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Neighborhoods of Hyperbolic Sets

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§ 1. Introduction

In this paper we study the asymptotic behavior of points near a compact hyperbolic set of a C^r diffeomorphism ($r \geqq 1$) $f: M \to M$, M being a compact manifold. The purpose of our study is to complete the proof of Smale's Ω-stability Theorem by demonstrating (2.1), (2.4) of [6].

Ω denotes the set of non-wandering points for f. Smale's Axiom A requires [5]:

(a) Ω has a hyperbolic structure,

(b) the periodic points are dense in Ω.

Hyperbolic structure, the stable manifold of Ω, and fundamental neighborhoods are discussed in §§ 2 and 5.

The result of [6] proved here is:

If f obeys Axiom A *then there exists a proper fundamental neighborhood V for the stable manifold of Ω such that the union of the unstable manifold of Ω and the forward orbit of V contains a neighborhood of Ω in M.*

As a consequence we have:

If f obeys Axiom A *then any point whose orbit stays near Ω is asymptotic with a point of Ω.*

Section 8 of the mimeographed version of [1] contains a generalization of the above results with an incorrect proof. A correct generalization is:

(1.1) Theorem. *If Λ is a compact hyperbolic set then $W^u(\Lambda) \cup 0_+ V$ contains a neighborhood U of Λ, where V is any fundamental neighborhood for $W^s(\Lambda)$ and $0_+ V = \bigcup_{n \geqq 0} f^n(V)$. If Λ has local product structure then a proper fundamental neighborhood may be found and any point whose forward orbit lies in U is asymptotic with some point of Λ.*

Theorem (1.1) is proved in § 5, local product structure is discussed in [5] and in § 2. In § 7 we prove the analogous theorems for flows.

Here is an example, due to Bowen, of a compact hyperbolic set Λ which does not have local product structure, has no proper fundamental neighborhood and for which there are points asymptotic to Λ without being asymptotic with any point of Λ.

Consider the Cantor set C as the space of sequences of zeros and ones with a decimal point:

$$x=(\dots x_{-2}\, x_{-1} \,.\, x_0\, x_1 \dots) \qquad x_j=0 \text{ or } 1.$$

The elements of C can be thought of as maps $\mathbf{Z}\to\{0,1\}$. The compact open topology makes C a metrizable compact space. A distance is given by

$$d(x,y)=\sum_{-\infty}^{\infty}\frac{|x_i-y_i|}{2^{|i|}}.$$

The shift map $\sigma: C\to C$ moves the decimal point one space to the right:

$$\sigma(\dots x_{-1}\,.\,x_0\, x_1\dots)=(\dots x_{-1}\, x_0\,.\,x_1\dots).$$

σ is a homeomorphism. By [4] (C,σ) is conjugate to the restriction of a diffeomorphism $f: S^2\to S^2$ to a certain compact invariant hyperbolic set K. Hence the topological properties of C reflect themselves exactly in K.

Consider the subset of C, and correspondingly of K,

$$\Lambda=\{x\in C: \text{ any finite maximal string of 0's is of even length}\}.$$

Clearly Λ is a compact σ-invariant subset of C and the periodic points are dense in Λ. However, there are points $c\in C$ such that

$$d(\sigma^n c,\Lambda)\to 0 \qquad \text{as } n\to\pm\infty$$

$$d(\sigma^n c,\sigma^n x)\not\to 0 \qquad \text{as } n\to\pm\infty \text{ for any } x\in\Lambda.$$

That is, there are points of C tending to Λ but not asymptotic with any point of Λ. Even worse, there are periodic points of $C-\Lambda$ whose entire orbits lie arbitrarily close to Λ. This behavior is opposite to that of (1.1).

Define such c as follows. Put odd maximal strings of zeros of increasing length on both sides of the decimal point

$$\dots 1000001000101\,.\,1010001000001\dots.$$

Given any $m>0$, $\sigma^n c$ has at most one entry of 1 in $[-m,m]$ for large enough $|n|$. The sequence x with zeros except at this entry belongs to Λ and

$$d(\sigma^n c,x)\leqq\sum_{|i|\geqq m}1/2^{|i|}=1/2^{m-1}$$

which is arbitrarily small when m is large. Hence $\sigma^n c\to\Lambda$. On the other hand, for any $x\in C$ if $d(\sigma^n c,\sigma^n x)\to 0$ as $|n|\to\infty$ then

$$x=c \text{ to the right of some entry if } n\to\infty$$

$$x=c \text{ to the left of some entry if } n\to-\infty.$$

Either way, x can not belong to Λ.

A periodic point $p \in C$ is a sequence which endlessly repeats a finite block. If this block is of the form $10 \ldots 0$ with $2m+1$ zeros then the orbit of p lies within a distance $1/2^{m-1}$ of Λ by the same reasoning as before. Thus, the existence of periodic points as claimed above is clear.

§ 2. Local Product Structure

Several notions and results from [1, 5] should be recalled.

A compact set $\Lambda \subset M$ is hyperbolic for the diffeomorphism $f\colon M \to M$ if $f\Lambda = \Lambda$ and Tf leaves invariant a continuous splitting $T_\Lambda M = E^s \oplus E^u$, expanding E^u and contracting E^s. That is

$$|Tf(v)| \leqq \tau |v| \qquad \text{if } v \in E^s$$

$$|Tf(v)| \geqq \tau^{-1} |v| \qquad \text{if } v \in E^u$$

for some constant $\tau, 0 < \tau < 1$, and some Riemannian metric on M. The constant τ is called the skewness.

