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The Ω-Stability Theorem for Flows

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§ 1. Introduction

Let X be a C^r tangent vector field, $r \geqq 1$, on a compact smooth Riemannian manifold M, without boundary, and let $\varphi = \{\varphi_t\}$ be the flow it generates. We think of φ as a C^r action of R on M. In [4, 5] Smale states and proves the Ω-stability theorem for diffeomorphisms of M – that is, C^r actions of Z on M. He states the corresponding theorem for flows and says "Presumably there is no difficulty in proving similar theorems for ordinary differential equations by the same method." This paper does just that and, assuming [2, 3] does it without much difficulty.

Ω-stability theorem (for flows). If φ obeys Axiom A′ and has the no cycle property then it is Ω-stable.

See § 2 for definitions of these terms.

Remark. In § 6 we handle global Ω-explosions in way different from Smale's.

§ 2. Definitions and Ω-Decompositions

We denote by Ω, the set of non wandering points of φ. Following [4] we say that φ obeys *Axiom A′ a* if the tangent flow $T\varphi = \{T\varphi_t\}$, $T\varphi_t$: $TM \to TM$, leaves invariant a continuous splitting $T_\Omega M = E^u \oplus E^\varphi \oplus E^s$ such that

(i) $T\varphi_t$ expands E^u, $t > 0$.

(ii) $T\varphi_t$ contracts E^s, $t > 0$.

(iii) $E_p^\varphi = \text{span}(X_p)$

for some Finsler on TM. Since $T\varphi_t(X_p) \equiv X_{\varphi_t p}$ for any smooth flow, invariance of E^φ is automatic.

A compact orbit of φ is either a circle or a point. The former is called a closed orbit, the latter a fixed point.

If p lies on a closed orbit of φ, $\mathcal{O}$, then $p \in \Omega$ and $E_p^\varphi = T_p\mathcal{O}$. By (i), (ii) $\mathcal{O}$ is hyperbolic – that is, the flow near $\mathcal{O}$ induces a transformation (the Poincaré map) on a hypersurface transverse to $\mathcal{O}$ at p and p is a hyperbolic

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fixed point of this transformation. If p is a zero of X then p is a fixed point of φ and belongs to Ω, but $E_p^\varphi = 0$. By (i), (ii) p is a hyperbolic fixed point. Hyperbolic fixed points are isolated, M is compact, and so $F = \{p : \varphi_t p \equiv p\} = \{p : X_p = 0\} = \{p : E_p^\varphi = 0\}$ is finite. Continuity of E^φ implies that F is disjoint from $\mathrm{Cl}(\Omega - F)$.

In particular closed orbits can not pass near fixed points. The second half of Axiom A' is *Axiom A' b.*

The closed orbits are dense in $\Omega - F$.

Now we can state one of the principal results.

(2.1) **Ω-Decomposition Theorem.** *If φ obeys Axiom A' then Ω decomposes uniquely as $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ where the Ω_i are disjoint, indecomposable components of Ω. On each Ω_i, φ is topologically transitive.*

"Indecomposable" means that Ω_i does not decompose as $\Omega_i' \cup \Omega_i''$ where Ω_i', Ω_i'' are disjoint components. "Component" is used in the usual sense a relatively open and closed non empty subset.

"Topologically transitive" means that the orbit, $\bigcup_{t \in R} \varphi_t S$, of any relatively open non empty set $S \subset \Omega_i$ is dense in Ω_i. Topological transitivity is equivalent to the existence of a point $p \in \Omega_i$ whose orbit $\{\varphi_t p : t \in R\}$ is dense in Ω_i [1].

Note that each φ-orbit $\mathcal{O} \subset \Omega$, being connected, lies in just one Ω_i. That is, the Ω_i are invariant. The remarkable assertion of (2.1) is that the process of successively dividing Ω into finite disjoint unions of components terminates at a finite stage. Given the other conditions "uniqueness" in (2.1) is clear. In § 4 we prove (2.1).

