ENDOMORPHISMS OF COMPACT DIFFERENTIABLE MANIFOLDS.*

By Michael Shub.

Introduction. A endomorphism of a differentiable manifold M is a differentiable function $f: M \to M$ of class C^r, $r \geqq 1$. An endomorphism f is an automorphism if f has a differentiable inverse: i. e., if f is a diffeomorphism. The orbit of $x \in M$ relative to the endomorphism f is the subset $(f^m(x) \mid m \in Z_+)$ where Z_+ denotes the integers $\geqq 0$. If $f^m(x) = x$ for some integer $m \geqq 0$, then x is called a periodic point and the minimal such m the period of x. If the period of x is 1, then x is a fixed point. In (20) Smale has exhibited very powerful techniques for the study of the global orbit structure of automorphisms of compact manifolds. It is the purpose of this paper to provide an introduction to a similar study of endomorphisms of compact manifolds. The concepts and terminology of (20) are used throughout. A connected finite dimensional C^∞ manifold will be simply called a manifold.

The strongest useful equivalence relation for the study of the orbit structure of endomorphisms appears to be topological conjugacy; $f: M \to M$ is topologically conjugate to $g: N \to N$ if there exists a homeomorphism $h: M \to N$ such that $hf = gh$. Let M be a compact manifold and $E^r(M)$ be the space of C^r endomorphism of M with the topology of uniform convergence of the first r derivatives; then $f \in E^r(M)$ is called structurally stable if there exists a neighborhood U of f in $E^r(M)$ such that any $g \in U$ is topologically conjugate to f. Some of the most natural endomorphisms to investigate which are not automorphisms are the endomorphisms of the complex numbers of absolute value one: $f_n: S^1 \to S^1$ defined by $f_n(z) = z^n$, $n \in Z$, $|n| > 1$. We may naturally ask, are these f_n structurally stable? The answer is yes, and is given by the following more general proposition.

Proposition. *Let $f: S^1 \to S^1$ be a C^1 endomorphism such that the absolute value of the derivative of f is greater than 1 everywhere, then f is topologically conjugate to f_n where $n = \deg f$.*

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The endomorphisms $f_n\colon S^1 \to S^1$ are examples of expanding endomorphisms and the above proposition is a special case of a more general theorem.

Definition. Let M be a complete Riemannian Manifold, $\partial M = \phi$, then a C^1 endomorphism $f\colon M \to M$ is expanding if $\exists c > 0$, $\lambda > 1$ such that $\| Tf^m v \| \geqq c\lambda^m \| v \|$ for all $v \in TM$ and for all $m \in Z_+$ such that $m > 0$. (Note that if M is compact this definition is independent of the metric.)

THEOREM (α). *Any two homotopic expanding endomorphisms of a compact manifold are topologically conjugate.*

As the expanding endomorphisms of a compact manifold M are clearly open in $E^1(M)$ we have the following corollary.

COROLLARY. *Any expanding endomorphism of a compact manifold is structurally stable.*

Examples of expanding endomorphisms as well as the proof of Theorem (α) are given in I. Anosov endomorphisms are also considered there, and new examples of Anosov diffeomorphisms are given. They are slight modifications of the general construction given in [20] and were suggested by the situation for expanding endomorphisms. Smale [20] has used the notion of expanding endomorphism to construct the *D-E* series of indecomposable pieces of non-wandering sets and R. F. Williams [24] has considered expanding endomorphisms of branched one-manifolds, called expansions, in the study of one dimensional non-wandering sets of diffeomorphisms.

In II an extension of the Kupka-Smale approximation theorem for diffeomorphisms (see [12], [22], and [17]) is given.

Definition. Let $x \in M$ be a periodic point of period p for the endomorphisms $f\colon M \to M$. Then x is called hyperbolic if $Tf^p\colon T_xM \to T_xM$ has no eigenvalue of absolute value 0 or 1.

If x is a hyperbolic periodic point of the endomorphism $f\colon M \to M$ then as in [22] for example we may define the local stable and local unstable manifolds of f at p $W_{\text{loc}}{}^s(p)$ and $W_{\text{loc}}{}^u(p)$ respectively.

Definition. Let p be a hyperbolic periodic point of period j of $f\colon M \to M$ then:

1) The stable manifold of p, $W^s(p) = (x \in M \mid \exists n$ such that $f^{nj}(x) \in W_{\text{loc}}{}^s(p))$; $n \in Z_+$.
2) The unstable manifold $W^u(p) = (x \in M \mid \exists n$ and $y \in W_{\text{loc}}{}^u(p)$ such that $f^{nj}(y) = x)$; $n \in Z_+$.