Through such a Λ pass families of smooth unstable and stable manifolds tangent to E^u, E^s at Λ [1]. The unstable manifold of size ε through $p \in \Lambda$ is called $W_\varepsilon^u(p)$, the stable one $W_\varepsilon^s(p)$. This "size ε" refers to the radius measured in the tangent space at p:

$$W_\varepsilon^u(p) = \exp_p(\text{graph } g_p)$$

where $g_p\colon E_p^u(\varepsilon) \to E_p^s(\varepsilon)$ is a smooth map whose graph has slope $\leqq 1$ with $g_p(0) = 0$, $T g_p(0) = 0$. $E_p^u(\varepsilon)$, $E_p^s(\varepsilon)$ are the ε-discs in E_p^u, E_p^s. Similarly for $W_\varepsilon^s(p)$.

The families $W_\varepsilon^u = \{W_\varepsilon^u(p) | p \in \Lambda\}$, $W_\varepsilon^s = \{W_\varepsilon^s(p) | p \in \Lambda\}$ are overflowing invariant in the following sense: $f^{-1} W_\varepsilon^u(p) \subset W_\varepsilon^u(f^{-1} p)$ and $f W_\varepsilon^s(p) \subset W_\varepsilon^s(f p)$. They are expanding from Λ and contracting to Λ in the sense that

$$\bigcap_{n \geqq 0} f^{-n} W_\varepsilon^u = \Lambda = \bigcap_{n \geqq 0} f^n W_\varepsilon^s$$

for $W_\varepsilon^u = \bigcup_{p \in \Lambda} W_\varepsilon^u(p)$ and $W_\varepsilon^s = \bigcup_{p \in \Lambda} W_\varepsilon^u(p)$. We call $W_\varepsilon^u = W_\varepsilon^u \Lambda$ the local unstable manifold of Λ. Similarly, $W_\varepsilon^s = W_\varepsilon^s \Lambda$ is called the local stable manifold of Λ. Notice that this terminology differs from that of Smale in [5].

If p is a hyperbolic fixed point (that is $\Lambda = p$ in the preceding discussion) and V is a local submanifold transverse to its stable manifold then locally f^n presses V toward the unstable manifold of p, $W^u(p)$. If V has the same dimension as $W^u p$ then as $n \to \infty$, $f^n V \to W^u(p)$ in the C^1

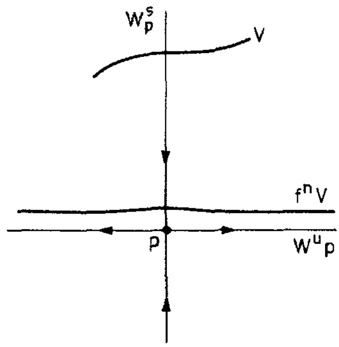

Fig. 1. f^n flattens V toward $W^u\,p$

sense, locally. This is the content of the λ-lemma [3] and is illustrated in Fig. 1.

A direct consequence of the λ-lemma is the

(2.1) *Cloud Lemma. If $f\colon M \to M$ is a diffeomorphism and p, q are hyperbolic periodic points with $W^u(p) \pitchfork W^s(q) \neq \emptyset$, $W^s(p) \pitchfork W^u(q) \neq \emptyset$ then these points of intersection are non-wandering.*

Proof [5]. See Fig. 2 for n so large that p, q are fixed points of f^n. The set U is a neighborhood of $x \in W^u(q) \cap W^s(p)$. We have drawn some iterates $f^{nk}\,U$ and shown how they must re-intersect U eventually, by the λ-lemma.

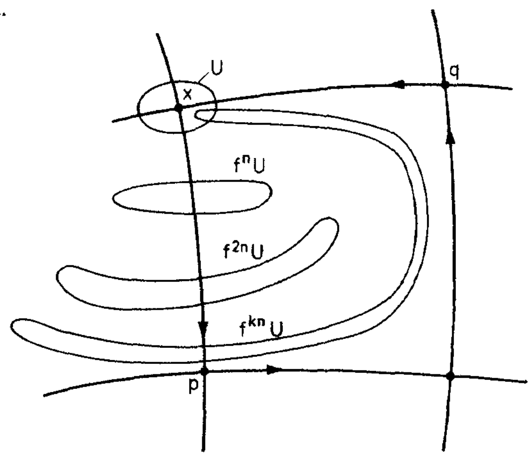

Fig. 2. $f^{kn}\,U$ re-intersects U

Definition. A hyperbolic set Λ has local product structure if, for some $\varepsilon > 0$,

$$W_\varepsilon^u(p) \cap W_\varepsilon^s(p') \subset \Lambda$$

for all $p, p' \in \Lambda$.

