Inventiones math. 10, 187–198 (1970)
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# Linearization of Normally Hyperbolic Diffeomorphisms and Flows

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## 1. Introduction

In this paper we linearize a diffeomorphism near an invariant submanifold in the presence of normal hyperbolicity.

**Definition.** *If $f\colon M\to M$ is a $C^1$ diffeomorphism of a Riemannian manifold $M$ leaving invariant the compact $C^1$ submanifold $V$, $fV=V$, then $f$ is normally hyperbolic at $V$ provided that its tangent $Tf\colon T_V M\to T_V M$ leaves invariant a continuous splitting $T_V M=N^u\oplus TV\oplus N^s$ and*

(a) $m(N^u f)>\|Vf\|$,

(b) $\|N^s f\|<m(Vf)$

*where $N^u f=Tf\,|\,N^u$, $Vf=Tf\,|\,TV$, $N^s f=Tf\,|\,N^s$ and*

$$m(N^u f)=\inf_{p\in V}\|N_p^u f^{-1}\|^{-1},\qquad \|Vf\|=\sup_{p\in V}\|V_p f\|,$$

$$\|N^s f\|=\sup_{p\in V}\|N_p^s f\|,\qquad m(Vf)=\inf_{p\in V}\|V_p f^{-1}\|^{-1}.$$

See [3] where normal hyperbolicity is discussed extensively. Conditions (a), (b) mean that the normal behavior dominates the tangent behavior.

**Definition.** *A $C^1$ flow $\{f^t\}$ on $M$ is normally hyperbolic at $V$ if $f^t V=V$ for all $t$ and $f^1$ is normally hyperbolic at $V$.*

From [3] we know that all $f^t$, $t\neq 0$, are normally hyperbolic at $V$ if one, say $f^1$, is. The splitting $N^u\oplus TV\oplus N^s$ is independent of $t$. If $V$ is a closed orbit of the flow $\{f^t\}$ then normal hyperbolicity at $V$ is equivalent to genericity of $V$ in the usual sense.

**Theorem 1.** *If $f$ is normally hyperbolic at $V$ then $f$ is conjugate to $Nf$ near $V$.*

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* Supported by the sloan Foundation and NSF GP 8007.

** Nato Postdoctoral fellow.

**Theorem 2.** *If* $\{f^t\}$ *is normally hyperbolic at* $V$ *then* $\{f^t\}$ *is conjugate to* $\{N f^t\}$ *near* $V$.

$N f$ is $Tf|N^u \oplus N^s$ and the conjugacy is a homeomorphism $h$ defined from a neighborhood of the zero section of $NV$ to a neighborhood of $V$ in $M$ such that $N(f)\,h = h\,f$.

A conjugacy between the flows $\{f^t\}$ and $\{N f^t\}$ is a single homeomorphism that conjugates each $f^t$ to $N f^t$ near $V$.

The proofs of Theorems 1, 2 were inspired by the geometric proof of Hartman's Theorem in Palis [6] and the proofs of Hartman's Theorem for a fixed point in a Banach space Palis [5] and Pugh [8]. For Hartman's Theorem is the case $V$= one point in Theorems 1, 2 [1a]. Theorem 2 for the case of a closed orbit was independently proven by Irwin [4]. The techniques used to prove Theorem 2 simplify when $V$ is a closed orbit as follows:

**Proposition.** *If* $\{f^t\}$ *is normally hyperbolic at a closed orbit* $V$ *then* $\{f^t\}$ *is conjugate to* $\{N f^t\}$ *near* $V$.

*Proof.* Let $V$ have period $\tau$. We need only find topological disks $D_0 \subset D$ transverse to $V$ at $x \in V$ such that $f^\tau(D_0) \subset D$, for then $f^\tau|D_0$ is conjugate to the Poincaré transformation on a differentiably transverse disc just by following the solution curves near $V$ and the Poincaré transformation is conjugate to $N f^\tau|N_x V$. Thus there is a local conjugacy $h\colon U \to D_0$. Now we may define the conjugacy $H$ by $H(x) = f^t\,h\,N f^{-t}(x)$ where $0 \leqq t < \tau$ and $N f^{-t}(x) \in D_0$. $H$ is defined and continuous in a neighborhood of $V$ and $HN f^t = f^t H$. The existence of the topologically transverse disc is a simple application of the proof of Theorem 1, extending $W^{uu}$ over $W^{ss}_\varepsilon x$ for $f^\tau$.