3) If q is another hyperbolic periodic point such that $W^s(q)$ is a 1-1 immersed submanifold of M then $W^u(p)$ is transversal to $W^s(q)$ [we abbreviate this as $W^u(p) \pitchfork W^s(q)$]; if $f^{nj} \mid W_{loc}{}^u(p)$ is transversal to $W^s(q)$ on $W_{loc}{}^u(p)$ for all $n \in Z_+$.

THEOREM (β). Let M *be a compact manifold without boundary and let* $KS^r(M)$ *be the set of* $f \in E^r(M)$ *which satisfy:*

1) *The periodic poits of* f *are all hyperbolic.*
2) *If* p *is a periodic point of* f, $W^s(p)$ *is a* 1-1 *immersed submanifold*
3) *If* p *and* q *are periodic points of* f, $W^u(p) \pitchfork W^s(q)$.
   of constant dimension.

Then $KS^r(M)$ is a Baire set in $E^r(M)$.

Since $\text{Diff}^r(M,M)$ is an open subset of $E^r(M)$ we have as an immediate corollary

COROLLARY. *The Kupka-Smale automorphisms are a Baire set in* $\text{Diff}^r(M,M)$.

This corollary is the Kupka-Smale approximation theorem for diffeomorphisms.

In III we consider non-wandering sets of endomorphisms.

The period during which this paper was written was a period of great activity in the study of global analysis at Berkeley which gave me the opportunity to speak to many mathematicians working on similar subjects. I would particularly like to thank the following mathematicians for valuable conversations, suggestions, and encouragements: D. Epstein, M. Hirsch, N. Kopell, I. Kupka, E. Lima, C. Moore, J. Palis, M. Peixoto, C. Pugh, J. Wolf, E. Zeeman, and especially my advisor S. Smale.

I. Expanding endomorphisms. The beginning of this section is devoted to a description of some of the geometric properties of expanding endomorphisms on compact manifolds.

THEOREM 1. *Let* M *be a compact manifold and* $f\colon M \to M$ *an expanding endomorphism. Then:*

a) f *has a fixed point.*
b) *The universal covering space of* M *is diffeomorphic to* R^n, $n = \dim M$.
c) *If* $V \subset M$ *is an open subset then* $\bigcup_{m \in Z_+} f^m(V) = M$.

d) *The stable manifold of any periodic point of f is dense in M.*

e) *The unstable manifold of any periodic point of f is all of M.*

f) *f has a dense orbit.*

g) *The periodic points of f are dense in M.*

LEMMA 1. *Let $f\colon N \to N$ be expanding; then f is a covering map. In particular, if N is simply connected, f is a diffeomorphism.*

Proof. This is a consequence of (26), for instance.

LEMMA 2 (*Contraction mapping Lemma*). *Let X be a complete metric space and $g\colon X \to X$ a continuous map such that $\exists k,\ u \in R\ u < 1$ such that $d(g^n(x), g^n(y)) \leqq k u^n d(x,y)$ for all $x,\ y \in X$ and for all $n > 0$. Then g has a unique fixed point.*

A g satisfying the hypotheses of Lemma 2 is called a contraction mapping.

LEMMA 3. *Let $f\colon N \to N$ be an expanding diffeomorphism. Then f has a unique fixed point.*

Proof. f is a diffeomorphism so it has an inverse f^{-1}. Since f is expanding $\exists c > 0,\ \lambda > 1$ such that $\| Tf^{-n}(v) \| \leqq \dfrac{1}{c\lambda^n} \| v \|$ for all $v \in TN$ and for all $n > 0$. Let $d(x,y)$ be the metric on N induced by the Riemannian metric on TN then $d(f^{-n}(x), f^{-n}(y)) \leqq \dfrac{1}{c\lambda^n} d(x,y)$ for all $x, y \in N$ and for all $n > 0$ by the following argument. Let $h\colon [0,1] \to M$ such that $h(0) = x$ and $h(1) = y$ be a minimal differentiable arc joining x and y, then $d(x,y) = \int_0^1 \| h'(t) \| \, dt$ and

$$d(f^{-n}(x), f^{-n}(y)) \leqq \int_0^1 \| Tf^{-n}h'(t) \| \, dt \leqq \frac{1}{c\lambda^n} \int_0^1 \| h'(t) \| \, dt.$$

Thus $f^{-1}\colon N \to N$ is a contraction mapping and f^{-1} has a unique fixed point which is obviously also a unique fixed point for f.