(2.2) *Local Product Structure Theorem. If f obeys Axiom* A *then f has local product structure on Ω.*

Proof. Let $\varepsilon > 0$ be small enough that the 3ε-local stable and unstable manifolds through points of Ω are given by the stable manifold theory of [1]. Let $x, x' \in \Omega$ have $y \in W^s_\varepsilon(x) \cap W^s_\varepsilon(x')$. By [1] the intersection is transverse, consists of a single point, and the same is true of the intersection $W^s_{2\varepsilon}(x) \cap W^u_{2\varepsilon}(x')$. (See Fig. 3.)

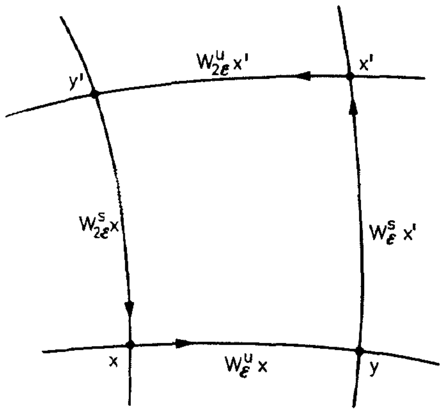

Fig. 3. Local product structure

Approximate x, x' by periodic points p, p'. By continuity of the stable and unstable manifolds, the intersections

$$W^u_{3\varepsilon}(p) \cap W^s_{3\varepsilon}(p'), \qquad W^s_{3\varepsilon}(p) \cap W^u_{3\varepsilon}(p')$$

continue to be transverse and to consist of single points, q, q' (see Fig. 4). By the Cloud Lemma, $q, q' \in \Omega$. As q is arbitrarily near y and Ω is a closed set, y belongs to Ω.

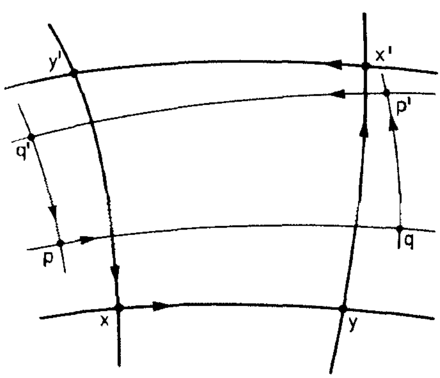

Fig. 4. Stable and unstable manifolds for periodic p, p' approximating x, x'

Corollary of the Proof. If Λ is any hyperbolic set isolated from $\Omega-\Lambda$ and the periodic points are dense in Λ then f has local product structure at Λ.

The following key lemma shows how local product structure simplifies the topology of $W_\varepsilon^s = \bigcup_{p\in\Lambda} W_\varepsilon^s(p)$.

(2.3) Lemma. *W_δ^s is a neighborhood of Λ in W_ε^s if $0<\delta\leqq\varepsilon$, ε is small, and Λ has local product structure.*

Proof. Suppose, on the contrary, that there existed a sequence of points $x_n\in W_\varepsilon^s(p_n)-W_\delta^s$ converging to some $p\in\Lambda$. As in (2.2), when ε is small this implies that $W_{2\varepsilon}^s(p_n) \pitchfork W_{2\varepsilon}^u(p)$ in a single point, say q_n. By local product structure $q_n\in\Lambda$ and $d(x_n,q_n)\to 0$. Thus $x_n\in W_\delta^s(q_n)$ when n is large, contradicting our assumption.

§3. Fundamental Domains

A fundamental domain for W_ε^s is a compact set $D\subset W_\varepsilon^s$ such that

$$W_\varepsilon^s-\Lambda\subset 0_+ D$$

where $0_+ D=\bigcup_{n\geqq 0} f^n D$, i.e. the forward orbit of D. A fundamental domain for W_ε^u is a compact set $D\subset W_\varepsilon^u$ such that

$$W_\varepsilon^u-\Lambda\subset 0_- D$$

where $0_- D=\bigcup_{n\geqq 0} f^{-n} D$. If in addition D is disjoint from Λ, we call D a *proper* fundamental domain.

(3.1) Lemma. *The set $D^s=Cl(W_\varepsilon^s-f W_\varepsilon^s)$ is a fundamental domain for W_ε^s and $D^u=Cl(W_\varepsilon^u-f^{-1} W_\varepsilon^u)$ is one for W_ε^u.*

Proof. $0_+ D^s\supset W_\varepsilon^s-\bigcap_{n\geqq 0} f^n W_\varepsilon^s=W_\varepsilon^s-\Lambda$. The proof for D^u is similar.

(3.2) Lemma. *If Λ has local product structure then D^s, D^u are proper fundamental domains.*

Proof. There is a δ, $0<\delta<\varepsilon$, such that $f W_\varepsilon^s\supset W_\delta^s$. By (2.3), W_δ^s is a neighborhood of Λ in W_ε^s. Therefore $D^s=Cl(W_\varepsilon^s-f W_\varepsilon^s)$ is disjoint from Λ. Similarly for D^u.

Question. Do the following conditions imply the existence of proper fundamental domain for $W^s\Lambda$:

(a) the periodic points are dense in Λ,

(b) there exists a neighborhood U of Λ such that if $x\in U$ and $0_+(x)\subset U$ then $\omega(x)\subset\Lambda$, where $\omega(x)$ is the ω-limit set of x.

§4. Semi Invariant Disc Families

In this section we state and prove the basic technical theorem required for (1.1).