Takens has recently shown that a differentiable $H$ exists generically for $V$ a closed orbit.

For such linearization theorems, the notion of normal hyperbolicity may be unnecessarily strong. For instance, suppose $N^u = 0$, $N^s = N$ is left invariant by $Tf$ and contracted, although perhaps not so sharply as is $TV$. We could then find a diffeomorphism $g$ leaving $V$ pointwise fixed such that $Tg$ leaves $NV$ invariant and contracts each fiber by constant-multiplication, $c > 0$. Then $g \circ f$ would be normally hyperbolic (purely contracting) if $c$ were small enough. Also it could be shown that $g \circ f$ is conjugate to $f$ near $V$ and $N(g \circ f)$ is conjugate to $N f$. By Theorem 1, $N(g \circ f)$ is conjugate to $g \circ f$ near $V$ and thus $N(f)$ is conjugate to $f$. This says that in the purely contracting case, we could weaken the normal hyperbolicity assumption to "0-normal hyperbolicity": the normal behavior, $N f$, dominates the zero-th power of $Tf$ on $TV$. Similarly in the purely expanding case. If we try the same trick in the true hyperbolic case, $N^u \neq 0 \neq N^s$, we would make $N^u g$ = multiplication by $C > 1$, $N^s g$ =

multiplication by $0<c<1$, then $g\circ f$ would be normally hyperbolic at $V$ for $c$ small and $C$ large. Although it can be seen that $N(g\circ f)$ is conjugate to $N(f)$, it is not clear whether $g\circ f$ is conjugate to $f$.

## 2. Linearization in Banach Bundles

The proofs of Theorems 1, 2 are not so similar as we would like. Both rely on forms of Hartman's Theorem for Banach bundles, but in the flow case we prove only a purely contracting Banach bundle theorem.

**(2.1) Theorem.** *Let*

$$\begin{array}{ccc} E & \xrightarrow{F} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X \end{array}$$

*be a hyperbolic Banach bundle automorphism covering the homeomorphism $f$. Let $F'$: $E\to E$ be continuous and obey*

(a) *$F'$ covers $f$,*

(b) *$L(F'_x-F_x)<\mu$ $x\in X$ and $|F'_x-F_x|<\mu$,*

(c) *$F'(O_x)=O_{fx}$ $x\in X$,*

*where $F_x$, $F'_x$ are $F$, $F'$ restricted to the fiber over $x$, $E_x$, and $O_x\leftrightarrow x$ is the origin of $E_x$. Then $F'$ is conjugate to $F$. The conjugacy leaves $X$ pointwise fixed and preserves $E$-fibers. It is the unique conjugacy covering the identity map $X\to X$ and at a finite distance from the identity map $E\to E$. The constant $\mu$ is determined as in* [8] *by $\mu<1-\tau$ where $\tau$ is the skewness.*

*Proof.* There are two ways to produce the conjugacy $h$ between $F'$ and $F$. One may just consider the functional analytic proof of Hartman's Theorem [5, 8] and observe that at no stage does the presence of the parameter $x\in X$ complicate the argument. Alternatively, one may let $F$, $F'$ induce hyperbolic automorphisms $F_b$, $F'_b$ on $\Sigma^b E$, the Banach space of bounded sections of $E$. The map $F'_b$ is within $\mu$ of the hyperbolic automorphism $F_b$. They are conjugate using Hartman's Theorem directly, say by $H$: $\Sigma^b E\to\Sigma^b E$. The characterization of $H$ can then be used, as in [2] and the deduction of stable manifold theory for hyperbolic sets from the theory for a point in a Banach space, to conclude that $H$ induces a conjugacy between $F'$ and $F$. It is defined by

$$h(y)=H(i_x\,y)(x)$$

for $y\in E_x$ and $i_x\,y$ the bounded section of $E$ which vanishes except at $x$ when it equals $y$.

