The $\Omega$-Stability Theorem for Flows

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§ 1. Introduction

Let $X$ be a $C^r$ tangent vector field, $r \geq 1$, on a compact smooth Riemannian manifold $M$, without boundary, and let $\varphi = \{\varphi_t\}$ be the flow it generates. We think of $\varphi$ as a $C^r$ action of $R$ on $M$. In [4, 5] Smale states and proves the $\Omega$-stability theorem for diffeomorphisms of $M$—that is, $C^r$ actions of $Z$ on $M$. He states the corresponding theorem for flows and says “Presumably there is no difficulty in proving similar theorems for ordinary differential equations by the same method.” This paper does just that and, assuming [2, 3] does it without much difficulty.

$\Omega$-stability theorem (for flows). If $\varphi$ obeys Axiom $A'$ and has the no cycle property then it is $\Omega$-stable.

See § 2 for definitions of these terms.

Remark. In § 6 we handle global $\Omega$-explosions in way different from Smale’s.

§ 2. Definitions and $\Omega$-Decompositions

We denote by $\Omega$, the set of non wandering points of $\varphi$. Following [4] we say that $\varphi$ obeys Axiom $A'$ if the tangent flow $T\varphi = \{T\varphi_t\}$, $T\varphi_t : TM \to TM$, leaves invariant a continuous splitting $T\varphi_t M = E^u \oplus E^s$ such that

(i) $T\varphi_t$ expands $E^u$, $t > 0$.

(ii) $T\varphi_t$ contracts $E^s$, $t > 0$.

(iii) $E^s_p = \text{span} (X_p)$

for some Finsler on $TM$. Since $T\varphi_t (X_p) = X_{\varphi_t p}$ for any smooth flow, invariance of $E^o$ is automatic.

A compact orbit of $\varphi$ is either a circle or a point. The former is called a closed orbit, the latter a fixed point.

If $p$ lies on a closed orbit of $\varphi$, $\mathcal{O}$, then $p \in \Omega$ and $E^o_p = T_p \mathcal{O}$. By (i), (ii) $\mathcal{O}$ is hyperbolic—that is, the flow near $\mathcal{O}$ induces a transformation (the Poincaré map) on a hypersurface transverse to $\mathcal{O}$ at $p$ and $p$ is a hyperbolic

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fixed point of this transformation. If \( p \) is a zero of \( X \) then \( p \) is a fixed point of \( \varphi \) and belongs to \( \Omega \), but \( E^\varphi_p = 0 \). By (i), (iii) \( p \) is a hyperbolic fixed point. Hyperbolic fixed points are isolated, \( M \) is compact, and so \( F = \{ p : \varphi_t p = p \} = \{ p : X_p = 0 \} = \{ p : E^\varphi_p = 0 \} \) is finite. Continuity of \( E^\varphi \) implies that \( F \) is disjoint from \( C^1(\Omega - F) \).

In particular closed orbits cannot pass near fixed points. The second half of Axiom \( A' \) is Axiom \( A'b \).

The closed orbits are dense in \( \Omega - F \).

Now we can state one of the principal results.

(2.1) **\( \Omega \)-Decomposition Theorem.** If \( \varphi \) obeys Axiom \( A' \) then \( \Omega \) decomposes uniquely as \( \Omega = \Omega_0 \cup \cdots \cup \Omega_m \) where the \( \Omega_i \) are disjoint, indecomposable components of \( \Omega \). On each \( \Omega_i \), \( \varphi \) is topologically transitive.

"Indecomposable" means that \( \Omega_i \) does not decompose as \( \Omega'_i \cup \Omega''_i \) where \( \Omega'_i, \Omega''_i \) are disjoint components. "Component" is used in the usual sense a relatively open and closed non empty subset.

"Topologically transitive" means that the orbit, \( \bigcup_{t \in R} \varphi_t S \), of any relatively open non empty set \( S \subset \Omega_i \) is dense in \( \Omega_i \). Topological transitivity is equivalent to the existence of a point \( p \in \Omega_i \) whose orbit \( \{ \varphi_t p : t \in R \} \) is dense in \( \Omega_i \) [1].

Note that each \( \varphi \)-orbit \( \varnothing \subset \Omega \), being connected, lies in just one \( \Omega_i \). That is, the \( \Omega_i \) are invariant. The remarkable assertion of (2.1) is that the process of successively dividing \( \Omega \) into finite disjoint unions of components terminates at a finite stage. Given the other conditions "uniqueness" in (2.1) is clear. In § 4 we prove (2.1).