The space $\mathrm{Emb}(D^u, M)$ of all embeddings of the closed u-disc into M may be thought of as a fiber bundle over M. The fiber at $x \in M$ is the set of all embeddings $e\colon D^u \to M$ such that $e(0)=x$. We put the uniform topology on $\mathrm{Emb}(D^u, M)$.

Definition. If $a\colon X \to \mathrm{Emb}(D^u, M)$ is a continuous section over $X \subset M$ then $\mathscr{A} = \{\text{image } a(x) \mid x \in X\}$ is a (continuous) u-disc family through X.

Clearly $W^u_\delta = \{W^u_\delta(p) \mid p \in \Lambda\}$ is a u-disc family for any small $\delta > 0$.

The following lemma may be thought of as a type of Inverse Function Theorem for certain u-disc families. By $R^m(\varepsilon)$ we mean the disc of radius ε in R^m, centered at the origin.

(4.1) Lemma. *If $u+s=m$ and $\mathscr{A} = \{A(y) \mid y \in R^s(\delta)\}$ is a u-disc family through $0 \times R^s(\delta) \subset R^m$ with $A(y) = \mathrm{graph}\, g_y$ for $g_y\colon R^u(\varepsilon) \to R^s$, $g_y(0)=y$ then $\bigcup_{y \in R^s} A(y)$ is a neighborhood of 0 in R^m.*

Proof. Continuity of $\mathscr{A}$ implies that $g\colon (x, y) \mapsto (x, g_y x)$ is a continuous map from $B = R^u(\varepsilon) \times R^s(\delta)$ into R^m. It suffices to prove that $g \mid \partial B$ is homotopic to the inclusion map $\partial B \hookrightarrow R^m - 0$. Such a homotopy is given by

$$G_t(x, y) = \big(x, (1-t)\, g_y + t\, y\big).$$

The curves $G_t(x, y)$ never pass through 0 when $(x, y) \in \partial B$ since $x=0$ implies $g_y(0)=y$ and so $G_t(x, y) = (0, y) \neq 0$. Thus, (4.1) is proved.

Now we may proceed to the main theorem of this section.

(4.2) Theorem. *For any small $\delta > 0$ there is a u-disc family $\tilde{W}^u_\delta$ through N, a neighborhood of Λ in M, which reduces to W^u_δ at Λ and is semi-invariant in the sense that*

$$\tilde{W}^u_\delta(f x) \subset f\, \tilde{W}^u_\delta(x) \qquad \text{for any } x \in N \cap f^{-1} N.$$

Moreover for any $p \in \Lambda$, $\bigcup_{y \in W^s_\varepsilon p} \tilde{W}^u_\delta(y)$ is a neighborhood of p in M.

Remark. If f is C^r we can make $\tilde{W}^u_\delta$ a continuous family of C^r u-discs.

Remark. We cannot expect any sort of uniqueness for $\tilde{W}^u_\delta$, as simple examples show with Λ taken to be one point. It is unknown whether $\tilde{W}^u_\delta$ can be chosen to foliate a neighborhood of Λ.

The proof of (4.2) is similar to the existence of $\{W^u_\varepsilon(p) \mid p \in \Lambda\}$ in [1]. All estimates needed here were proved in [1]. We must deal with graph transforms induced by maps of one space to another instead of to itself. We outline what must be re-interpreted.

Let $E=E_1\times E_2$, $F=F_1\times F_2$ be Banach spaces, equiped with the product norms

$$|(x_1,x_2)|=\max(|x_1|,|x_2|).$$

Define $\mathscr{G}_\varepsilon(E)$ to be the space of maps $g\colon E_1(\varepsilon)\to E_2$ such that

$$g(0)=0, \qquad L(g)\leqq 1$$

where L denotes the Lipschitz constant and $E(\varepsilon)$ denotes the closed ball of radius ε. The metric in $\mathscr{G}_\varepsilon(E)$ is $|g-g'|=\sup_x |g\,x-g'\,x|$ and is complete. Define $\mathscr{G}_\varepsilon(F)$ similarly.

Let $f\colon E(\varepsilon)\to F$, $f(0)=0$ be a map. It might happen that for every $g\in\mathscr{G}_\varepsilon(E)$ there exists $h\in\mathscr{G}_\varepsilon(F)$ such that

$$\text{graph}\, h\subset f(\text{graph}\, g).$$

If so, h is uniquely determined by g and we call h the graph transform of g by f,

$$h=\Gamma_f\, g.$$

The next theorem was proved in [1, §4].

(4.3) Theorem. *Let $T\colon E\to F$ be an isomorphism sending E_1 onto F_1, E_2 onto F_2 when $E_1\times E_2=E$, $F_1\times F_2=F$ as above. Suppose $\|T_2\|\leqq\tau$, $\|T_1^{-1}\|\leqq\tau$, $0<\tau<1$ for $T_i=T|E_i$, $i=1,2$. If $f\colon E(\varepsilon)\to F$ is sufficiently near $T|E(\varepsilon)$ then $\Gamma_f\colon \mathscr{G}_\varepsilon(E)\to\mathscr{G}_\varepsilon(F)$ is defined and has $L(\Gamma_f)<1$. The precise condition on f is*

$$f(0)=0 \quad \textit{and} \quad L(f-T)<\frac{1-\tau}{1+\tau}.$$

Before proving (4.2) we need a simple extension lemma.