Though the second proof is somehow more satisfying, it is considerably longer. Both are straightforward.

The functional analytic proof of Hartman's Theorem gives uniqueness. In practice we must deal with a local situation.

**(2.2) Theorem.** *Let $F$, $F'$ be as in* (2.1) *except that $F'$ is defined only on $U$, a uniform neighborhood of a closed $f$-invariant subset $X_0 \subset X$ and satisfies*

(a) *$F'$ covers $f$, equaling $f$ on $X \cap U$,*

(b) $L(F'_x - F_x) < \mu/2 \;\; x \in X \cap U$

*the function $F'_x - F_x$ being defined only on $U \cap E_x$. Then restrictions of $F'$ and $F$ to neighborhoods of $X_0$ are conjugate. The conjugacy equals the identity on $X$ and preserves fibers.*

*Proof.* It is merely a matter of extending $F'$, or a restriction of $F'$, to all of $E$ while preserving (a), (b). Then we apply the global Theorem (2.1) to $F$ and this extension. The resulting conjugacy, restricted to a neighborhood of $X_0$, works.

Let $\varepsilon > 0$ be chosen so that

$$U \supset U_\varepsilon = \{y \in E_x \colon d(x, X_0) \leqq \varepsilon, \, |y| \leqq \varepsilon\}$$

and let $\varphi$ be a continuous bump function on $X$, $0 \leqq \varphi \leqq 1$, vanishing off $U \cap X$ and equalling 1 on $U_\varepsilon \cap X$.

Let $\rho \colon E \to E(\varepsilon)$ be the radial retraction defined by

$$\rho(y) = \begin{cases} y & \text{if } |y| \leqq \varepsilon \\ \varepsilon \, y/|y| & \text{if } |y| \geqq \varepsilon. \end{cases}$$

Then define

$$\bar{F}(y) = \varphi(x) \cdot (F' - F) \circ \rho(y) + F(y)$$

for $y \in E_x$. When $x \notin U$, this makes $\bar{F}_x = F_x$. Note that $\bar{F}$ covers $f$, equals $F'$ on $U_\varepsilon$, and

$$L(\bar{F}_x - F_x) = L(\varphi(x) \cdot (F'_x - F_x) \circ \rho_x) \leqq \varphi(x) \, L(F'_x - F_x) \, L(\rho_x).$$

In the next lemma, we check that $L(\rho_x) \leqq 2$. Hence $\bar{F}$ verifies the hypotheses of (2.1) and (2.2) is proved.

**(2.3) Lemma.** *$L(\rho) \leqq 2$ where $\rho \colon E \to E(\varepsilon)$ is the radial retraction of the normed space $E$ onto its closed $\varepsilon$-ball $E(\varepsilon)$.*

*Proof.* Let $y_1, y_2 \in E$. We must show that

$$|\rho(y_1) - \rho(y_2)| \leqq 2 \, |y_1 - y_2|.$$

If $y_1, y_2 \in E(\varepsilon)$ then there is nothing to prove. This leaves the cases $|y_1| \leqq \varepsilon \leqq |y_2|$, $\varepsilon \leqq |y_1| \leqq |y_2|$. The first is a consequence of the second: a norm is continuous so there is a point $y_3$ on the segment $[y_1, y_2]$ having

$|y_3|=\varepsilon$. Then

$$|\rho(y_1)-\rho(y_2)|\leqq|\rho(y_1)-\rho(y_3)|+|\rho(y_3)-\rho(y_1)|$$
$$\leqq|y_1-y_3|+2|y_3-y_2|\leqq 2[|y_1-y_3|+|y_3-y_2|]=2|y_1-y_2|.$$

It remains to consider: $\varepsilon\leqq|y_1|\leqq|y_2|$.

$$|\rho(y_1)-\rho(y_2)|=\left|\frac{\varepsilon y_1}{|y_1|}-\frac{\varepsilon y_2}{|y_2|}\right|=\frac{\varepsilon}{|y_1|}\left|y_1-\frac{|y_1|}{|y_2|}y_2\right|$$
$$\leqq\left|y_1-\frac{|y_1|}{|y_2|}y_2\right|\leqq|y_1-y_2|+\left|y_2-\frac{|y_1|}{|y_2|}y_2\right|$$
$$\leqq|y_1-y_2|+\left|\frac{|y_2|-|y_1|}{|y_2|}\right||y_2|\leqq 2|y_1-y_2|.$$

*Remark.* This estimate is the sharpest possible for the Lipschitz constant of $\rho$. For $L(\rho)=2$ when $E=R^2$ and $|(x,y)|=\max(|x|,|y|)$.

