_j$ iff $W^u \Lambda_i$ meets $W^s \Lambda_j$ off $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$. A cycle is a chain $\Lambda_{i_1} \prec \cdots \prec \Lambda_{i_n} = \Lambda_{i_1}$, $n \geqq 2$. The following result completes the proof of our Main Theorem.

(5.1) **Theorem.** *Let $V_1, \ldots, V_m$ be any neighborhoods of $\Lambda_1, \ldots, \Lambda_m$. If there are no cycles among the Λ_i then any G-action C^0 near φ has nonwandering set contained in $V = V_1 \cup \cdots \cup V_m$.*

Proof. The idea of the proof is the same as the one in [15]. The details are considerably harder due to the use of (4.15) instead of obvious properties of $\mathbb{R}$. We shall assume (5.1) is false and produce an arbitrarily long chain of unrepeated Λ_i's.

Let U be the seed of G, $\tau: G \to \mathbb{Z}$ be the map, and c_k, C_k, K the constants constructed in (4.15). We think of τ as "time along G from $\mathfrak{e}_-$ toward $\mathfrak{e}_+$". We write $\varphi(g)(x) = \varphi(g, x)$ when convenient.

Let $V_1, \ldots, V_m$ be neighborhoods of $\Lambda_1, \ldots, \Lambda_m$ in M. Let W_i be a small neighborhood of $\varphi(U^4, \bar{V}_i) = \{\varphi(g, x): g \in U^4, x \in \bar{V}_i\}$ and let X_i be a small neighborhood of $\bar{W}_i$. Without loss of generality we assume the V_i are open, the W_i are compact, the X_i are open, disjoint, and prove that the nonwandering set of the nearby action lies in $X = X_1 \cup \cdots \cup X_m$. For $\varphi(U^4, \bar{V}_i)$ shrinks to Λ_i as V_i does since $\varphi(G, \Lambda_i) = \Lambda_i$.

Let $N_i = W_i - V_i$, a compact set disjoint from Λ. We claim that N_i acts as a sort of fundamental neighborhood for Λ_i. Precisely, we assert that if φ' is a G-action near φ and $x \in M$ then

$$\left.\begin{array}{l} \varphi(a, x) \in V_i \\ \varphi(b, x) \in M - W_i \\ \tau(a) < \tau(b) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \varphi'(g, x) \in N_i \text{ for some } g \in G \text{ with } \tau(a) < \tau(g) \leqq \tau(b) \\ \text{and } \varphi'(g', x) \in \varphi(U^3, V_i) \text{ for all } g' \\ \text{with } \tau(a) \leqq \tau(g') < \tau(g). \end{array}\right. \tag{1}$$

When $G = \mathbb{R}$ (1) says that a trajectory leaving V_i must traverse N_i on its way out. The proof of (1) is a sort of least upper bound argument, using (4.15(3)).

For any φ', a, b, x as in (1) consider

$$\begin{aligned} T_x = \{t \in G: &\ \tau(a) \leqq \tau(t) \leqq \tau(b) \text{ and} \\ &\text{if } \tau(a) \leqq \tau(t') \leqq \tau(t) \text{ then } \varphi'(t', x) \in \varphi'(U^3, V_i)\}. \end{aligned}$$

We observe that $a \in T_x$ since any t' with $\tau(a) \leqq \tau(t') \leqq \tau(b)$ has $\tau(t') = \tau(a)$ and hence by (4.15) $t' a^{-1} \in U^3$. (This is the inequality $|\tau a - \tau a'| \leqq c_3 \Rightarrow a' a^{-1} \in U^3$.) Hence

$$\varphi'(t', x) = \varphi'(t' a^{-1} a, x) = \varphi'(t' a^{-1}, \varphi'(a, x)) \in \varphi'(U^3, V_i).$$

Thus $a \in T_x$. In particular $T_x \neq \emptyset$ and is τ-bounded above by $\tau(b)$, so we can choose some $t_1 \in T_x$ such that $\tau(t_1) \geqq \tau(t)$ for all $t \in T_x$. By (4.15) there is a $g \in \mathscr{H}_U(t_1)$ such that $\tau(g) = \tau(t_1) + 1$. This says $g \notin T_x$, $g = u t_1$ for some $u \in U$, and so

$$\begin{aligned}\varphi'(g, x) &= \varphi'(u t_1, x) \\ &= \varphi'(u, \varphi'(t_1, x)) \in \varphi'(U, \varphi'(U^3, \bar{V}_i)) \subset \varphi'(U^4, \bar{V}_i).\end{aligned}$$