Now suppose \( \varphi \) obeys Axiom \( A' \) and has \( \Omega \)-decomposition \( \Omega = \Omega_0 \cup \cdots \cup \Omega_m \) as in (2.1). If \( \alpha(x) \subset \Omega_i \) and \( \omega(x) \subset \Omega_j \), for some \( x \in M \), we write \( \Omega_i \leq \Omega_j \). The sets \( \alpha(x), \omega(x) \) are the usual \( \alpha \)- and \( \omega \)-limit sets of the trajectory through \( x \):

\[
\alpha(x) = \bigcap_{\tau < 0} C^1 \{ \varphi_t x : t \leq \tau \} \quad \omega(x) = \bigcap_{\tau > 0} C^1 \{ \varphi_t x : t \geq \tau \}.
\]

\( \varphi \) is said to obey the no cycle property if there is no cycle respecting this partial order \( \leq \). A cycle would be \( \Omega_{i_0} \leq \Omega_{i_1} \leq \Omega_{i_2} \leq \cdots \leq \Omega_{i_k} \leq \Omega_{i_0} \) with \( i_0, \ldots, i_k \) distinct and \( k \geq 1 \).

To complete the explanation of (1.1) we must recall the idea of \( \Omega \)-stability. Two flows are \( \Omega \)-conjugate if there is a homeomorphism of the nonwandering set of the first onto that of the second, sending sensed trajectories onto sensed trajectories. The flow \( \varphi \) is said to be \( \Omega \)-stable if it is \( \Omega \)-conjugate to each flow \( \varphi' \) which is \( C^1 \) near it.

Strictly speaking an \( \Omega \)-conjugacy is not a conjugacy because it is not required to preserve the parameterization of trajectories; only their sense (= direction). A sequence of flows \( \varphi^1, \varphi^2, \varphi^3, \ldots \) tends to \( \varphi \) in the
C' sense if $\varphi^n|_{M \times [0, 1]} \to \varphi|_{M \times [0, 1]}$ uniformly in the $C'$ sense. Since $\varphi$ is completely determined by $\varphi|_{M \times [\alpha, \beta]}$ for any $\alpha < \beta$, this defines the $C'$ topology on the set of $C'$ flows in a way compatible with our intuition.

The proof of the $\Omega$-stability theorem (1.1), breaks up into three parts: the $\Omega$-decomposition theorem, purely local $\Omega$-explosions, and global $\Omega$-explosions.

§ 3. Stable Manifold Theory for Flows

Here we amplify what appears in [2, § 7]. Let $\varphi$ obey Axiom $A'$. The time one map, $\varphi_1$, is a diffeomorphism leaving invariant the compact set $\Omega$. Its tangent leaves $E^u \oplus E^s \oplus E^s$ invariant. We may apply [3, 3 B.5] to $\varphi_1$ and likewise to $\varphi_{-1}$ -- giving

(3.1) Theorem. If $\varphi$ obeys Axiom $A'$ then through each point $p \in \Omega$ pass $C'$ injectively immersed manifolds $W^u(p), W^s(p)$ tangent to $E^u, E^s_p$ at $p$. They are characterized by

$$W^u(p) = \{x \in M : \text{for some } c < 0 \quad d(\varphi_t x, P_p) e^{ct} \to 0 \quad \text{as } t \to -\infty\},$$

$$W^s(p) = \{y \in M : \text{for some } c > 0 \quad d(\varphi_t y, P_p) e^{ct} \to 0 \quad \text{as } t \to \infty\}.$$

$W^u(q)$ and $W^u(p)$ either coincide or are disjoint. The families $W^u = \{W^u(p) : p \in \Omega\}, W^s = \{W^s(p) : p \in \Omega\}$ are invariant by $\varphi : \varphi_1(W^u p) = W^u(\varphi_1 p), \varphi_{-1}(W^s p) = W^s(\varphi_{-1} p)$.

$W^u$ is locally smoothly continuous in the sense that if $q \to p$ then any compact smooth disc in $W^u(q)$ is uniformly approached in the $C^1$ sense by smooth discs in $W^u(p)$. Similarly for $W^s$.