Now suppose φ obeys Axiom A' and has Ω-decomposition $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ as in (2.1). If $\alpha(x) \subset \Omega_i$ and $\omega(x) \subset \Omega_j$, for some $x \in M$, we write $\Omega_i \leqq \Omega_j$. The sets $\alpha(x)$, $\omega(x)$ are the usual α- and ω-limit sets of the trajectory through x:

$$\alpha(x) = \bigcap_{\tau < 0} \mathrm{Cl}\{\varphi_t x : t \leqq \tau\} \quad \omega(x) = \bigcap_{\tau > 0} \mathrm{Cl}\{\varphi_t x : t \geqq \tau\}.$$

φ is said to obey the *no cycle property* if there is *no cycle* respecting this partial order $\leqq$. A cycle would be $\Omega_{i_0} \leqq \Omega_{i_1} \leqq \Omega_{i_2} \leqq \cdots \leqq \Omega_{i_k} \leqq \Omega_{i_0}$ with $i_0, \ldots, i_k$ distinct and $k \geqq 1$.

To complete the explanation of (1.1) we must recall the idea of Ω-stability. Two flows are Ω-conjugate if there is a homeomorphism of the nonwandering set of the first onto that of the second, sending sensed trajectories onto sensed trajectories. The flow φ is said to be *Ω-stable* if it is Ω-conjugate to each flow φ' which is C^1 near it.

Strictly speaking an Ω-conjugacy is not a conjugacy because it is not required to preserve the parameterization of trajectories; only their sense (=direction). A sequence of flows $\varphi^1, \varphi^2, \varphi^3, \ldots$ tends to φ in the

C^r sense if $\varphi^n|M\times[0,1]\to\varphi|M\times[0,1]$ uniformly in the C^r sense. Since φ is completely determined by $\varphi|M\times[\alpha,\beta]$ for any $\alpha<\beta$, this defines the C^r topology on the set of C^r flows in a way compatible with our intuition.

The proof of the Ω-stability theorem (1.1), breaks up into three parts: the Ω-decomposition theorem, purely local Ω-explosions, and global Ω-explosions.

§ 3. Stable Manifold Theory for Flows

Here we amplify what appears in [2, § 7]. Let φ obey Axiom A'. The time one map, φ_1, is a diffeomorphism leaving invariant the compact set Ω. Its tangent leaves $E^u\oplus E^\varphi\oplus E^s$ invariant. We may apply [3, 3B.5] to φ_1 – and likewise to φ_{-1} – giving

(3.1) **Theorem.** *If φ obeys Axiom A' then through each point $p\in\Omega$ pass C^r injectively immersed manifolds $W^u(p)$, $W^s(p)$ tangent to E^u_p, E^s_p at p. They are characterized by*

$$W^u(p)=\{x\in M: \text{for some} \quad c<0 \quad d(\varphi_t x,\varphi_t p)\,e^{ct}\to 0 \quad \text{as } t\to-\infty\},$$

$$W^s(p)=\{y\in M: \text{for some} \quad c>0 \quad d(\varphi_t y,\varphi_t p)\,e^{ct}\to 0 \quad \text{as } t\to\infty\}.$$

$W^u(q)$ and $W^u(p)$ either coincide or are disjoint. The families $\mathscr{W}^u=\{W^u(p):p\in\Omega\}$, $\mathscr{W}^s=\{W^s p:p\in\Omega\}$ are invariant by $\varphi:\varphi_t(W^u p)=W^u(\varphi_t p)$, $\varphi_t(W^s p)=W^s(\varphi_t p)$.

$\mathscr{W}^u$ is locally smoothly continuous in the sense that if $q\to p$ then any compact smooth disc in $W^u(p)$ is uniformly approached in the C^1 sense by smooth discs in $W^u(q)$. Similarly for $\mathscr{W}^s$.

For any set $S\subset\Omega$, we write $W^u S=\bigcup_{p\in S} W^u(p)$, $W^s S=\bigcup_{p\in S} W^s(p)$. When $S=\Omega$, we just write W^u, W^s. For $p\in M$ and $\delta>0$, we denote by $\mathcal{O}_\delta(p)$ the δ-orbit of p, $\{\varphi_t p: -\delta\leqq t\leqq\delta\}$ and by $\mathcal{O}(p)$ its total orbit, $\{\varphi_t p: t\in R\}$.