Since $\bar{V}_i$ and U^4 are compact, this last set, $\varphi'(U^4, \bar{V}_i)$, will lie in W_i for φ' near φ. Thus, $\varphi'(g, x) \in W_i$ but since $g \notin T_x$, $\varphi'(g, x) \notin V_i$. This says $\varphi'(g, x) \in N_i$ and proves (1). [Note how important it is that g be $u t_1$, not $t_1 u$. That is, we must use a left-end theory here.]

Suppose that (5.1) is false. Then there are G-actions φ_n converging to φ in $A^0(G, M)$ as $n \to \infty$, such that Ω_{φ_n} meets $M - X$. By compactness of $M - X$ there exists a limit point $x \in M - X$ of the Ω_{φ_n}. Using a diagonal process, a sequence $g_n \in G$ can be selected so that $\{g_n\}$ has no limit point in G and

$$\begin{gathered}\varphi_n(g_n, x_n) = y_n \to x \leftarrow x_n \qquad n \to \infty \\ x_n, y_n, x \in M - W.\end{gathered}$$

Since G has only two ends, $e_\pm$, g_n must be getting near one or both of them as $n \to \infty$. By choosing a subsequence and possibly interchanging x_n with y_n we may assume $g_n \to e_+$. In what follows we choose subsequences freely without relabelling them.

Since $x \in \Lambda$ and there are no cycles, $\partial_- O_x \subset \Lambda_i$, $\partial_+ O_x \subset \Lambda_{i_2}$, and $i_1 \neq i_2$. Since $\varphi_n \to \varphi$ in $A(G, M)$ and $x_n \to x$ in M we can find a sequence $g_n \in G$ such that

$$0 < \tau(g_n') < \tau(g_n) \qquad \varphi_n(g_n', x_n) = x_n' \to \lambda_{i_2} \in \Lambda_{i_2}.$$

To see this, choose any sequence $a_k \to e_+$ in G. For k fixed, $\varphi_n(a_k, x_n) \xrightarrow[n]{} \varphi(a_k, x)$. As $k \to \infty$, $\varphi(a_k, x) \to \lambda_{i_2} \in \Lambda_{i_2}$ (for a subsequence). Thus $g_n' = a_{k(n)}$, $n \gg k \to \infty$ suffices,

In particular, $x_n' \in V_{i_2}$, n large. Since $y_n = \varphi_n(g_n, x_n) \in M - W$, (1) gives some $g_n'' \in G$ with

$$\begin{aligned}\varphi_n(g_n'', x_n) &\in N_{i_2} && \text{with } \tau(g_n') \leqq \tau(g_n'') \leqq \tau(g_n) \\ \varphi_n(g, x_n) &\in \varphi_n(U^3, V_{i_2}) && \text{if} \quad g \in \tau^{-1}[\tau g_n', \tau g_n'').\end{aligned} \tag{2}$$

We observe that

$$\tau(g_n'') - \tau(g_n') \to \infty. \tag{3}$$

For otherwise we could apply (4.15) and get a subsequence with $g_n''(g_n')^{-1} \to g_* \in G$. But then

$$\begin{aligned}\varphi_n(g_n'', x_n) &= \varphi_n(g_n''(g_n')^{-1} g_n', x_n) \\ &= \varphi_n(g_n''(g_n')^{-1}, \varphi_n(g_n', x_n)) \\ &= \varphi_n(g_n''(g_n')^{-1}, x_n') \to \varphi(g_*, \lambda_{i_2}) \in \Lambda_{i_2}\end{aligned}$$

contradicting the fact that $\varphi_n(g_n'', x_n) \in N_{i_2}$, a compact set disjoint from Λ.