For any set $S \subset \Omega$, we write $W^u S = \bigcup_{p \in S} W^u(p), W^s S = \bigcup_{p \in S} W^s(p)$. When $S = \Omega$, we just write $W^u, W^s$. For $p \in M$ and $\delta > 0$, we denote by $O_\delta(p)$ the $\delta$-orbit of $p$, $\{\varphi_t p : -\delta \leq t \leq \delta\}$ and by $O(p)$ its total orbit, $\{\varphi_t p : t \in \mathbb{R}\}$.

From (3.1) $W^u \cap, W^s \cap$ are smooth submanifolds, invariant by the flow, and smoothly fibered by the $W^u(p), W^s(p) \in O$. At $O$, they are tangent to $E^u \oplus E^s, E^s \oplus E^s$. Either $W^u \cap, W^s \cap$ coincide or are disjoint. Likewise $W^s \cap, W^u \cap$.

For small $\varepsilon > 0$ we denote by $W^u_\varepsilon(p)$ the closed $\varepsilon$ disc in $W^u p$, centered at $p$. Precisely, $W^u_\varepsilon(p)$ is the connected component of $\exp(T_p M(\varepsilon)) \cap W^u(p)$ containing $p$. The set $T_p M(\varepsilon)$ is the closed $\varepsilon$-ball in the tangent space of $M$ and $\exp$ is the smooth exponential on $M$. Similarly, we define $W^s_\varepsilon(p), W^u_\varepsilon S, W^s_\varepsilon S, W^u_\varepsilon, W^s_\varepsilon$.

We say that $\Omega$ has local product structure if

$$W^u_\varepsilon \cap W^s_\varepsilon = \Omega$$

for some $\varepsilon > 0$. The next theorem repeats [2, 7.2].
(3.2) **Theorem.** If \( \varphi \) obeys Axiom \( A' \) then \( \Omega \) has local product structure.

**Proof.** The fixed point set \( F \) is bounded away from \( \Omega - F \) and so, for small \( \varepsilon > 0 \), we must just prove that if \( x, x' \in \Omega - F \) have \( y \in W^u_\varepsilon(x) \cap W^s_\varepsilon(x') \) then \( y \in \Omega \). Let \( \tau \) be the smallest period of a closed orbit and choose \( \delta, 0 < \delta < \tau / 2 \). For small \( \varepsilon > 0 \), (3.1) implies that the intersections \( W^u_\varepsilon(x) \cap W^s_\varepsilon(\mathcal{O}_\delta x') \) and \( W^u_\varepsilon(x') \cap W^s_\varepsilon(\mathcal{O}_\delta x) \) are transverse and consist of single points, say \( y, y' \). No other intersections occur when \( \varepsilon \) is replaced by \( 3 \varepsilon \).

By Axiom \( A'b \) we approximate \( x, x' \) by \( p, p' \) with \( \mathcal{O}(p), \mathcal{O}(p') \) compact.

By (3.1), \( W^u_{3\varepsilon}(p) \cap W^s_{3\varepsilon}(\mathcal{O}_\delta p') \) in a point \( q \) near \( y \) and \( W^u_{3\varepsilon}(p') \cap W^s_{3\varepsilon}(\mathcal{O}_\delta p) \) in a point \( q' \) near \( y' \). (See Fig. 1.) By the usual Cloud lemma \([2, 4]\) \( q, q' \in \Omega \).

As \( \Omega \) is a closed set, \( y \) and \( y' \) also belong to \( \Omega \), proving the theorem.

Fig. 1. Local product structure

§ 4. **\( \Omega \)-Decomposition**

Here we prove (2.1) and show how \( M \) is divided into the stable manifolds of the \( \Omega_i \).

**Proof of** (2.1). By (3.2) choose \( \varepsilon > 0 \) with \( W^u_{2\varepsilon} \cap W^s_{2\varepsilon} = \Omega \) and \( 0 < \delta < \) half the least period of any closed orbit. For each \( p \in \Omega \) let

\[
N(p) = \{ z = W^u_\varepsilon(y) \cap W^s_\varepsilon(\mathcal{O}_\delta x) : x \in W^u_\varepsilon p, y \in W^s_\varepsilon(\mathcal{O}_\delta p) \}.
\]

By (3.1), \( N(p) \) contains a neighborhood of \( p \) in \( \Omega \). For any \( z \in \Omega \) near \( p \) has \( W^u_\varepsilon(p) \cap W^s_\varepsilon(\mathcal{O}_\delta z) = \{ x \} \), \( W^u_\varepsilon(z) \cap W^s_\varepsilon(\mathcal{O}_\delta p) = \{ y \} \) and so \( z \in N(p) \). By (3.2), \( N(p) \subset \Omega \).