From (3.1) $W^u\mathcal{O}$, $W^s\mathcal{O}$ are smooth submanifolds, invariant by the flow, and smoothly fibered by the $W^u p$, $W^s p$, $p\in\mathcal{O}$. At $\mathcal{O}$, they are tangent to $E^u\oplus E^\varphi$, $E^\varphi\oplus E^s$. Either $W^u\mathcal{O}$, $W^u\mathcal{O}'$ coincide or are disjoint. Likewise $W^s\mathcal{O}$, $W^s\mathcal{O}'$.

For small $\varepsilon>0$ we denote by $W^u_\varepsilon(p)$ the closed ε disc in $W^u p$, centered at p. Precisely, $W^u_\varepsilon(p)$ is the connected component of $\exp(T_p M(\varepsilon))\cap W^u(p)$ containing p. The set $T_p M(\varepsilon)$ is the closed ε-ball in the tangent space of M and exp is the smooth exponential on M. Similarly, we define $W^s_\varepsilon(p)$, $W^u_\varepsilon S$, $W^s_\varepsilon S$, W^u_ε, W^s_ε.

We say that Ω has local product structure if

$$W^u_\varepsilon\cap W^s_\varepsilon=\Omega$$

for some $\varepsilon>0$. The next theorem repeats [2, 7.2].

(3.2) **Theorem.** *If φ obeys Axiom A' then Ω has local product structure.*

Proof. The fixed point set F is bounded away from $\Omega - F$ and so, for small $\varepsilon > 0$, we must just prove that if $x, x' \in \Omega - F$ have $y \in W_\varepsilon^u(x) \cap W_\varepsilon^s(x')$ then $y \in \Omega$. Let τ be the smallest period of a closed orbit and choose δ, $0 < \delta < \tau/2$. For small $\varepsilon > 0$, (3.1) implies that the intersections $W_\varepsilon^u(x) \cap W_\varepsilon^s(\mathcal{O}_\delta x')$ and $W_{2\varepsilon}^u(x') \cap W_{2\varepsilon}^s(\mathcal{O}_\delta x)$ are transverse and consist of single points, say y, y'. No other intersections occur when ε is replaced by 3ε. By Axiom $A'b$ we approximate x, x' by p, p' with $\mathcal{O}(p), \mathcal{O}(p')$ compact. By (3.1), $W_{3\varepsilon}^u(p) \pitchfork W_{3\varepsilon}^s(\mathcal{O}_\delta p')$ in a point q near y and $W_{3\varepsilon}^u(p') \pitchfork W_{3\varepsilon}^s(\mathcal{O}_\delta p)$ in a point q' near y'. (See Fig. 1.) By the usual Cloud lemma [2, 4] $q, q' \in \Omega$. As Ω is a closed set, y and y' also belong to Ω, proving the theorem.

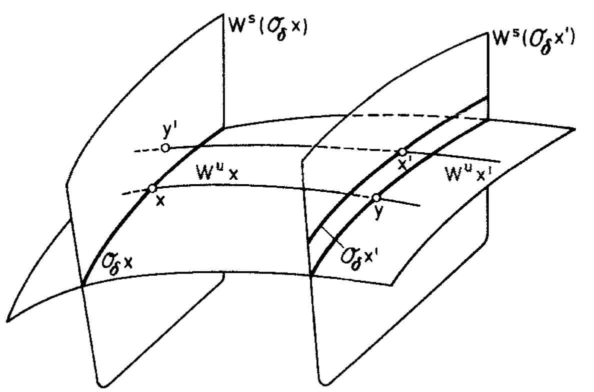

Fig. 1. Local product structure

§ 4. Ω-Decomposition

Here we prove (2.1) and show how M is divided into the stable manifolds of the Ω_i.

Proof of (2.1). By (3.2) choose $\varepsilon > 0$ with $W_{2\varepsilon}^u \cap W_{2\varepsilon}^s = \Omega$ and $0 < \delta <$ half the least period of any closed orbit. For each $p \in \Omega$ let

$$N(p) = \{z = W_\varepsilon^u(y) \cap W_\varepsilon^s(\mathcal{O}_\delta x) : x \in W_\varepsilon^u p, \ y \in W_\varepsilon^s(\mathcal{O}_\delta p)\}.$$

By (3.1), $N(p)$ contains a neighborhood of p in Ω. For any $z \in \Omega$ near p has $W_\varepsilon^u(p) \pitchfork W_\varepsilon^s(\mathcal{O}_\delta z) = \{x\}$, $W_\varepsilon^u(z) \pitchfork W_\varepsilon^s(\mathcal{O}_\delta p) = \{y\}$ and so $z \in N(p)$. By (3.2), $N(p) \subset \Omega$.