(4.4) Lemma. *Let E, M be manifolds and $\pi\colon E\to M$ a fiber bundle. If $s\colon X\to E$ is a section over a closed $X\subset M$ then s extends to $\bar{s}\colon N\to E$, a section over a neighborhood of X.*

Proof. Since E is an ANR (absolute neighborhood rectract) and M is a normal space, s extends to a map $s_0\colon N_0\to E$ when N_0 is a neighborhood of X in M. Let d be a metric on M. Choosing small neighborhoods $N\subset N_0$ of X makes the function $d(\pi\, s_0\, x, x)$ small if $x\in N_0$. Because M is an ANR, this makes $\pi\, s_0|N\colon N\to M$ homotopic, rel X, to the inclusion $i_N\colon N\hookrightarrow M$. The Covering Homotopy Theorem [7] shows that $s_0|N$ is homotopic, rel X, to a map $\bar{s}\colon N\to E$ such that $\pi\,\bar{s}=i_N$. This proves the lemma.

Proof of (4.2). Let the skewness of $Tf|T_\Lambda M$ be $<\tau$. By (4.4) there exists a neighborhood N_0 of Λ and extensions $\bar{E}^u$, $\bar{E}^s$ of the bundles E^u, E^s to N_0 for they are sections of the Grassmann bundles $G^u(M)$, $G^s(M)$ restricted to Λ. Taking a smaller N_0 if necessary we have $\bar{E}^u\oplus\bar{E}^s=T_{N_0}M$ since $E^u\oplus E^s=T_\Lambda M$.

Applying (4.4) again we can find a "connector" θ, that is a continuous family of isomorphisms $\theta(x,x')\colon M_x \to M_{x'}$ defined for all $(x,x') \in M \times M$ sufficiently near the diagonal Δ of $\Lambda \times \Lambda$ such that

$$\theta(x,x)=I_x, \qquad \theta(x,x')\bar{E}^u_x=\bar{E}^u_{x'}, \qquad \theta(x,x')\bar{E}^s_x=\bar{E}^s_{x'}.$$

In fact $GL(\bar{E}^u)$ is a bundle over $N_0 \times N_0$ whose fiber over (x,x') is the space of all isomorphisms $\bar{E}^u_x \to \bar{E}^u_{x'}$ and the map $(x,x) \to I_x$ is a section of $GL(\bar{E}^u)$ over Δ. By (4.4) this section extends giving a section θ^u. Similarly we get a section θ^s. Combining the two as $\theta^u \oplus \theta^s$ we get the connector θ.

Of course $\bar{E}^u \oplus \bar{E}^s$ is not likely to be Tf-invariant. Let $(Tf)_x\colon M_x \to M_{fx}$ be defined, with respect to this splitting, by the matrix

$$(Tf)_x=\begin{pmatrix} A_x & B_x \\ C_x & K_x \end{pmatrix}$$

for $x \in f^{-1}N_0 \cap N_0$. Consider the linear map $T_x\colon M_x \to M_{fx}$ defined on $f^{-1}N_0 \cap N_0$ by

$$T_x=\begin{pmatrix} A_x & 0 \\ 0 & K_x \end{pmatrix}.$$

At Λ the entries B and C vanish and $T=Tf$.

For any $N \subset f^{-1}N_0 \cap N_0$ the map $T\colon T_N M \to T_{fN} M$ defined by $T|M_x = T_x$ is a bundle isomorphism leaving $\bar{E}^u \oplus \bar{E}^s$ invariant and equal Tf at Λ.

Let $\bar{f}=\exp^{-1} \circ f \circ \exp\colon M(\delta) \to TM$, where $M(\delta)=\bigcup_{x\in M} M_x(\delta)$. For small enough δ and a small enough neighborhood $N \subset \bar{N}_0$ of Λ, $\bar{f}$ is well defined and

$$\sup_{x\in N} L((\bar{f}-T)\,|\,M_x(\delta))=\xi$$

satisfies the inequality of (4.3), $\xi<(1-\tau)/(1+\tau)$. For when $s \to 0$ and $N \to \Lambda$ we are merely seeing how well Tf approximates $\bar{f}$ at Λ. By (4.3) $\Gamma_{\bar{f}}\colon \mathscr{G}_\delta(T_N M) \to \mathscr{G}_\delta(T_{fN} M)$ is defined and has Lipschitz constant <1.

We may assume N chosen to be a smooth manifold with boundary. If V is a neighborhood of fN with $V \subset N_0$ and $\rho\colon V \to fN$ a retraction then $\theta(\rho x, x)$ will be defined for V small enough and ρ near enough the inclusion $i_V\colon V \hookrightarrow M$.