Exactly as in \([4]\), we let \( \Omega(p) \) be the closure of the orbit of \( N(p) \), \( C1 \{ \varphi : z \in N(p) \} \). This decomposes \( \Omega \) as \( \bigcup_{p \in \Omega} \Omega(p) \). We claim that these \( \Omega(p) \) suffice for (2.1). Clearly \( \Omega(p) \) is closed and invariant.
As in [4] we notice that $O(p)$ is unchanged if $N(p)$ is replaced by any nonempty relatively open subset of $\Omega$ in $N(p)$, say $V$. Indeed if $z$ is any point of $N(p)$ we can approximate it by $q$ with $O(q)$ compact and we can also find $q'$ in $V$ with $O(q')$ compact. By (3.1) the intersections

$$W^u_{2\varepsilon}(q) \subset W^u_{\varepsilon}(O(q)) \quad \text{and} \quad W^u_{\varepsilon}(q') \subset W^u_{\varepsilon}(O(q))$$

are single points. By the Cloud lemma [2, 4], $\varphi_t$ presses $V$, the neighborhood of $q'$, toward $W^u_{2\varepsilon}(q)$. In particular, the orbit of $V$ contains $q$ in its closure. As $q$ approximates $z$, it has $z$ in its closure, too. Thus $C(\varphi V) \supset N(p)$, which implies by invariance that $C(\varphi V) = O(p)$.

This lets us establish the required properties for $\mathcal{U}_0(p)$. If $q \in \Omega$ lies near $\Omega(p)$ then the orbit of $N(p)$ meets $N(q)$ and so, for some nonempty relatively open $V \subset N(q)$ and some $\tau \in \mathcal{R}$, $\varphi_\tau V \subset N(p)$. Since $\Omega(p) = C(\varphi V)$ and $\Omega(q) = C(\varphi V)$ they are equal. Thus the $\Omega(p)$ are open in $\Omega$ and either coincide or are disjoint. Since $\Omega$ is compact, there are only finitely many disjoint $\Omega(p)$.

Since $\Omega(p) = C(\varphi V)$ for any nonempty open $V \subset \Omega(p)$, topological transitivity is obvious. Topological transitivity implies indecomposability and the proof of (2.1) is complete.

Since the $\alpha$ limit of any point $x \in M$ is a connected subset of $\Omega$, it belongs to exactly one $\Omega_i$. Thus $M$ is disjointly decomposed as $\bigcup \Omega_i$ and similarly as $\bigcup W^s_{\varepsilon} \Omega_i$.

Next we prove a lemma to be used in § 6.

(4.1) Lemma. $W^u_{\varepsilon} \Omega_i \cap W^s_{\varepsilon} \Omega_i = \Omega_i$ if $\varphi$ obeys Axiom $A'$.

Proof. Recall that local product structure (3.2), says only that $W^u_{\varepsilon} \Omega_i \cap W^s_{\varepsilon} \Omega_i = \Omega_i$. If $z \in W^u(y) \cap W^s(x)$ with $x, y \in \Omega_i$ we must show that $z \in \Omega_i$.

To prove that $z$ is non wandering we use the topological transitivity of $\varphi$ on $\Omega_i$, (2.1).

Let $U$ be a neighborhood of $z$ in $M$ and choose neighborhoods $X, Y$ of $x, y$ so small that $x' \in X \cap \Omega_i, y' \in Y \cap \Omega_i$ implies that $W^u(y') \cap U \neq \emptyset, W^s(x') \cap U \neq \emptyset$. By topological transitivity $\varphi(X \cap \Omega_i) \cap (Y \cap \Omega_i) \neq \emptyset$. Since the compact orbits are dense, we may therefore choose $\varphi$, a compact orbit in $\Omega_i$ meeting $X$ and $Y$. Thus $W^u \varphi$ and $W^s \varphi$ both meet $U$. By the Cloud lemma [2, 4] $\varphi_t U \cap U \neq \emptyset$ for large $t$ and so $z \in \Omega$. If $z$ were in $\Omega_j \neq \Omega_i$ then so would be $\alpha(z) \cup \omega(z)$. But this contradicts our assumption that $z \in W^u_{\varepsilon} \Omega_i \cap W^s_{\varepsilon} \Omega_i$.