Exactly as in [4], we let $\Omega(p)$ be the closure of the orbit of $N(p)$, $\mathrm{Cl}\{\varphi_t z : z \in N(p)\}$. This decomposes Ω as $\bigcup_{p \in \Omega} \Omega(p)$. We claim that these $\Omega(p)$ suffice for (2.1). Clearly $\Omega(p)$ is closed and invariant.

As in [4] we notice that $\Omega(p)$ is unchanged if $N(p)$ is replaced by any nonempty relatively open subset of Ω in $N(p)$, say V. Indeed if z is any point of $N(p)$ we can approximate it by q with $\mathcal{O}(q)$ compact and we can also find q' in V with $\mathcal{O}(q')$ compact. By (3.1) the intersections

$$W^u_{2\varepsilon}(q) \pitchfork W^s_{2\varepsilon}(\mathcal{O}_\delta q') \quad \text{and} \quad W^u_{2\varepsilon}(q') \pitchfork W^s_{2\varepsilon}(\mathcal{O}_\delta q)$$

are single points. By the Cloud lemma [2, 4], φ_t presses V, the neighborhood of q', toward $W^u_{2\varepsilon}(q)$. In particular, the orbit of V contains q in its closure. As q approximates z, it has z in its closure, too. Thus $\mathrm{Cl}(\mathcal{O}V) \supset N(p)$, which implies by invariance that $\mathrm{Cl}(\mathcal{O}V) = \Omega(p)$.

This lets us establish the required properties for $\bigcup \Omega(p)$. If $q \in \Omega$ lies near $\Omega(p)$ then the orbit of $N(p)$ meets $N(q)$ and so, for some nonempty relatively open $V \subset N(q)$ and some $\tau \in R$, $\varphi_\tau V \subset N(p)$. Since $\Omega(p) = \mathrm{Cl}(\mathcal{O}(\varphi_\tau V))$ and $\Omega(q) = \mathrm{Cl}(\mathcal{O}V)$ they are equal. Thus the $\Omega(p)$ are open in Ω and either coincide or are disjoint. Since Ω is compact, there are only finitely many disjoint $\Omega(p)$.

Since $\Omega(p) = \mathrm{Cl}(\mathcal{O}V)$ for any nonempty open $V \subset \Omega(p)$, topological transitivity is obvious. Topological transitivity implies indecomposability and the proof of (2.1) is complete.

Since the α limit of any point $x \in M$ is a connected subset of Ω, it belongs to exactly one Ω_i. Thus M is disjointly decomposed as $\bigcup_i W^u \Omega_i$ and similarly as $\bigcup W^s \Omega_i$.

Next we prove a lemma to be used in § 6.

(4.1) **Lemma.** $W^u \Omega_i \cap W^s \Omega_i = \Omega_i$ *if* φ *obeys Axiom* A'.

Proof. Recall that local product structure (3.2), says only that $W^u_\varepsilon \Omega_i \cap W^s_\varepsilon \Omega_i = \Omega_i$. If $z \in W^u(y) \cap W^s(x)$ with $x, y \in \Omega_i$ we must show that $z \in \Omega_i$.

To prove that z is non wandering we use the topological transitivity of φ on Ω_i, (2.1).

Let U be a neighborhood of z in M and choose neighborhoods X, Y of x, y so small that $x' \in X \cap \Omega_i$, $y' \in Y \cap \Omega_i$ implies that $W^u(y') \cap U \neq \emptyset$, $W^s(x') \cap U \neq \emptyset$. By topological transitivity $\mathcal{O}(X \cap \Omega_i) \cap (Y \cap \Omega_i) \neq \emptyset$. Since the compact orbits are dense, we may therefore choose $\mathcal{O}$, a compact orbit in Ω_i meeting X and Y. Thus $W^u \mathcal{O}$ and $W^s \mathcal{O}$ both meet U. By the Cloud lemma [2, 4] $\varphi_t U \cap U \neq \emptyset$ for large t and so $z \in \Omega$. If z were in $\Omega_j \neq \Omega_i$ then so would be $\alpha(z) \cup \omega(z)$. But this contradicts our assumption that $z \in W^u \Omega_i \cap W^s \Omega_i$.