(2.4) *Note.* The hypotheses (b) of (2.2) is verified on a small enough neighborhood of $X_0$ if

(b′) $F'_x$ is $C^1$, its derivative is uniformly continuous at $X_0$, and $DF'_x=F_x$ at $X_0$.

This differentiability of $F'_x$ means the derivative of $F'$ exist along fibers; its continuity at $X_0$ means that $(DF'_{x'})_y$ tends uniformly to $(DF_x)_0$ as $|y|\to 0$ and $x'\to x\in X_0$. Since $L(F_x-F'_x)\leqq\sup_y\|F_x-(DF'_x)_y\|$, (2.2b) holds on a small neighborhood of $X_0$.

## 3. Proof of Theorem 1

*Case* 1. $N^s=0$. That is, $f$ is purely expanding in the normal direction, $N=N^u$. According to [3], there is homeomorphism $g$: $N_\varepsilon\to W^u_\varepsilon$, the unstable manifold of $V$, carrying the linear fiber $N_\varepsilon(p)$ to the strong unstable fiber $W^{uu}_\varepsilon(p)$. The map g, restricted to each $N_\varepsilon(p)$, is $C^1$, uniformly in $p$, and

$$T(g|N_\varepsilon(p))\colon\ T_p(N(p))\to T_pM$$

is the inclusion $N(p)\hookrightarrow T_pM$. Since $N(p)$ is linear $T_p(N(p))$ is canonically isomorphic to $N(p)$.

We consider $f'=g^{-1}fg\colon N_\delta\to N$ for $\delta\leqq\varepsilon$. Then $f'$ and $N^u f=Nf$ are fiber preserving maps of $N$, both covering the same map $f|V$.

$$\begin{array}{ccc} N & \xrightarrow{f',Nf} & N \\ \big\downarrow & & \big\downarrow \\ V & \xrightarrow{f|V} & V \end{array}$$

Of course $f'$ is just defined locally. Since $T(g|N_\varepsilon(p))$ is the inclusion at $p$, $f'$ is tangent to $Nf$ along fibers at $V$. This lets us apply (2.4), (2.2) to produce a conjugacy $h$,

$$f'h = h \circ Nf.$$

Then $gh = H$ conjugates $Nf$ to $f$ since $H \circ Nf = g \circ h \circ Nf = g \circ f' \circ h = f \circ H$.

*Case* 2. $N^u = 0$. Apply Case 1 to $f^{-1}$.

*Case* 3. $N^u \neq 0 \neq N^s$. Restricting $f$ to the $C^1$ submanifold $W^s$, the stable manifold of $V$, we may apply Case 2 to produce a conjugacy of $N^s f$ to $f|W^s$ near $V$

$$h_s \colon N^s_\varepsilon \to W^s_\varepsilon, \qquad h_s \circ N^s f = f|W^s_\varepsilon \circ h_s.$$

At present $W^u$ is fibered over $V$ by the strong unstable fibers $W^{uu}$. We shall extend the fibration, $f$-invariantly, to be over $W^s_\varepsilon$.

Indeed let $G$ be a fundamental domain of $W^s V$. That is, let $G$ be a compact subset of $W^s_\varepsilon$ satisfying

$$f^2 G \cap G = \emptyset,$$

$$\bigcup_{n \geqq 0} f^n G \cup V \quad \text{is a neighborhood of } V \text{ in } W^s V.$$

Such domains exist by [6, 7]. Alternatively one could observe that a linear contractive bundle automorphism certainly has a fundamental domain: the closed unit disc bundle minus the image of the open unit disc bundle. Thus $N^s f$ has a fundamental domain and we could use $h_s$ to give us one for $f|W^s_\varepsilon$.