§ 5. Local $\Omega$-Explosions

Here we prove they can't occur.

(5.1) Theorem. Let $\varphi$ satisfy Axiom $A'$ and have $\Omega$-decomposition $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$. There exist neighborhoods $U_i$ of $\Omega_i$ and a $C^1$-neighbor-
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hood $\mathcal{U}$ of $\varphi$ such that if $\varphi' \in \mathcal{U}$ has $\Omega' = \Omega(\varphi')$ then $\Omega' \cap U_i$ contains a maximal closed invariant subset $\Omega'_i$ of $U_i$ which is hyperbolic and there exists an orbit preserving homeomorphism $h: \Omega_i \to \Omega'_i$.

This means that $\Omega_i$ can not locally explode to an $\Omega'_i$ essentially different from $\Omega_i$. It remains possible, however, for $\Omega'$ to be essentially larger than $\Omega$ because some new orbits of $\Omega'$ might be near no single $\Omega_i$ that is, might not be local.

Proof of (5.1). Fix $\tau \neq 0$. Then $\Omega_i$ is a normally hyperbolic laminated set for $\varphi$, and, since $\varphi$ is of class $C^1$, the lamination is sufficiently smooth to guarantee "plaque-expansiveness" [3, § 4]. If $\varphi'_i$ is $C^1$ close to $\varphi$, then there is a canonical $\varphi'$-invariant lamination near $\Omega_i$, $\Lambda'_i$ [3, § 4], and there is a canonical homeomorphism $h: \Omega_i \to \Lambda'_i$ near the inclusion $\Omega_i \to M$, sending lamina to lamina. (Actually $h$ is just unique up to the composition of homeomorphisms $\Omega_i \to \Omega_i$, near the inclusion, sending each laminum onto itself.) It therefore remains only to show that $\Lambda'_i$ equals $\Omega'_i$, the maximal $\varphi'$-invariant subset of $U_i$.

Since the compact orbits are dense in $\Omega_i$, they are dense in $\Lambda'_i$ so $\Lambda'_i \subseteq \Omega'_i$. We must show $\Lambda'_i = \Omega'_i$, i.e. $h \Omega_i = \Omega'_i$. When $U_i$ is a sufficiently small neighborhood of $\Omega_i$ and $\varphi'$ is $C^1$ near enough $\varphi$, $\Omega'_i$ must at least be a normally hyperbolic laminated set for $\varphi'_i$ [3, § 4]. By the uniformities of [3, § 3 c], $\varphi$ lies in the neighborhood of $\varphi'_i$ having canonical laminated sets that correspond to $\Omega'_i$. That is there exists a canonical (unique up to small translations along lamina) homeomorphism $h': \Lambda'_i \to \Lambda$ when $\Lambda$ is a $\varphi'_i$-invariant normally hyperbolic laminated set. By [2] there are no $\varphi$-invariant sets near, but not contained in, $\Omega_i$. Thus when no perturbation at all of $\varphi$ is made, the canonical homeomorphism of $\Omega_i$ into $M$ is the inclusion and so by uniqueness [3, § 4] $h' \circ h: \Omega_i \to \Lambda'_i \to \Omega_i$ is a homeomorphism sending each orbit onto itself. Therefore $\Lambda'_i = \Omega'_i$ and (5.1) is proved.

§ 6. Global $\Omega$-Explosions in Topological Dynamics

Let $A_0, \ldots, A_m$ be compact invariant disjoint subsets of $M$ such that

$$\Omega \subseteq A_0 \cup \ldots \cup A_m.$$ 

Define $A = A_0 \cup \ldots \cup A_m$ and

$$W^u A_i = \{ x \in M : \alpha(x) \subseteq A_i \} \quad W^s A_i = \{ x \in M : \omega(x) \subseteq A_i \}.$$ 

Here $M$ can be any compact metric space and $\varphi$ any continuous flow. Since each $\alpha(x)$, $\omega(x)$ is a connected invariant subset of $\Omega$, we have the disjoint decompositions

$$\bigcup_{i=0}^{m} W^u A_i = M = \bigcup_{i=0}^{m} W^s A_i.$$
We say that \( A_i \leq A_j \) if there exists \( x \in M - A \) such that \( \alpha(x) \subset A_i, \omega(x) \subset A_j \).