§ 5. Local Ω-Explosions

Here we prove they can't occur.

(5.1) **Theorem.** *Let* φ *satisfy Axiom* A' *and have* Ω*-decomposition* $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$. *There exist neighborhoods* U_i *of* Ω_i *and a* C^1*-neighbor-*

hood $\mathscr{U}$ of φ such that if $\varphi' \in \mathscr{U}$ has $\Omega' = \Omega(\varphi')$ then $\Omega' \cap U_i$ contains a maximal closed invariant subset Ω'_i of U_i which is hyperbolic and there exists an orbit preserving homeomorphism $h: \Omega_i \to \Omega'_i$.

This means that Ω_i can not locally explode to an Ω'_i essentially different from Ω_i. It remains possible, however, for Ω' to be essentially larger than Ω because some new orbits of Ω' might be near no single Ω_i that is, might not be local.

Proof of (5.1). Fix $\tau \neq 0$. Then Ω_i is a normally hyperbolic laminated set for φ_τ and, since φ is of class C^1, the lamination is sufficiently smooth to guarantee "plaque-expansiveness" [3, § 4]. If φ'_τ is C^1 close to φ_τ then there is a canonical φ'-invariant lamination near Ω_i, Λ'_i [3, § 4], and there is a canonical homeomorphism h: $\Omega_i \to \Lambda'_i$ near the inclusion $\Omega_i \to M$, sending lamina to lamina. (Actually h is just unique up to the composition of homeomorphisms $\Omega_i \to \Omega_i$, near the inclusion, sending each laminum onto itself.) It therefore remains only to show that Λ'_i equals Ω'_i, the maximal φ'-invariant subset of U_i.

Since the compact orbits are dense in Ω_i, they are dense in Λ'_i so $\Lambda'_i \subset \Omega'_i$. We must show $\Lambda'_i \supset \Omega'_i$, i.e. $h\Omega_i \supset \Omega'_i$. When U_i is a sufficiently small neighborhood of Ω_i and φ' is C^1 near enough φ, Ω'_i must at least be a normally hyperbolic laminated set for φ'_τ [3, § 4]. By the uniformities of [3, § 3c], φ_τ lies in the neighborhood of φ'_τ having canonical laminated sets that correspond to Ω'_i. That is there exists a canonical (unique up to small translations along lamina) homeomorphism $h': \Omega'_i \to \Lambda$ when Λ is a φ_τ-invariant normally hyperbolic laminated set. By [2] there are no φ-invariant sets near, but not contained in, Ω_i. Thus when no perturbation at all of φ_τ is made, the canonical homeomorphism of Ω_i into M is the inclusion and so by uniqueness [3, § 4] $h' \circ h$: $\Omega_i \to \Lambda'_i \to \Omega_i$ is a homeomorphism sending each orbit onto itself. Therefore $\Lambda'_i = \Omega'_i$ and (5.1) is proved.

§ 6. Global Ω-Explosions in Topological Dynamics

Let $\Lambda_0, \ldots, \Lambda_m$ be compact invariant disjoint subsets of M such that

$$\Omega \subset \Lambda_0 \cup \ldots \cup \Lambda_m.$$

Define $\Lambda = \Lambda_0 \cup \cdots \cup \Lambda_m$ and

$$W^u \Lambda_i = \{x \in M: \alpha(x) \subset \Lambda_i\} \quad W^s \Lambda_i = \{x \in M: \omega(x) \subset \Lambda_i\}.$$