Over a neighborhood of $G$ in $W^s_\varepsilon$ introduce a $C^1$ fibration whose fibers meet $W^s V$ transversally at $G$, $\pi^u \colon U \to G$. As in [6, 7] the fibration $\pi^u$ and $f_* \pi^u = f \circ \pi^u \circ f^{-1}$ can be averaged over a neighborhood of $G \cap fG$ so that an $f$-invariant fibration over $G$ is achieved, call it again $\pi^u$. Then extend $\pi^u$ to an entire neighborhood of $V$ except $W^u_\varepsilon V$ by $f$-iteration as in [6, 7]. By the $\lambda$-lemma [6] $\pi^u$ extends regularly to the $W^{uu}$ fibration: the union of the $\pi^u$ fibers and the $W^{uu}$ fibers forms a continuous fibration over $W^s_\varepsilon$ and the tangent spaces to its fibers very continuously, Call $\bar{\pi}^u$ the union of these two fibrations:

$\bar{\pi}^u | W^u$ is the $W^{uu}$ fibration,
$\bar{\pi}^u$ is $f$-invariant regular,
$\bar{\pi}^u \colon \bar{U} \to W^s_\varepsilon\, \bar{\pi}^u | W^s_\varepsilon$ is the identity.

For $y \in N^s(p)$ define $\tilde{N}^u(y)$ to be the affine translate of $N^u(p)$ to $y$ in $N(p)$. This lets us look at $N$ as a bundle over $N^s$ and look at $Nf$ as a bundle expansion over the base homeomorphism $N^s f$

$$\begin{array}{ccc} N & \xrightarrow{Nf} & N \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ N^s & \xrightarrow{N^s f} & N^s \end{array}$$

For the fibers of $\tilde{\pi}$ are the $\tilde{N}^u$'s and they are invariantly expanded by $Nf$. We intend to apply (2.2) when $N^s = X$, $V = X_0$. To lift $f$ to $N$ in such a way that it also covers $N^s f$ requires a nice parameterization of the $\bar{\pi}^u$ fibers.

Define a homeomorphism $g: N_\varepsilon \to \overline{U}$ such that

(i) $g(y) = h_s(y)$ if $y \in N^s(p)$,

(ii) $g | \tilde{N}_\varepsilon^u(y)$ is a $C^1$ diffeomorphism onto the fiber $(\bar{\pi}^u)^{-1}(h_s y)$. Its derivative is continuous on $N_\varepsilon$ and equals the inclusion $N^u(p) \hookrightarrow T_p M$ at $y = p \in V$.

That such a $g$ exists will be seen in a moment. Now look at

$$f' = g^{-1} \circ f \circ g$$

which lifts $f$ to $N$, near $V$, as a fiber preserving map locally covering $N^s f$

$$\begin{array}{ccc} N_\varepsilon & \xrightarrow{f'} & N \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ N^s & \xrightarrow{N^s f} & N^s \end{array}$$

By (ii), $Nf$ and $f'$ are tangent along the $\tilde{\pi}$-fibers at $V$. That is, $D(f' | \tilde{N}^u(p))_p = Nf | N^u(p) = N^u f(p)$. Restricting to a small neighborhood of $V$, as in (2.4), we can apply (2.2) and find a conjugacy

$$h \circ Nf = f' \circ h.$$

Put $H = g \circ h$. Then $H \circ Nf = g \circ h \circ Nf = g \circ f' \circ h = f \circ H$ so $H$ is a conjugacy between $Nf$ and $f$ near $V$.