Since N_{i_2} is compact we may assume $x_n'' = \varphi_n(g_n'', x_n) \to x^2 \in N_{i_2}$. We claim that $\varphi(Q_-, x^2) \subset W_{i_2}$ for some $Q_- \subset e_-$. In fact consider $Q_- = \{g \in G: \tau(g) \leqq -K\}$ where K is the "translation constant" in (4.15(2)). Suppose that $\varphi(g, x^2) \in M - W_{i_2}$

for some g with $\tau(g) \leqq -K$. Then

$$\varphi_n(g, x_n'') \to \varphi(g, x^2).$$

But $\varphi_n(g, x_n'') = \varphi_n(g g_n'', x_n)$ and by (4.15(2)),

$$\tau(g g_n'') < \tau(g_n'').$$

By (4.15(1)) $|\tau(g_n'') - \tau(g g_n'')| \leqq C_k$ where $g g_n''(g_n'')^{-1} \in U^k$, i.e. where $g \in U^k$. This k is fixed and so

$$\tau(g_n'') - C_k \leqq \tau(g g_n'') < \tau(g_n'').$$

By (3)

$$\tau(g_n') < \tau(g g_n'') < \tau(g_n'')$$

for large n. This says that $x_n'' = \varphi_n(g g_n'', x_n) \in \varphi_n(U^3, V_i)$ since all $a \in G$ with $\tau(g_n') < \tau(a) < \tau(g_n'')$ have this property according to (2). But this contradicts $\varphi_n(g, x_n'') \to \varphi(g, x^2) \in M - W_{i_2}$. Hence such a g does not exist and $\varphi(Q_-, x^2) \subset W_{i_2}$. Therefore $\partial_-(O_{x^2}) \subset \Lambda_{i_2}$.

Let $\partial_+(O_{x^2}) \subset \Lambda_{i_3}$. Since there are no cycles, i_1, i_2, i_3 are distinct. We shall proceed with x^2 as we did with x. We do not claim $x^2 \in \lim_n \Omega_{\varphi_n}$. Rather we shall use the same x_n, y_n we used above, $x_n, y_n \to x$. This makes it slightly harder to find x^3 and i_4.

First observe that

$$\tau(g_n) - \tau(g_n'') \to \infty. \tag{4}$$

Otherwise by (4.15(2)) and a subsequence we could assume $g_n(g_n'')^{-1} \to g \in G$. Then

$$\begin{aligned} x \leftarrow y_n = \varphi_n(g_n, x_n) &= \varphi_n(g_n(g_n'')^{-1} g_n'', x_n) \\ &= \varphi_n(g_n(g_n'')^{-1}, x_n'') \to \varphi(g_*, x^2). \end{aligned}$$

Hence x and x^2 are on the same orbit so $\partial_- O_x = \partial_- O_{x^2}$, contradicting $i_1 \neq i_2$.

Since $x_n'' \to x^2$ in M and $\varphi_n \to \varphi$ in $A(G, M)$, we can find a sequence $g_n''' \to e_+$ such that

$$\begin{aligned} &\tau(g'') < \tau(g_n''') < \tau(g_n) \\ &\varphi(g_n''', x_n) = x_n''' \to \lambda_{i_3} \in \Lambda_{i_3}. \end{aligned} \tag{5}$$

To see this, choose any sequence $a_k \to e_+$ in G, say $a_k \in U^k$. For k fixed

$$\varphi_n(a_k, x_n'') \xrightarrow[n]{} \varphi(a_k, x^2).$$

As $k \to \infty$, $\varphi(a_k, x_2)$ tends to some $\lambda_{i_3} \in \Lambda_{i_3}$ (for a subsequence). Consider $g_n''' = a_k g_n''$ where $k = k(n)$, $n \gg k \to \infty$. Clearly we can assume $\varphi_n(g_n''', x_n) = \varphi_n(a_k, x_n'') \to \lambda_{i_3}$. By (4.15(2)), $\tau(g_n''') > \tau(g_n'')$ as soon as $\tau(a_k) \geqq K$. By (4.15(1))

$$|\tau(g_n'') - \tau(g_n''')| \leqq C_k$$

when $g_n'''(g_n'')^{-1} \in U^k$. But $g_n'''(g_n'')^{-1} = a_k \in U^k$. Thus, $\infty \leftarrow k(n) \ll n$ and (4) let us make $\tau(g_n) > \tau(g_n''')$, completing the proof of (5).