Here M can be any compact metric space and φ any continuous flow. Since each $\alpha(x)$, $\omega(x)$ is a connected invariant subset of Ω, we have the disjoint decompositions

$$\bigcup_{i=0}^{m} W^u \Lambda_i = M = \bigcup_{i=0}^{m} W^s \Lambda_i.$$

We say that $\Lambda_i \leqq \Lambda_j$ if there exists $x \in M - \Lambda$ such that $\alpha(x) \subset \Lambda_i$, $\omega(x) \subset \Lambda_j$. A cycle is a chain $\Lambda_{i_0} \leqq \Lambda_{i_1} \leqq \cdots \leqq \Lambda_{i_k}$ with $k \geqq 1$ and $i_k = i_0$. (Note that now we permit a "self cycle" $\Lambda_{i_0} \leqq \Lambda_{i_0}$ if $W^u \Lambda_{i_0} \cap W^s \Lambda_{i_0} \neq \Lambda_{i_0}$. The definition in § 2 does not. Otherwise they are the same.)

(6.1) **Theorem.** *Let $U_0, \ldots, U_m$ be any neighborhoods of $\Lambda_0, \ldots, \Lambda_m$. If there are no cycles among the Λ_i then any flow C^0 close to φ has non wandering set contained in $U_0 \cup \cdots \cup U_m$.*

Proof. Without loss of generality we assume $U_0, \ldots, U_m$ are open, $\bar{U}_0, \ldots, \bar{U}_m$ are disjoint, and prove that the non wandering set of the nearby flow must lie in $\bar{U}_0 \cup \cdots \cup \bar{U}_m$.

By the no cycle assumption,

$$W^u \Lambda_i \cap W^s \Lambda_i = \Lambda_i .$$

For if $x \notin \Lambda_i$ were a point of intersection then $x \notin \Lambda_j$, $j \neq i$ (because $\omega(x)$ is contained in Λ_i, not in Λ_j) and so we would have a cycle from Λ_i to itself.

Now suppose the theorem is false. Then there are flows $\varphi_n \to \varphi$ which have non wandering points in $M - U$, $U = U_0 \cup \cdots \cup U_m$. Let us write, for clarity, $\varphi(t, \cdot)$ instead of $\varphi_t(\cdot)$ and $\varphi_n(t, \cdot)$ instead of $\varphi_{n,t}(\cdot)$. Since $M - U$ is compact, we may assume without loss of generality that arbitrarily near some $x \in M - U$, there occur non-wandering points of φ_n. Thus, there exist x_n, y_n tending to x such that $\varphi_n(t_n, x_n) = y_n$ for some sequence $t_n \to \infty$.

Consider $\alpha(x) \subset \Lambda_{i_0}$, $\omega(x) \subset \Lambda_{i_1}$. Since we have no cycles $i_0 \neq i_1$. Since $\omega(x) \subset \Lambda_{i_1}$, by continuity there is a sequence $t_n^1 \to \infty$ such that

$$0 < t_n^1 < t_n \qquad \varphi_n(t_n^1, x_n) \to \Lambda_{i_1} .$$

Thus there is a sequence $T_n^1 \to \infty$ such that

$$t_n^1 < T_n^1 < t_n \qquad \varphi_n(T_n^1, x_n) \in \partial U_{i_1},$$

$$\varphi_n(t, x_n) \in U_i \qquad \text{on } [t_n^1, T_n^1],$$

$$T_n^1 - t_n^1 \to \infty .$$

We simply let T_n^1 be defined as the first time $t > t_n^1$ that $\varphi_n(t, x_n) \in \partial U_{i_1}$.

Since $\varphi_n(t_n, x_n) = y_n \notin U_{i_1}$, this T_n^1 occurs before t_n.

Since Λ_{i_1} is φ-invariant $T_n^1 - t_n^1$ is forced to tend to ∞. (Otherwise, by continuity and compactness of Λ_{i_1}, there would be a point x_* of Λ_{i_1} having $\varphi(t, x_*) \in \partial U_{i_1}$ for some finite t. But $\varphi(t, x_*)$ must remain always in Λ_{i_1}.)

Since ∂U_{i_1} is compact, we may assume that

$$\varphi_n(T_n^1, x_n) \to x^1 \in \partial U_{i_1} .$$

Since $T_n^1 - t_n^1 \to \infty$, $\mathcal{O}_-(x^1) \subset U_{i_1}$.