Define $h\colon V \to N$ by $h=f^{-1} \circ \rho$. For each $x \in V$ define $(\theta,\bar{f})_x\colon M_{hx}(\delta) \to M_x$ by

$$(\theta,\bar{f})_x=\theta(\rho x, x) \circ \bar{f}\,|\,M_{hx}(\delta)$$

and define $(\theta,T)_x\colon M_{hx} \to M_x$ by

$$(\theta,T)_x=\theta(\rho x, x) \circ T_{hx}.$$

Since θ is continuous, and equals I_x at Λ, $(\theta, T)_x$ may be made arbitrarily near $Tf \mid T_\Lambda M$ by taking N small, V near N, and ρ near i_V. That is

$$\sup_{x \in V} L\big((\theta, \bar{f})_x - (\theta, T)_x\big) = \bar{\xi}$$

can be made near ξ and in particular can be made $<(1-\tau)(1+\tau)$. But (θ, T) preserves $\bar{E}^u$ and $\bar{E}^s$ because both θ and T do. Taking N small enough assures (θ, T) has skewness $\leqq \tau$ because $Tf \mid T_\Lambda M$ has skewness $<\tau$. Hence we may apply (4.3) to conclude that $(\theta, \bar{f})$ defines a graph-transform $\Gamma_{(\theta, \bar{f})}: \mathscr{G}_\delta(T_N M) \to \mathscr{G}_\delta(T_V M)$ with Lipschitz constant <1.

Let $\varphi: M \to [0, 1]$ be continuous with $\varphi = 0$ off V and $\varphi = 1$ on fN. Define $F: \mathscr{G}_\delta(T_N M) \to \mathscr{G}_\delta(T_N M)$ by sending $g \in \mathscr{G}_\delta(T_N M)$ onto the element of $\mathscr{G}_\delta(T_N M)$ which, when restricted to $\bar{E}^u_x(\delta)$ equals

$$\begin{array}{ll} 0 & \text{if } x \in N - V, \\ \varphi(x)\, \Gamma_{(\theta, \bar{f})_x}(g) & \text{if } x \in V \cap N. \end{array}$$

If $x \in \partial V \cap N$, we know that $\varphi(x) = 0$ and so the two definitions agree. Note that $\Gamma_{(\theta, \bar{f})_x}(g)$ is a map $\bar{E}^u_x(\delta) \to \bar{E}^s_x(\delta)$ and so our definition of F is well stated. Also note that F is *not* globally a graph transform: it does not cover a map $N \to N$.

Clearly F carries $\mathscr{G}_\delta(T_N M)$ into itself because $\Gamma_{(\theta, f)}$ carries $\mathscr{G}_\delta(T_N M)$ into $\mathscr{G}_\delta(T_V M)$ and multiplication by φ, $0 \leqq \varphi \leqq 1$, can only help. In the computation of the Lipschitz size of the map $F(g)_x: \bar{E}^u_x(\delta) \to \bar{E}^s_x(\delta)$, φ is constant. For the same reason, $L(F) \leqq L(\Gamma_{(\theta, f)}) < 1$. Hence F has a unique fixed point, say g^*.

Put $\tilde{W}^u_\delta(x) = \exp_x(\text{graph}(g^* \mid M_x(\delta)))$, where $x \in f^{-1} N \cap N$. For such x, it is clear that $f \tilde{W}^u_\delta(x) \supset \tilde{W}^u_\delta(fx)$ by the definition of F. On Λ we have just gone through the construction of W^u_δ in [1] again and so $\tilde{W}^u_\delta(p) = W^u_\delta(p)$ by uniqueness of $\{W^u_\delta(p) \mid p \in \Lambda\}$. Thus $\tilde{W}^u_\delta$ is a semi-invariant family of u-discs extending W^u_δ. It remains to show that $\bigcup_{y \in W^s_\varepsilon(p)} \tilde{W}^u_\delta(y)$ is a neighborhood of p for $p \in \Lambda$.

There is a diffeomorphism $j: M_p(\varepsilon) \to M_p(\varepsilon)$ such that $(Tj)_p = I_p$ and $j(\exp_p^{-1}(W^s_\varepsilon p)) = E^s_p(\varepsilon)$, $j(\exp_p^{-1}(W^u_\varepsilon p)) = E^u_p(\varepsilon)$. So

$$\{j \circ \exp^{-1}(\tilde{W}^u_\delta y) \mid y \in W^s_\mu p\}$$

is a u-disc family through $E^s_p(\mu)$. Its radius may be slightly less than δ, but it is greater than $\delta/2$, say, if μ is small enough. Also, for small δ these u-discs will be graphs of maps $E^u_p(\delta/2) \to E^s_p(\varepsilon)$ having slope $\leqq 2$. By (4.3) this u-disc family includes a neighborhood of p in $M_p(\varepsilon)$ and hence its $\exp_p \circ j^{-1}$ image $\{\tilde{W}^u_\delta(y) \mid y \in W^s_\mu p\}$ contains a neighborhood of p in M, since $\exp_p \circ j^{-1}$ is a local diffeomorphism at p. This proves (4.2).

Remark. In a similar manner we could obtain an invariant family of linear graphs $G_x: \bar{E}^u_x \to \bar{E}^s_x$. These determine a subbundle of $T_N M$ semi-invariant under Tf, and thus a semi-invariant extension of $E^s \oplus E^u$ to $\tilde{E}^u \oplus \tilde{E}^s = T_N M$. Proceeding with the construction of $\tilde{W}^u_\delta$ we find that $\tilde{W}^u_\delta(y)$ is tangent to $\tilde{E}^u_y$ at y for y near Λ. In fact, it can be shown that $\tilde{W}^u_\delta(y)$ is C^r when f is C^r by imitating the proof of smoothness of unstable manifolds in either [1] or [2]. In this way one can prove.