In particular, $x_n''' = \varphi_n(g_n''', x_n) \in V_{i_3}$, n large, and (1) implies the analogue of (2)

$$\begin{aligned} &\varphi_n(g_n'''', x_n) \in N_{i_3} && \text{with } \tau(g_n''') \leqq \tau(g_n'''') \leqq \tau(g_n) \\ &\varphi_n(g, x_n) \in \varphi_n(U^3, V_{i_3}) && \text{if } \quad g \in \tau^{-1}[\tau(g_n'''), \tau(g_n'''')). \end{aligned} \tag{6}$$

By exactly the same reasoning as above we get $\varphi_n(g_n'''', x_n) = x_n'''' \to x^3 \in N_{i_3}$ with $\partial_-(O_{x^3}) \subset A_{i_3}$. Again the no cycles assumption implies $\partial_+(O_{x^3}) \subset A_{i_4}$ with i_1, i_2, i_3, i_4 distinct. Continuing this was (the succeeding steps are exactly the same as $i_3 \to i_4$) we produce an arbitrarily long chain of unrepeated A_i's which is ridiculous since there are only m of them. Hence (5.1) and the Main Theorem are proved.

§ 6. Examples

(i) *It is Reasonable to Assume G is Compactly Generated.* Let A be the standard linear Anosov diffeomorphism of T^2, $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Let G be the infinite direct sum $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ with the discrete topology. Elements of G are infinite strings $(n_1, n_2, \ldots)$ with all but finitely many entries equal to zero. G is *not* compactly generated but otherwise is a perfectly good Lie group. Let G act on T^2 by

$$\varphi(n_1, n_2, \ldots)\colon\ x \mapsto A^{2(n_1+n_2+\cdots)}(x).$$

The φ-orbits are clearly the A^2-orbits, the compact (=finite) A^2-orbits are dense in T^2, and $f = A^2$ is hyperbolic to the φ-orbit lamination. The laminae are points and the orbits are finite or countable sets of points. Thus, φ is an Axiom A G-action with hyperbolic elements $f = \varphi(1, 0, 0, \ldots) = A^2$. Since $\Omega_\varphi = M$, φ satisfies the no cycle condition vacuously. It also satisfies Axiom B [16].

However, φ is not Ω-stable. Any given neighborhood U of φ in $A^r(G, M)$ contains a smaller neighborhood of the form

$$\{\psi \in A^r(G, M)\colon d_r(\psi(n_1, n_2, \ldots), \varphi(n_1, n_2, \ldots)) < \varepsilon$$
$$\text{for all } (n_1, n_2, \ldots) \text{ with } n_l = 0\ \forall\, l \geqq k\}.$$

The numbers ε and k depend on U. The d_r is a metric on $\mathrm{Diff}^r(T^2)$. In particular, we can choose ψ to be

$$\psi(n_1, n_2, \ldots) = A^{2(n_1+\cdots+n_{k-1})+n_k}$$

and ψ will lie in U. The ψ-orbits are the A-orbits. Since A^2 has more one-point-orbits than A has, A^2 and A are not orbit-conjugate. Hence ψ is not orbit conjugate to φ and so φ is not Ω-stable.

(ii) *Why do we Assume the Hyperbolic Element f central?* Let F_2 be the free group on two generators, a and b. Give F_2 the discrete topology. F_2 is compactly generated. Let A be an Anosov diffeomorphism of M. Then

$$\begin{aligned} \varphi\colon F_2 &\to \mathrm{Diff}(M) \\ a &\mapsto A \\ b &\mapsto id \end{aligned}$$

gives an F_2-action on M and $f = \varphi(a)$ is normally hyperbolic to the orbit lamination. (The φ-orbits are the A-orbits.) But $\psi\colon F_2 \to \mathrm{Diff}(M)$ defined by $a \mapsto A$, $b \mapsto g \doteq id$, is an action near ψ whose orbits, in all likelihood, are totally different than the φ-orbits. Hence φ is not Ω stable. The same example shows why M. Hirsch needs to assume G/G_1 is connected in [6].