M. Hirsch pointed out the following lemma to me.

LEMMA 4. *Let $f\colon N \to N$ be an expanding diffeomorphism and let c, λ and d be as is Lemma 3. Let $x \in N$ and $B_r(x) = (y \in N \mid d(x,y) \leqq r)$; then $f^n(B_r(x)) \supset B_{c\lambda^n r}(f^n(x))$.*

Proof. Let $h\colon [0,1] \to f^n(B_r(x))$ be a minimal differentiable arc from $f^n(x)$ to $\partial(f^n(B_r(x)))$. Then $f^{-n}h\colon [0,1] \to B_r(x)$ is an arc from x to $\partial B_r(x)$ so $\int_0^1 \| Tf^{-n}h'(t) \| \, dt \geqq r$ and $\int_0^1 \| h'(t) \| \, dt \geqq c\lambda^n r$.

LEMMA 5. *Let $f\colon N \to N$ be an expanding diffeomorphism. Then N is diffeomorphic to R^n, $n = \dim N$.*

Proof. Let x_0 be the unique fixed point for both f and f^{-1}. Then the eigenvalues of $Tf_{x_0}^{-1}\colon T_{x_0}N \to T_{x_0}N$ are all less than one in absolute value; see (18, p. 425). Thus there is an embedding of the closed unit disk $D^n \subset R^n$, $i\colon D^n \to N$, such that $i(0) = x_0$ and $f^{-1} \circ i(D^n) \subset$ interor $i(D^n)$. By Lemma 4 $\bigcup_{n \geqq 0} f^n(f^{-1} \circ i(D^n))$ is both open and closed in N, hence all of N, and N is the expanding union of differentiably embedded disks, thus N is diffeomorphic to R^n.

Notation. For $j \in R_+$ let $m_j\colon R^n \to R^n$ be the map $m_j(x) = jx$ and $\hat{m}_j\colon R^n \to R^n$ be the map $m_j \circ r$ where r is an orientation reversing reflection of R^n.

The proof of the following proposition may essentially be found in [28] or [22].

PROPOSITION 1. *Let $f\colon N \to N$ be an expanding diffeomorphism and $j > 1$. Then f is topologically conjugate to m_j if f is orientation preserving, or to $\hat{m}_j$ if f is orientation reversing.*

Notation. If M is a Riemannian manifold, its universal covering space will be denoted by $\bar{M}$ with covering map $P\colon \bar{M} \to M$ and the Riemannian metric which is the pull back of the Riemannian metric on M by P. Thus the covering transformations of $\bar{M}$ are isometries. If M is complete so is $\bar{M}$. Given a continuous function $\alpha\colon N \to M$ a lifting of α is a continuous function $\bar{\alpha}\colon \bar{N} \to \bar{M}$ such that

$$\begin{tikzcd} \bar{N} \arrow[r, "\bar{\alpha}"] \arrow[d, "P_N"] & \bar{M} \arrow[d, "P_M"] \\ N \arrow[r, "\alpha"] & M \end{tikzcd}$$ commutes.

It is well known from covering space theory that any $\alpha\colon N \to M$ has a lifting and if $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are both liftings of α then there exists a unique covering transformation ϕ of $\bar{M}$ such that $\bar{\alpha}_1 = \phi\bar{\alpha}_2$.

Let $D_{\bar{N}}$ ($D_{\bar{M}}$) denote the group of covering transformations of $\bar{N}(\bar{M})$. Then a continuous function $\beta\colon \bar{N} \to \bar{M}$ is a lifting if and only if there is a group homomorphism $h\colon D_{\bar{N}} \to D_{\bar{M}}$ such that $\beta\phi = h(\phi)\beta$ for all $\phi \in D_{\bar{N}}$, in which case h will be denoted by $\beta^{\#}$. A compact subset C of $\bar{M}$ will be called a covering domain if C is the closure of its interior and $P(C) = M$.

LEMMA 6. *Let $f\colon M \to M$ be expanding and let $\bar f\colon \bar M \to \bar M$ be a lifting of f, then $\bar f$ is expanding.*

PROPOSITION 2. *Let $f\colon M \to M$ be expanding. Then the universal covering space of M is diffeomorphic to R^n, $n = \dim M$, and f has a fixed point.*

Proof. Lemmas 1, 3, 5, and 6, and the fact that if x_0 is a fixed point for a lifting $\bar f$ of f then $P(x_0)$ is a fixed point for f.