(4.5) Theorem. *The semi-invariant family* $\tilde{W}^u_\delta$ *of* (4.2) *can be chosen to be a continuous family of* C^r *u-discs when* f *is* C^r.

§ 5. Fundamental Neighborhoods and the Proof of the Main Theorem

As before, let Λ be a hyperbolic set for $f \in \mathrm{Diff}(M)$. A *fundamental neighborhood* for W^s_ε is a compact neighborhood of a fundamental domain of W^s_ε. Similarly for W^u_ε.

The following theorem was stated in § 8 of the mimeographed version of [1] but was proved incorrectly.

(5.1) Theorem. *If* V *is a fundamental neighborhood for* W^s_ε *then* $W^u_\varepsilon \cup 0_+ V$ *is a neighborhood of* Λ *in* M.

From (5.1) we can finish the proof of (1.1). By (3.1), (3.2) it suffices to prove the following

(5.2) Theorem. *If* $W^u_\varepsilon \Lambda$ *has a proper fundamental domain* D *then* Λ *has a neighborhood* U *such that if* $0_+ x \subset U$ *then* $x \in W^s_\varepsilon(p)$ *for some* $p \in \Lambda$.

Proof. Let V be a compact neighborhood of D disjoint from Λ. Set

$$U = (W^s_\varepsilon \cup 0_- V) - V$$

where $0_- V = \bigcup_{n \geqq 0} f^{-n} V$. By (5.1) applied to f^{-1}, U is a neighborhood of Λ. If $f^n(x) \in U$ for all $n \geqq 0$ then $x \notin 0_- V$ for otherwise $f^n(x) \in V$ for some $n \geqq 0$, but $f^n(x) \in U \subset M - V$ which is a contradiction. Therefore $x \in W^s_\varepsilon$, proving the theorem.

Proof of (5.1). Let D be a fundamental domain for W^s_ε and let V be a neighborhood of it. We must show that $W^u_\varepsilon \cup 0_+ V$ is a neighborhood of Λ in M.

Consider the semi-invariant u-disc family $\tilde{W}^u_\delta$ constructed in (4.2) where δ has been chosen so small that $\tilde{W}^u_\delta(y) \subset V$ for any $y \in D$ and $0 < \delta \leqq \varepsilon$.

Clearly

$$W^u_\varepsilon \cup 0_+ V \supset W^u_\varepsilon \cup \bigcup_{y \in D} \tilde{W}^u_\delta(0_+ y)$$

by the semi-invariance of $\tilde{W}^u_\delta$. But $W^s_\varepsilon = \Lambda \cup \bigcup_{y \in D} (0_+ y)$ and so

$$W^u_\varepsilon \cup \bigcup_{y \in D} \tilde{W}^u_\delta(0_+ y) \supset \bigcup_{y \in W^s_\varepsilon} \tilde{W}^u_\delta(y).$$

But for each $p \in \Lambda$, $\bigcup_{y \in W^s_\varepsilon} \tilde{W}^u_\delta(y) \supset \bigcup_{y \in W^s_\varepsilon p} \tilde{W}^u_\delta(y)$ which is a neighborhood of p in M by (4.2). Hence $W^u_\varepsilon \cup 0_+ V$ is a neighborhood of Λ and (5.1) is proved.

§ 6. The Pseudo-Hyperbolic Case

The goal of this section is to generalize (4.2) to pseudo hyperbolic invariant sets of diffeomorphisms. We will apply this theorem to extend to flows the main results of the earlier sections about diffeomorphisms.

Recall that a compact subset Λ of M is pseudo hyperbolic for $f \in \mathrm{Diff}(M)$ if $f\Lambda = \Lambda$ and Tf leaves invariant a continuous spitting $T_\Lambda M = E_1 \oplus E_2$ such that $\|(Tf)^{-1} | E_1\| < k$, $\|Tf | E_2\| < \ell$, and $k\ell < 1$. For $f \in \mathrm{Diff}^r(M)$, $r \geqq 1$, Theorem 3B.5 of [2] says

(6.1) Theorem. *If $k < 1$ then there exists $\varepsilon > 0$ and a unique f-invariant C^r regular family $W_{1,\varepsilon} = \{W_{1,\varepsilon}(p) : p \in \Lambda\}$ of submanifolds of size ε such that $W_{1,\varepsilon}(p)$ is tangent to $E_1(p)$ for each $p \in \Lambda$. The manifold $W_{1,\varepsilon}(p)$ is characterized by $x \in W_{1,\varepsilon}(p)$ if and only if*

$$d(f^{-n}x, f^{-n}p) \leqq \varepsilon \qquad \text{for all } n \geqq 0$$

$$\lim_{n\to\infty} \frac{d(f^{-n}x, f^{-n}p)}{k^n} = 0.$$

We extend W_1 to a disc family through a neighborhood of Λ as in § 4.