It remains to explain how the map $g: N_\varepsilon \to \overline{U}$ was defined. We just work backwards. As in [3] we have a smooth exponential map, exp. For $y \in N^s(p)$, the $\pi^u$-fiber through $h_s(y)$ is pulled back by $\exp_p^{-1}$ to the graph of a $C^1$ map $\gamma: N_\varepsilon^u(p) \to N^s(p) \oplus T_p V$. Let $\rho$ be the projection $T_p M \to N^u(p)$ along $N^s(p) \oplus T_p V$. Then $g$ can be defined as the composition

$$\tilde{N}_\varepsilon^u(y) \xrightarrow{\rho} N_\varepsilon^u(p) \xrightarrow{\text{graph}\,\gamma} T_p M \xrightarrow{\exp} (\bar{\pi}^u)^{-1}(h_s y).$$

When $y = p$, $g$ maps $N_\varepsilon^u(p)$ onto $W_\varepsilon^{uu}(p)$ the same as in [3] and its derivative is the inclusion required in (ii).

This completes the proof of Theorem 1.

## 4. Flows

Turning to the case of flows, we should like to imitate (2.2), (2.4). Unfortunately, we cannot do this, but we can prove a theorem sufficiently general for our needs.

**(4.1) Theorem.** *Let $\{F^t\}$ be a continuous linear flow on a Banach bundle $\pi\colon E\to X$ covering the flow $\{f^t\}$ on the base $X$. Suppose the time one map is a uniform contraction on fibers:*

$$\|F^1|E_x\|\leqq\alpha<1 \qquad \textit{for all } x\in X.$$

*Let $U$ be a negatively invariant nonempty subset of $X$. Let $\{G^t\}$ be a local flow over $U$ also covering $\{f^t\}$, leaving the zero section of $E$ invariant and being close to $\{f^t\}$:*

$$L(F^t-G^t|E_x)\leqq\mu<\min(\tfrac{1}{3}\beta,1-\alpha) \qquad 0\leqq t\leqq 1 \;\; x\in U$$

*for $\beta=\inf\{m(F^t|E_x)\colon x\in X, 0\leqq t\leqq 1\}$. Then $\{F^t\}$ and $\{G^t\}$ are conjugate over $U$ near the zero section. Similarly for a uniform expansion.*

*Remark.* By "negatively invariant" we mean

$$f^t(x)\in U \qquad \text{for all } t\leqq 0 \text{ and } x\in U.$$

By "local flow over $U$" we mean that $G^t(y)$ is defined, continuous, and has the group property, $G^{t+s}(y)=G^t(G^s(y))$, on its domain. This must include

$$\{(t,y)\colon 0\leqq t\leqq 1, |y|\leqq\varepsilon, \Pi y\in U\}\cup\{(t,G_s(y))\colon s\geqq 0, -s\leqq t\leqq 0\}$$

for some $\varepsilon>0$.

*Remark* 2. It is the problem of extending a non-smooth flow on a non-smooth manifold that forces us back to a local proof of this theorem, in contrast to the Banach space case [8].

*Remark* 3. If $F^1$ is hyperbolic, instead of purely contracting, we do not know whether the theorem holds.

**(4.2) Lemma.** *If $F\colon E\to E'$ is a Banach space isomorphism and $h\colon U\to E'$ for $U\subset E$ satisfies*

$$h(0)=0, \qquad L(h)<\tfrac{1}{3}m(F)$$

*then $|F+h|$ is radially monotone.*

*Proof.* For $x\in E$, $x\neq 0$, we must show that $|(F+h)(\lambda x)|$ increases as $\lambda$ increases, $\lambda>0$. When $0<\lambda<1$,

$$\begin{aligned}
|(F+h)(\lambda x)| &= \lambda\left|Fx+\frac{1}{\lambda}h(\lambda x)\right| \\
&= \lambda\left|Fx+h(x)+\frac{1}{\lambda}\left[(h(\lambda x)-h(x))+(h(x)-\lambda h(x))\right]\right| \\
&\leqq \lambda|(F+h)(x)|+|h(\lambda x)-h(x)|+|h(x)-\lambda h(x)| \\
&\leqq \lambda|(F+h)(x)|+L(h)(1-\lambda)|x|+(1-\lambda)|h(x)|.
\end{aligned}$$

Thus

$$|(F+h)(x)|-|(F+h)(\lambda x)|\geqq(1-\lambda)[|(F+h)(x)|-L(h)|x|-|h(x)|]$$
$$\geqq(1-\lambda)[m(F)-3L(h)]|x|>0$$

which proves the lemma.

