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STRUCTURALLY STABLE DIFFEOMORPHISMS ARE DENSE

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Let M be a C^∞ compact manifold without boundary. Let $f \in \mathrm{Diff}^r(M)$, $1 \leqq r \leqq \infty$. f is structurally stable in $\mathrm{Diff}^r(M)$ if there exists a neighborhood U_f of f in $\mathrm{Diff}^r(M)$ such that given $g \in U_f$ there exists a homeomorphism $h: M \to M$ such that $hf = gh$. The structurally stable diffeomorphisms are known not to be dense in $\mathrm{Diff}^r(M)$, unless M is the circle (see [4], [5], etc.). On the other hand I will prove below the mixed result that the structurally stable diffeomorphisms are always dense in $\mathrm{Diff}^r(M)$ with the C^0 topology. The main tool is the theorem of Smale [6], that every diffeomorphism is isotopic to a structurally stable diffeomorphism. Theorem 2 improves this theorem by producing an isotopy which is arbitrarily small in the C^0 topology. I expect to prove a corresponding theorem for vector fields with M. W. Hirsch. It is a pleasure to acknowledge helpful conversations with M. W. Hirsch and S. Smale. Let $m = \dim M$.

THEOREM 1. *Let $1 \leqq r \leqq \infty$. Then the structurally stable diffeomorphisms are dense in* $\mathrm{Diff}^r(M)$ *with the C^0 topology.*

A sharper version of this theorem is

THEOREM 2. *Let $1 \leqq r \leqq \infty$. Let $f \in \mathrm{Diff}^r(M)$. Then f is C^r isotopic to a structurally stable diffeomorphism g by an isotopy which is arbitrarily small in the C^0 topology.*

The following proposition is actually part of the proof of the main theorem of [6].

PROPOSITION (SMALE). *Let $f \in \mathrm{Diff}^r(M)$. Let $M = H_m \supset H_{m-1} \supset \cdots \supset H_1 \supset H_0$ be a handle body decomposition of M (corresponding to a "nice" Morse function). Suppose $f(H_i) \subset$ interior H_i for all i. Then f is C^r isotopic to a structurally stable diffeomorphism.*

The proof of this proposition in [6] is essentially complete, one need only take a little care in keeping track of the stable and unstable manifolds. Also, the C^0 size of the isotopy may be made small if the i-handles, $0 \leqq i \leqq m$, are small.

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Smale fixes a handle body decomposition of M and isotopes any f to satisfy the hypotheses of the proposition. The idea for Theorem 2 is to pick a fine handle body decomposition so that this first isotopy may also be made C^0 small.

PROOF OF THEOREM 2. Let T be a triangulation of M with small mesh (see [2]). Let T_i, $0 \leqq i \leqq m$, be the i-skeleton of T. $f(T_{m-1})$ misses a point in the interior of each m simplex. By pushing away from this point we may isotope f such that $f(T_{m-1})$ is contained in an arbitrarily small neighborhood of T_{m-1}. By downward induction we may isotope f to g so that $g(T_i)$ is contained in an arbitrarily small neighborhood of T_i, $0 \leqq i \leqq m-1$. Now think of a small neighborhood U_0 of T_0 as the 0-handles of a handle body decomposition of M, a small neighborhood U_1 of $T_1 - U_0$ as the 1-handles, etc. Make U_i so small that the $g(U_i)$ are already contained in prescribed small neighborhoods of the T_i. Now by upward induction on i we may further isotope g to preserve this handle body in the sense of the proposition. The C^0 size of the isotopy depends only on the mesh of T and m.

Combining this theorem with [1] and [3], we see

THEOREM 3. *Let* $1 \leqq r < \infty$. *Then there exists an open and dense set* $U \subset \mathrm{Diff}^r(M)$ *with the* C^0 *topology and a dense set of structurally stable diffeomorphisms* $V \subset U$, *such that the diffeomorphisms in* V *are locally minimizing for the topological entropy of the diffeomorphisms in* U.

This theorem may be of some interest to ecologists and others who sometimes try to achieve stability by maximizing the entropy.

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