Thus $\alpha(x^1) \subset \Lambda_{i_1}$. Since $x^1 \notin \Lambda$ and there are no cycles, $\omega(x^1) \subset \Lambda_{i_2}$ with $i_2 \neq i_1, i_0$.

Now we proceed with x^1 similarly. Since $\omega(x^1) \subset \Lambda_{i_2}$ by continuity there is a sequence $t_n^2 \to \infty$ such that

$$T_n^1 < t_n^2 < t_n, \qquad \varphi_n(t_n^2, x_n) \to \Lambda_{i_2}.$$

Thus there is a sequence $T_n^2 \to \infty$ such that

$$t_n^2 < T_n^2 < t_n, \qquad \varphi_n(T_n^2, x_n) \in \partial U_{i_2},$$

$$\varphi_n(t, x_n) \in U_{i_2} \qquad \text{on } [t_n^2, T_n^2],$$

$$T_n^2 - t_n^2 \to \infty.$$

We simply let T_n^2 be defined as the first time $t > t_n^2$ that $\varphi_n(t, x_n) \in \partial U_{i_2}$. Since $\varphi_n(t_n, x_n) = y_n \notin U_{i_2}$, this T_n^2 occurs before t_n. Since Λ_{i_2} is φ-invariant $T_n^2 - t_n^1$ is forced to tend to ∞. Just as before, this produces $x^2 \in \partial U_{i_2}$ such that $\alpha(x^2) \subset \Lambda_{i_2}$, $\omega(x^2) \subset \Lambda_{i_3}$, $i_3 \neq i_0, i_1, i_2$.

We may proceed with x^2 exactly as with x^1 and produce, by iteration, a chain of Λ_i's with no repetitions and length greater than m. This is ridiculous and the theorem is proved.

As a consequence we have the Ω-stability theorem. That is

(6.2) **Corollary.** *If φ is a C^1 flow obeying Axiom A′ and its Ω-decomposition $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ has no cycles in the sense of* § 2 *then it is Ω-stable.*

Proof. By (4.2) there are no self cycles in the Ω-decomposition, so we may let $\Lambda_i = \Omega_i$ and apply (6.1). The non-wandering set of any φ' C^0 close to φ lies in $U_0 \cup \cdots \cup U_m$ where U_i is the neighborhood of Ω_i constructed in (5.1). Thus, if φ' is also C^1 near φ, (5.1) says that φ' is Ω-conjugate to φ, proving the corollary.

§ 7. Other Treatments of Global Ω-Explosions

Instead of the topological arguments given in § 6, we could have imitated Smale's filtration arguments applied to the time-one map φ_1. Saturating the resulting filtration by letting φ_t act, $0 \leqq t \leqq 1$ would have been necessary to have φ_t' take the elements of the filtration into their interiors for all $t \geqq 2$.

Alternatively, and more in the spirit of flows, we could have defined a smooth gradient function $f\colon M \to [0, m]$. Let $\Omega_0 < \Omega_1 < \cdots < \Omega_m$ be a relabelling of the Ω_i in a simple order subordinate to $\leqq$. Then f will obey

a) $f^{-1}[0, i] = \bigcup_{j \leqq i} W^s \Omega_j \overset{\text{def}}{=} \tilde{\Omega}_i \quad f^{-1}(i) \cap \tilde{\Omega}_i = \Omega_i$.

b) f increases strictly on each φ trajectory off Ω.

Using the smooth Lyapunov theory of [6] it is not hard to show that f exists because $\tilde{\Omega}_i$ is a uniform repellor (a uniformly asymptotically unstable compact invariant set for φ). In fact [6] produces a smooth Lyapunov function f_i: $M \to [0, 1]$ such that

a$_i$) $f_i^{-1}(0) = \tilde{\Omega}_i \quad f_i^{-1}(1) = M - W^u \tilde{\Omega}_i$.

b$_i$) *f_i strictly increases on each φ trajectory in $W^u \tilde{\Omega}_i - \tilde{\Omega}_i$.*

Then $f = \sum_0^m f_i$ is our gradient function and the usual arguments show that the trajectory of any φ' which is C^1 near φ also rises through the f-level surfaces away from Ω. Hence $\Omega(\varphi')$ can not occur far from Ω.

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