A cycle is a chain \( A_{i_0} \leq A_{i_1} \leq \cdots \leq A_{i_k} \) with \( k \geq 1 \) and \( i_k = i_0 \). (Note that now we permit a "self cycle" \( A_{i_0} \leq A_{i_0} \) if \( W^u A_{i_0} \cap W^s A_{i_0} \neq A_{i_0} \). The definition in §2 does not. Otherwise they are the same.)

(6.1) **Theorem.** Let \( U_0, \ldots, U_m \) be any neighborhoods of \( A_0, \ldots, A_m \). If there are no cycles among the \( A_i \) then any flow \( \phi^0 \) close to \( \phi \) has non wandering set contained in \( U_0 \cup \cdots \cup U_m \).

**Proof.** Without loss of generality we assume \( U_0, \ldots, U_m \) are open, \( \overline{U_0}, \ldots, \overline{U_m} \) are disjoint, and prove that the non wandering set of the nearby flow must lie in \( \overline{U_0} \cup \cdots \cup \overline{U_m} \).

By the no cycle assumption,

\[
W^u A_i \cap W^s A_i = A_i.
\]

For if \( x \notin A_i \) were a point of intersection then \( x \notin A_j \), \( j \neq i \) (because \( \omega(x) \) is contained in \( A_i \), not in \( A_j \)) and so we would have a cycle from \( A_i \) to itself.

Now suppose the theorem is false. Then there are flows \( \varphi_n \to \varphi \) which have non wandering points in \( M - U \), \( U = U_0 \cup \cdots \cup U_m \). Let us write, for clarity, \( \varphi(t, \cdot) \) instead of \( \varphi_n(t, \cdot) \) and \( \varphi_n(t, \cdot) \) instead of \( \varphi_n(t, \cdot) \). Since \( M - U \) is compact, we may assume without loss of generality that arbitrarily near some \( x \in M - U \), there occur non-wandering points of \( \varphi_n \).

Thus, there exist \( x_n, y_n \) tending to \( x \) such that \( \varphi_n(t_n, x_n) = y_n \) for some sequence \( t_n \to \infty \).

Consider \( \alpha(x) \subset A_{i_0}, \omega(x) \subset A_{i_1} \). Since we have no cycles \( i_0 = i_1 \). Since \( \omega(x) \subset A_{i_1} \), by continuity there is a sequence \( t_n \to \infty \) such that

\[
0 < t_n < t_n \quad \varphi_n(t_n, x_n) \to A_{i_1}.
\]

Thus there is a sequence \( T_n \to \infty \) such that

\[
t_n < T_n < t_n \quad \varphi_n(T_n, x_n) \in \partial U_{i_1},
\]

\[
\varphi_n(t, x_n) \in U_i \quad \text{on } [t_n, T_n],
\]

\[
T_n - t_n \to \infty.
\]

We simply let \( T_n \) be defined as the first time \( t > t_n \) that \( \varphi_n(t, x_n) \in \partial U_{i_1} \).

Since \( \varphi_n(t_n, x_n) = y_n \notin U_i \), this \( T_n \) occurs before \( t_n \).

Since \( A_{i_1} \) is \( \varphi \)-invariant \( T_n - t_n \) is forced to tend to \( \infty \). (Otherwise, by continuity and compactness of \( A_{i_1} \), there would be a point \( x_0 \) of \( A_{i_1} \) having \( \varphi(t, x_0) \in \partial U_{i_1} \) for some finite \( t \). But \( \varphi(t, x) \) must remain always in \( A_{i_1} \).)

Since \( \partial U_i \) is compact, we may assume that

\[
\varphi_n(T_n, x_n) \to x' \in \partial U_i.
\]
Since $T_n^1 - t_n^1 \to \infty$, $\partial_-(x^1) \subset U_i$.

Thus $\alpha(x^1) \in \Lambda_{i_1}$. Since $x^1 \notin \Lambda$ and there are no cycles, $\omega(x^1) \in \Lambda_{i_2}$ with $i_2 \neq i_1, i_0$.