*Proof of* (4.1). Consider the functions defined for $y\in E(\varepsilon)\cap\pi^{-1}U$ by

$$g(y)=\int_0^1|G^t(y)|\,dt,\qquad f(y)=\int_0^1|F^t y|\,dt.$$

We claim that

(i) The flow $\{G^t\}$ is topologically transverse to the level surface of $g$ off the zero section.

(ii) $g^{-1}(\varepsilon\mu)$ cuts every radius of $E(\varepsilon)\cap\Pi^{-1}U$ exactly once.

(iii) $g\circ G^t y$ and $|G^t(y)|$ become small, as $t$ grows large, at controlled rates until, if ever, $f^t(\pi y)$ leaves $U$.

(iv) Similarly for $f$, $\{F^t\}$.

Note that (i, ii, iii) $\Rightarrow$ (iv) by taking $G^t\equiv F^t$.

For $0<|y|\leqq\varepsilon$, $\pi y\in U$, and $0<t<1$,

$$g(G^t y)=\int_0^1|G^{t+s}(y)|\,ds=\int_t^{t+1}|G^s(y)|\,ds=\int_0^t|G^1G^s(y)|\,ds+\int_t^1|G^s(y)|\,ds$$
$$\leqq(\alpha+\mu)\int_0^t|G^s(y)|\,ds+\int_t^1|G^s(y)|\,ds$$

so that $g\circ G^s(y)$ decreases as $t$ increases. This proves (i).

By (4.2); the integrand $|G^t(\lambda y)|$, for any fixed $(t,y)$ is monotone in $\lambda$, $0\leqq\lambda\leqq1$. Clearly its integral also is: $g(y)$ is radially monotone. If $|y|=\varepsilon$ then

$$g(y)\geqq(\beta-\mu)|y|>\mu\varepsilon$$

so that for some unique $\lambda$, $0<\lambda<1$, $g(\lambda y)=\varepsilon\mu$. This proves (ii).

(iii) follows at once from the estimates on $\{F^t\}$.

By (ii), (iv) we can define a homeomorphism

$H\colon g^{-1}(\varepsilon\mu)\to f^{-1}(\varepsilon\mu)$

by sliding along radii. Then we extend $H$ by setting

$$\overline{H}|G^t(g^{-1}(\varepsilon\mu))=F^t\circ H\circ G^{-t},\qquad 0\leqq t\leqq\infty,$$
$$\overline{H}(y)=y\qquad\text{for }\pi y\in U,\ |y|=0$$

(i) insures $\overline{H}$ is well defined. Since $F^t$, $G^t$ cover the same map on $X$, $\overline{H}$ preserves fibers and covers 1. By (iii), (iv) $\overline{H}$ is defined on all of

$g^{-1}([0, \varepsilon\mu])$ and is continuous at 0. The inverse of $\bar{H}$ is given by

$$\bar{H}^{-1}|F^t(f^{-1}(\varepsilon\mu)) = G^t \circ H^{-1} \circ F^{-t} \qquad 0 \leqq t \leqq \infty$$

and for the same reasons is well defined, continuous, and defined on all of $f^{-1}([0, \varepsilon\mu])$. By construction $\bar{H}$ conjugates the two flows. The expanding case is treated by taking reverses of all flows, $\{F^{-t}\}$, $\{G^{-t}\}$, etc.

In order to produce smooth fundamental domains for flows we must use some Lyapunov theory.

**(4.3) Theorem.** *If $\{f^t\}$ is a $C^1$ flow on a finite dimensional smooth manifold $M$ and if $V$ is a uniform attractor then there is a smooth Lyapunov function for the flow at $V$, $H: M \to R$:*

$$H^{-1}(0) = V,$$

*$H$ decreases along trajectories near $V$.*

*Proof* [9]. By "uniform attractor" we mean that for some compact neighborhood $K$ of $V$

$$\bigcap_{t \geqq 0} K_t = V \quad \text{where } K_t = \{f^s(x)\colon\ x \in K,\ s \geqq t\}.$$

Now we are ready to prove Theorem 2 that a normally hyperbolic flow can be linearized.