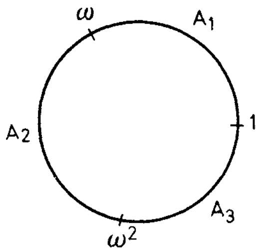

Fig. 3. Arcs in a circle

(iii) *J. Stallings' Example of a Hyperbolic Group which is Not the Direct Product: Finite* $\times \mathbb{Z}$. Let F_2 be the free group on the generators a, b. Let G be F_2 divided out by the relations

$$a^3 = 1 \qquad bab^{-1} = a^2.$$

We continue to denote by a, b the cosets containing a, b. It is easy (and it is left for the reader) to check that the center of G is $\{b^{2k}\}_{k \in \mathbb{Z}}$. Suppose $G \approx K \times \mathbb{Z}$. Then $\mathbb{Z} \subset$ center (G), and some b^{2l} generates $\mathbb{Z}$. But b^l-itself must be expressible as

$$b^l = (k, b^{2nl}) \qquad \text{for some } k \in K, \ n \in \mathbb{Z}.$$

This implies $b^{2l} = (k^2, b^{4nl})$, so $2n = 1$. Since $\frac{1}{2} \notin \mathbb{Z}$, no such factorization $G = K \times \mathbb{Z}$ exists.

Any element of G can be written in one of the forms

$$b^m \quad ab^m \quad a^2 b^m \qquad m \in \mathbb{Z}.$$

It is then easy to see that G has two ends (they are approached as $m \to \pm \infty$ in the above expressions) and that right G-multiplication leaves the ends invariant. Hence G is hyperbolic.

(iv) *How Stallings' Group Acts Faithfully on* S^1, *Satisfying Axiom A*. An action φ: $G \to \mathrm{Diff}(M)$ is *faithful* iff φ is injective. Thus, we think of an action as a representation of G in $\mathrm{Diff}(M)$. If φ is not faithful then it can be replaced by ψ: $G/\ker \varphi \to \mathrm{Diff}(M)$ which *is* faithful. The orbit decompositions of M by ψ and φ are equal, so faithful actions are the only interesting ones for us.

Let $\omega = e^{2\pi i/3}$ and let A_1, A_2, A_3 be the counterclockwise arcs of S^1 from 1 to ω, ω to ω^2, ω^2 to 1. See Fig. 3. Let g: $S^1 \to S^1$ be rotation by $2\pi/3$. Thus, $g\omega = \omega^2$, $g\omega^2 = 1$, $g1 = \omega$. Let h_1: $A_1 \to A_3$ be a diffeomorphism fixing 1 and having $T_1 h_1 = (T_{\omega^2} g) \circ (T_\omega h_1) \circ (T_1 g)$. Thus, h_1 reverses orientation and, up to translation, has equal derivative at 1 and ω. Put

$$h(z) = \begin{cases} h_1(z) & z \in A_1 \\ g^2 h_1 g^2(z) & z \in A_2 \\ g h_1 g(z) & z \in A_3. \end{cases}$$

Then h is a diffeomorphism of S^1 which sends $A_1 \to A_3$, $A_3 \to A_1$, $A_2 \to A_2$. Since $T_1 h = (T_{\omega^2} g) \circ (T_\omega h) \circ (T_1 g)$, the left and right derivatives of h match up at $1, \omega, \omega^2$.

Now h_1 can be chosen, subject to the above condition on its derivative at 1 and ω, so that h is a Morse-Smale diffeomorphism of S^1. For instance, requiring $|T_1 h_1| < 1$ forces sinks at $1, \omega, \omega^2$; the rest of h_1 may be defined to give three sources for h in between the sinks. For such an h define

$$\begin{aligned} \varphi\colon F_2 &\to \mathrm{Diff}(M) \\ a &\mapsto g \\ b &\mapsto h \end{aligned}$$

where F_2 is the free group generated by a, b. Then $\ker(\varphi)$ is generated by $a^3 = 1$ and $ba = a^2 b$. That is, φ induces a faithful action of Stallings' group G on S^1. This ψ satisfies Axiom A because, as is easily checked, $\Omega_\psi =$ the sources and sinks of h. Thus b^2 is a hyperbolic element of G. (Note that b is not a hyperbolic element since $b \notin \mathrm{center}(G)$.)

(v) *Question.* Can Axiom Ab be weakened to

Ab': φ-orbits with noncompact isotropy group are dense in Ω_φ

so that $Aa + Ab' \Rightarrow \Omega$-stability? It seems to us that the Cloud Lemma is the main obstacle here.

(vi) *Question.* If G is a hyperbolic group and M is a smooth compact manifold, when is there a faithful Axiom A G-action on M? When $G = \mathbb{Z}$ or $\mathbb{R}$ the answer is "always". For any manifold supports Morse-Smale diffeomorphisms and flows.

(vii) *Question.* If φ is an Axiom A action with strong transversality is φ structurally stable?

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