Proof of Theorem 1. (a) and (b) are special cases of Proposition 2. Since the local stable manifold of a periodic point of an expanding endomorphism is the periodic point itself and the local unstable manifold is a neighborhood of the point (c) implies (d) and (e). (c) also implies (f) see Birkhoff [7]. It remains to show (c) and (g). Let $C \subset \bar M$ be a covering domain and $V \subset M$ an open set. Then $V' = P^{-1}(V) \cap$ interior of $C \neq \emptyset$. By Lemmas 4 and 6 for any lifting $\bar f$ of f, there exists an $n > 0$, a $\phi \in D_{\bar M}$ and a closed ball $B \subset V'$ such that (1) $\bar f^n(B) \supset \phi(C)$, which proves (c). Moreover, by (1) $B \supset \bar f^{-n}\phi(C)$ and in particular $B \supset \bar f^{-n}\phi(B)$ which implies $\exists x \in B$ such that $x = \bar f^{-n}\phi(x)$ or equivalently $\bar f^n(x) = \phi(x)$. Thus $f^n(P(x)) = P(x)$, which proves (g).

PROPOSITION 3. *Let $f\colon M \to M$ be an expanding endomorphism and $m \in M$. Then:*

(a) *M is a $K(\pi_1(M,m),1)$.*

(b) *$\pi_1(M,m)$ is torsion free.*

(c) *$f_\#\colon \pi_1(M,m) \to \pi_1(M,f(m))$ is injective. Moreover, if M is compact $f_\#(\pi_1(M,(m))$ is of finite index in $\pi_1(M,f(m))$.*

(d) *If M is compact then $X(M)$ the Euler characteristic of M equals zero.*

Proof. (a) By Proposition 2, $\bar M$ is diffeomorphic to R^n, $n = \dim M$ so for $i > 1$ $\pi_i(M,m) \approx \pi_i(\bar M, \bar m) \approx \pi_i(R^n, 0) = 0$.

(b) By a theorem of P. A. Smith Z_p cannot act freely on R^n so $\pi_1(M,m)$ cannot have a torsion subgroup. An alternate proof is given in [15, p. 103].

(c) Lemma 1 says that f is a covering map.

(d) If M is compact, by Lemma 6 f is a k sheeted covering map $k > 1$ and consequently $X(M) = kX(M) = 0$.

LEMMA 7. *Let G be a group and $\phi\colon G \to G$ a monomorphism. Let $G_\infty = \bigcap_{m=0}^{\infty} \phi^m(G)$. Then $\phi(G_\infty) = G_\infty$.*

Proof. If $h \in G_\infty$ *then* $h \in \phi^m(G)$ *for all* m, *and hence* $\phi(h) \in \phi^{m+1}(G)$ for all $m \geqq 0$. As $\phi^i(G) \subset \phi^j(G)$ for $j < i$, $\phi(G_\infty) \subset \bigcap_{m=0}^{\infty} \phi^m(G) = G_\infty$. Now if $y \in G_\infty$ there exists $x_i \in \phi^i(G)$ such that $\phi(x_i) = y$, but as ϕ is injective $x_i = x_j$ for all $i, j \geqq 0$ and $x_i \in G_\infty$.

PROPOSITION 4. *Let* $f\colon M \to M$ *be expanding and* x_0 *a fixed point of* f. *Then* $\pi_\infty = \bigcap_{m=0}^{\infty} f_\#{}^m(\pi_1(M, x_0)) = 1$.

Proof. Let M_{π_∞} be the covering space of M corresponding to π_∞. By Proposition 3(c) $f_\#$ is a monomorphism and by above Lemma $f_\#(\pi_\infty) = \pi_\infty$. Therefore f lifts to an expanding diffeomorphism of M_{π_∞} and and by Lemma 5, M_{π_∞} is diffeomorphic to R^n. Thus $\pi_\infty = 1$.

COROLLARY 1. $f_\#\colon \pi_1(M, x_0) \to \pi_1(M, x_0)$ *leaves no proper subgroup invariant and in particular the identity is its unique fixed point.*

Conjugacy theorems for expanding endomorphisms. I would like to thank J. Tate who communicated to me a proof of Theorem (α) of the introduction for the case where the compact manifold is S^1, which I had proven geometrically. Tate's methods were more amenable to generaization and the proofs presented below are generalizations of his method. Throughout this sction N and M will be manifolds; N will be compact and M will be complete.