*Proof of Theorem* 2. $\{f^t\}$ is a $C^1$ flow on $M$ leaving $V$ invariant. One $f^a$ is normally hyperbolic at $V$. By [3] all $f^t$, $t \neq 0$, are and the splitting $T_V M = N^u \oplus TV \oplus N^s$ is independent of $t$. By [3] the stable, strong stable, unstable, and strong unstable manifolds for $f^a$ are the same as those for $f^t$, $t\,a > 0$.

As in Theorem 1 we have three cases.

*Case* 1. $N^s = 0$. Using (4.1) instead of (2.1), the proof is the same as for Case 1 of Theorem 1.

*Case* 2. $N^u = 0$. Apply Case 1 to $\{f^{-t}\}$.

*Case* 3. $N^u \neq 0 \neq N^s$. In $W_\varepsilon^s$ there exists a $C^1$ submanifold $B$ of codimension one bounding a neighborhood of $V$, and across which $X = \frac{d}{dt} f^t \Big|_{t=0}$ points transversally inwards. $B$'s existence is assured by the Lyapunov Theorem (4.3).

Over $B$ we erect a $C^1$ fibration $\pi^u$ whose fibers are transverse to $W_\varepsilon^s$ as in [6, 7]. Then let $f^t$ act on $\pi^u$ by

$$f^t \circ \pi^u \circ f^{-t}.$$

Since $B$ and $\pi^u$ are $C^1$ and the flow is *differentiably* transverse to $B$ this produces, locally, a $C^1$ fibration, also called $\pi^u$, over a neighborhood of $B$ in $W_\varepsilon^s$. By the $\lambda$-lemma it tends regularly to the $W^{uu}$-fibration of $W_\varepsilon^u$ as $t \to \infty$. The union of the $\pi^u$ and the $W^{uu}$ fibrations is a fibration, $\bar{\pi}^u$, over $W_\varepsilon^s$ whose fibers have continuously varying tangent planes. (Of course, we always cut the $\bar{\pi}^u$-fibers down to a fixed neighborhood of $V$.) As in Theorem 1

$$\bar{\pi}^u | W^u \text{ is the } W^{uu}\text{-fibration,}$$

$$\bar{\pi}^u \text{ is } f^t \text{ invariant, } t \geqq 0, \text{ and regular,}$$

$$\bar{\pi}^u\colon \bar{U} \to W_\varepsilon^s,\ \bar{\pi}^u | W_\varepsilon^s \text{ is the identity.}$$

By Case 1, there is a homeomorphism $h_s\colon N_\varepsilon^s \to W_\varepsilon^s$ conjugating $\{N^s f^t\}$ and $\{f^t | W_\varepsilon^s\}$.

As in Theorem 1 define $\tilde{N}^u(y)$ to be the affine translate of $N^u(p)$ to $y$ in $N(p)$ when $y \in N^s(p)$. Then

$$\begin{array}{ccc} N & \xrightarrow{Nf^t} & N \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ N^s & \xrightarrow{N^s f^t} & N^s \end{array}$$

is a purely expanding bundle flow over the base flow $\{N^s f^t\}$. The $\tilde{\pi}$-fibers are the $\tilde{N}^u$.

Let $g$ be the same parameterization of the $\bar{\pi}^u$-fibration as in Theorem 1. Putting

$$G^t = g^{-1} \circ f^t \circ g$$

lifts $\{f^t\}$ to $N_\varepsilon$ as a fiber preserving flow locally covering $\{N^s f^t\}$.

$$\begin{array}{ccc} N_\varepsilon & \xrightarrow{G^t} & N \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ N^s & \xrightarrow{N^s f^t} & N^s \end{array}$$

As in Theorem 1, $N f^t$ and $G^t$ are tangent along the $\tilde{\pi}$-fibers at $V$. Restricting to a small enough neighborhood of $V$, as in (2.4), we may apply (4.7) to produce a conjugacy $h$ between $\{G^t\}$ and $\{N f^t\}$ near $V$. As in Theorem 1, $g \circ h$ conjugates $\{N f^t\}$ to $\{f^t\}$ near $V$.

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*(Received March 2, 1970)*