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FUTURE STABILITY IS NOT GENERIC

MICHAEL SHUB AND R. F. WILLIAMS

In [2] S. Smale gave an example to show that structurally stable systems are not dense in the space of all systems. His argument plays two “invariants” against each other: The stable and unstable manifolds. The purpose of this note is to give a new argument for this result; the novelty here is that only one of these invariants is used. Thus “future stable” systems are not dense. Future stability was introduced in a recent lecture of S. Smale in which he expressed some, if not much, hope that it would be a generic property.

We will consider only diffeomorphisms of a manifold. Each diffeomorphism corresponds (via “suspension”, e.g. [1, p. 797]) to a system on a manifold of one higher dimension. In this setting we give the following:

Definitions [after Smale]. For a diffeomorphism \( f : M \to M \) we say two points \( x, y \in M \) are stably equivalent \( x \sim_y \) provided \( \lim_{n \to \infty} \rho(f^n(x), f^n(y)) = 0 \). This is an equivalence relation and partitions \( M \) into stable sets; the stable set containing \( x \) is called \( W^s(x, f) \), or \( W^s(x) \). \( W^s(x, f) \) is by definition \( W^s(x, f^{-1}) \). \( f \) is future stable if for all nearby pertubations \( f' \) of \( f \) there is a (topological) homeomorphism \( h : M \to M \) which sends stable sets of \( f \) into stable sets of \( f' \); that is, \( h(W^s(x, f)) = W^s(h(x), f') \), for all \( x \in M \).

We also need a more technical definition. Suppose some of the stable sets of \( f \) are smooth manifolds of codimension one, which smoothly foliate an open set. Then locally they are the level sets of a real valued function, say \( \phi : U \to \mathbb{R} \). One says that an unstable set \( W^u(p) \) hooks at \( W^u(q) \) provided that there is \( x_0 \in W^u(p) \cap W^u(q) \cap U \) where \( \phi(x_0) \) is an isolated local maximum of the function \( W^u(p) \to \mathbb{R} \) given by \( x \to \phi(x) \). Let \( T^2 \) be a two-dimensional torus.

Theorem. There is an open set \( \mathcal{U} \subset \text{Diff}^r(T^2) \), \( r \geq 2 \), and dense subsets \( \mathcal{A}, \mathcal{B} \) of \( \mathcal{U} \) such that each \( f \in \mathcal{U} \) has an attractor \( \Sigma \) (a 1-dimensional generalized solenoid) and a hyperbolic fixed point \( p \), where

1. \( W^u(x), W^s(x) \) are smooth copies of \( \mathbb{R} \), for \( x \in \Sigma \).
2. \( \Sigma = \bigcup_{x \in \Sigma} W^u(x) \).
3. A neighborhood of \( \Sigma \) is foliated by \( \{ W^u(x) : x \in \Sigma \} \).
4. \( W^u(p) \) hooks at, and only at, \( W^s(q) \), where

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(a) $q$ is a periodic point, if $f \in \mathbb{A}$;
(b) $q$ is aperiodic and has a dense orbit, if $f \in \mathbb{B}$.

This result is due to S. Smale, who proved it (with appropriate modification) for $T^2$; for $T^2$, see [3]. It is also true for the $C^1$-topology if one allows finitely many points to play the role of $q$.

Now for $f \in \mathcal{U}$, let $B(f)$ be the compact set $p \cup W^u(p) \cup \Sigma(f)$. $B(f)$ is the union of a certain solenoid, the point $p$ and a line, $W^u(p)$.

Claim. If $f \in \mathcal{U}$, then $f^{-1}$ is not future stable. For otherwise there would be nonempty open sets $\mathcal{V} \subset \mathcal{U}$ such that each two $B(f), f \in \mathcal{V}$ would be homeomorphic. But

**Lemma.** For $f \in \mathbb{A}, g \in \mathbb{B}$, $B(f)$ and $B(g)$ are not homeomorphic.

**Proof.** Let $S$ be the closure of the graph of $\sin(1/x)$, $x \in \mathbb{R}$. Then the line interval joining $(0, 1)$ to $(0, -1)$ is in $S$. Now finitely many (as many as the period of $g$) points of $\Sigma(f)$ have small neighborhoods which contain a topological copy of a small neighborhood in $S$ of $(0, 1)$. All other points of $\Sigma(f)$ have small neighborhoods of the form (some set) $[0, 1]$, i.e. no arc doubles back, near them.

However, as the points in $B(g)$ which correspond to points at which $W^u(p)$ hooks at $W^s(q)$, converge to all of $\Sigma(g), \Sigma(g)$ has no point of this second type.

**Bibliography**


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