(6.2) Theorem. *If Λ is as in (6.1) then for any small $\delta > 0$ there is a disc family $\tilde{W}_{1,\delta}$ through N, a neighborhood of Λ, which reduces to $W_{1,\delta}$ at Λ and is semi-invariant in the sense that*

$$\tilde{W}_{1,\delta}(fx) \subset f\tilde{W}_{1,\delta}(x) \qquad \text{for } x \in N \cap f^{-1}N.$$

Proof. The proof of (6.2) is essentially the same as that of (4.2), except that a new complete metric on the function spaces $\mathscr{G}_\varepsilon(E)$, $\mathscr{G}_\varepsilon(F)$, etc. must be used. It was introduced in [2, § 3B] and is

$$|g - g'|_* = \sup_{x \neq 0} |gx - g'x| / |x|.$$

The estimates demonstrating the analogue of (4.3) in the pseudo hyperbolic case with $k < 1$ are contained in [2, § 3B] and are similar to those in [1]. Once this analogue of (4.3) is proved, the proof of (6.2) is exactly the same as that of (4.2).

§ 7. Flows

Let $\{\varphi_t\}$ be a C^r flow on M, $r \geqq 1$. A compact invariant subset $\Lambda \subset M$ is said to be hyperbolic for $\{\varphi_t\}$ if, for every $t>0$, $T\varphi_t$ leaves invariant a continuous splitting

$$T_\Lambda M = E^u \oplus E^\varphi \oplus E^s$$

expanding E^u and contracting E^s, where E^φ is the tangent bundle to the orbits of the flow. If X is the vector field generating the flow then E^φ_x is the 0-dimensional or 1-dimensional space spanned by X_x. The equality $(T\varphi_t)_x(X_x) = X_{\varphi_t x}$ is valid for any smooth flow $\{\varphi_t\}$, so invariance of E^φ is automatic. In [5] and [2] details and equivalent definitions are given.

Note that Λ is a pseudo hyperbolic set for φ_t, $t \neq 0$. If $t>0$ then $E_1 = E^u$ and $E_2 = E^\varphi \oplus E^s$. If $t<0$ then $E_1 = E^s$ and $E_2 = E^u \oplus E^\varphi$. In either case we can choose $k<1$ where $\|(T\varphi_t | E_1)^{-1}\| < k$.

Applying (6.1) to φ_1, we get a unique φ_1-invariant family through Λ, $W^u_\varepsilon = \{W^u_\varepsilon p \,|\, p \in \Lambda\}$ tangent to E^u. By uniqueness, as in [2], W^u_ε is locally φ_t invariant for all t. Applying (6.1) to φ_{-1}, we get a unique φ_{-1} invariant family through Λ, $W^s_\varepsilon = \{W^s_\varepsilon p : p \in \Lambda\}$ tangent to E^s and W^s_ε is φ_t-invariant for all t. For any orbit $\mathcal{O} \subset \Lambda$ we put

$$W^u_\varepsilon \mathcal{O} = \bigcup_{p \in \mathcal{O}} W^u_\varepsilon p, \qquad W^s_\varepsilon \mathcal{O} = \bigcup_{p \in \mathcal{O}} W^s_\varepsilon p.$$

(7.1) Theorem. *The $\{W^u_\varepsilon \mathcal{O}\}$, $\{W^s_\varepsilon \mathcal{O}\}$ form invariant families of C^r manifolds. The families $\{W^u_\varepsilon p \,|\, p \in \mathcal{O}\}$, $\{W^s_\varepsilon p : p \in \mathcal{O}\}$ form C^r fibrations of $W^u_\varepsilon \mathcal{O}$, $W^s_\varepsilon \mathcal{O}$.*

Proof. Since $\varphi_t W^u_\varepsilon p \supset W^u_\varepsilon(\varphi_t p)$ for $t>0$ and $\varphi_t W^s_\varepsilon p \supset W^s_\varepsilon(\varphi_t p)$ for $t<0$, (7.1) is a direct consequence of (6.1).

We say that Λ has local product structure if

$$W^s_\varepsilon \cap W^u_\varepsilon = \Lambda$$

for some $\varepsilon > 0$. (2.3) and (3.1) generalize at once to

(7.2) Proposition. *Let $\{\varphi_t\}$ be a flow satisfying Axiom A' [5], i.e. the nonwandering set Ω is hyperbolic and the periodic orbits are dense in Ω. Then Ω has local product structure.*

(7.3) Lemma. *If $t>0$ and Λ is hyperbolic for $\{\varphi_t\}$ then the set $D^s = Cl(W^s_\varepsilon - \varphi_t W^s_\varepsilon)$ is a fundamental domain for $W^s_\varepsilon \Lambda$. If Λ has local product structure, D^s is proper.*

We also have

(7.4) Theorem. *If $D \subset W^s_\varepsilon \Lambda$ is a fundamental domain for some φ_τ, $\tau > 0$, and V is a neighborhood of D. Then the set*

$$W^u_\varepsilon \Lambda \cup 0_+ V$$

is a neighborhood of Λ in M.

Proof. Extend the disc family W_ε^u to a disc family $\tilde{W}_\varepsilon^u$, semi-invariant by φ_τ, through a neighborhood of Λ. This can be done by (6.2). Then apply (4.1) as in (4.2). Since $0_+ V = \bigcup_{t \geq 0} \varphi_t V \supset \bigcup_{n \geq 0} \varphi_{n\tau} V$, the theorem is proved.

(7.2, 7.3, 7.4) provide the analogue of (1.1) for flows.

Remark. (7.4) holds for normally hyperbolic foliations as defined in [2]. The proof is the same.

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