Now we proceed with $x^1$ similarly. Since $\omega(x^1) \subset \Lambda_{i_2}$ by continuity there is a sequence $t_n^2 \to \infty$ such that

$$T_n^1 < t_n^2 < t_n^1, \quad \varphi_n(t_n^2, x_n) \to \Lambda_{i_2}.$$ 

Thus there is a sequence $T_n^2 \to \infty$ such that

$$t_n^2 < T_n^2 < t_n^1, \quad \varphi_n(T_n^2, x_n) \in \partial U_{i_2},$$

$$\varphi_n(t, x_n) \in U_{i_2} \quad \text{on } [t_n^2, T_n^2],$$

$$T_n^2 - t_n^2 \to \infty.$$ 

We simply let $T_n^2$ be defined as the first time $t > t_n^2$ that $\varphi_n(t, x_n) \in \partial U_{i_2}$. Since $\varphi_n(t_n, x_n) = y \notin U_{i_2}$, this $T_n^2$ occurs before $t_n$. Since $\Lambda_{i_2}$ is $\varphi$-invariant $T_n^2 - t_n^1$ is forced to tend to $\infty$. Just as before, this produces $x^2 \in \partial U_{i_2}$ such that $\alpha(x^2) \subset \Lambda_{i_2}$, $\omega(x^2) \subset \Lambda_{i_3}$, $i_3 \neq i_0, i_1, i_2$.

We may proceed with $x^2$ exactly as with $x^1$ and produce, by iteration, a chain of $\Lambda_i$'s with no repetitions and length greater than $m$. This is ridiculous and the theorem is proved.

As a consequence we have the $\Omega$-stability theorem. That is

(6.2) **Corollary.** If $\varphi$ is a $C^1$ flow obeying Axiom A' and its $\Omega$-decomposition $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ has no cycles in the sense of §2 then it is $\Omega$-stable.

**Proof.** By (4.2) there are no self cycles in the $\Omega$-decomposition, so we may let $\Lambda_i = \Omega_i$ and apply (6.1). The non-wandering set of any $\varphi'$ $C^0$ close to $\varphi$ lies in $U_0 \cup \cdots \cup U_m$ where $U_i$ is the neighborhood of $\Omega_i$ constructed in (5.1). Thus, if $\varphi'$ is also $C^1$ near $\varphi$, (5.1) says that $\varphi'$ is $\Omega$-conjugate to $\varphi$, proving the corollary.

§ 7. Other Treatments of Global $\Omega$-Explosions

Instead of the topological arguments given in §6, we could have imitated Smale's filtration arguments applied to the time-one map $\varphi_1$. Saturating the resulting filtration by letting $\varphi_t$ act, $0 \leq t \leq 1$ would have been necessary to have $\varphi_t$ take the elements of the filtration into their interiors for all $t \geq 2$.

Alternatively, and more in the spirit of flows, we could have defined a smooth gradient function $f: M \to [0, m]$. Let $\Omega_0 < \Omega_1 < \cdots < \Omega_m$ be
a relabelling of the $\Omega_i$ in a simple order subordinate to $\leq$. Then $f$ will obey

\[ a) \ f^{-1}[0, i] = \bigcup_{\Omega_j \leq i} W^s \Omega_j = \bar{\Omega}_i \quad f^{-1}(i) \cap \bar{\Omega}_i = \Omega_i. \]

\[ b) \ f \text{ increases strictly on each } \phi \text{ trajectory off } \Omega. \]

Using the smooth Lyapunov theory of [6] it is not hard to show that $f$ exists because $\bar{\Omega}_i$ is a uniform repellor (a uniformly asymptotically unstable compact invariant set for $\phi$). In fact [6] produces a smooth Lyapunov function $f_i : M \rightarrow [0, 1]$ such that

\[ a_i) \ f_i^{-1}(0) = \bar{\Omega}_i \quad f_i^{-1}(1) = M - W^u \bar{\Omega}_i. \]

\[ b_i) \ f_i \text{ strictly increases on each } \phi \text{ trajectory in } W^u \bar{\Omega}_i - \bar{\Omega}_i. \]

Then $f = \sum_{0}^{m} f_i$ is our gradient function and the usual arguments show that the trajectory of any $\phi'$ which is $C^1$ near $\phi$ also rises through the $f$-level surfaces away from $\Omega$. Hence $\Omega(\phi')$ can not occur far from $\Omega$.

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References


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