Definition. Let $\alpha\colon N \to M$ be a continuous function and $\bar{\alpha}\colon \bar{N} \to \bar{M}$ a lifting of α. Then $V_{\bar{\alpha}}$ will be the set of continuous functions $B\colon \bar{N} \to \bar{M}$ which are liftings of continuous functions from N to M such that $B^\# = \bar{\alpha}^\#$. If $A, B \in V_{\bar{\alpha}}$ then $D(A, B) = \sup_{n \in N}(d(A(n), B(n)))$.

PROPOSITION 5. *D is a complete metric on* $V_{\bar{\alpha}}$.

Proof. (a) $D(A, B) < \infty$: let C be a covering domain in $\bar{N}$. Then for any $x \in \bar{N}$ there exists a $\phi \in D_{\bar{N}}$ such that $\phi(x) \in C$. Thus $d(A(x), B(x)) = d(A\phi^{-1}\phi(x), B\phi^{-1}\phi(x))$ and as $A^\# = B^\# = \bar{\alpha}^\#$,

$$d(A\phi^{-1}\phi(x), B\phi^{-1}\phi(x)) = d(\bar{\alpha}^\#(\phi^{-1})A\phi(x), \bar{\alpha}^\#(\phi^{-1})B\phi(x)).$$

But $\bar{\alpha}^\#(\phi^{-1}) \in D_{\bar{M}}$ and is an isometry of $\bar{M}$ so $d(A(x), B(x)) = d(A\phi(x), B\phi(x))$ and $\sup_{n \in N}(A(n), B(n)) = \sup_{x \in C}(A(x), B(x))$. As C is compact, $D(A, B) < \infty$.

(b) D is complete: if k_i is a Cauchy sequence in $V_{\bar{\alpha}}$ then k_i converges to a continuous function $k\colon \bar{N} \to \bar{M}$, and $k_i\phi \to k\phi$ for any $\phi \in D_{\bar{N}}$. But

$k_i\phi = k_i^{\#}(\phi)k_i = \bar{\alpha}^{\#}(\phi)k_i$. Thus $\bar{\alpha}^{\#}(\phi)k_i \to k\phi$ and as $k_i \to k$; $\bar{\alpha}^{\#}(\phi)k = k\phi$ and so $k \in V_{\bar{\alpha}}$.

THEOREM 2. *Let $f: M \to M$ be expanding and let $g: N \to N$ and $\alpha: N \to M$ be continuous. If there exists liftings $\bar{f}: \bar{M} \to \bar{M}$, $\bar{g}: \bar{N} \to \bar{N}$, and $\bar{\alpha}: \bar{N} \to \bar{M}$ of f, g, and α respectively, such that $\bar{f}^{\#}\bar{\alpha}^{\#} = \bar{\alpha}^{\#}\bar{g}^{\#}$ then there exists a unique $B \in V_{\bar{\alpha}}$ such that $\bar{f}B = B\bar{g}$.*

Proof. For $A \in V_{\bar{\alpha}}$ define $T(A) = \bar{f}^{-1}A\bar{g}$. It suffices to show that T is a contraction mapping on $V_{\bar{\alpha}}$. B is then its unique fixed point.

(a) $T(V_{\bar{\alpha}}) \subset V_{\bar{\alpha}}$: let $A \in V_{\bar{\alpha}}$ and $\phi \in D_{\bar{N}}$. Then

$$\bar{f}^{-1}A\bar{g}\phi = \bar{f}^{-1}(A^{\#}\bar{g}^{\#}(\phi))A\bar{g}.$$

Since $\bar{f}^{\#}$ is injective and

$$A^{\#}\bar{g}^{\#}(\phi) = \bar{\alpha}^{\#}\bar{g}^{\#}(\phi) \in \operatorname{Im}\bar{f}^{\#}; (\bar{f}^{\#-1}A^{\#}\bar{g}^{\#}(\phi))\bar{f}^{-1}A\bar{g} = \bar{f}^{-1}(A^{\#}\bar{g}^{\#}(\phi))A\bar{g}.$$

So $\bar{f}^{\#-1}A^{\#}\bar{g}^{\#}(\phi)\bar{f}^{-1}A\bar{g}\phi. = \bar{f}^{-1}A\bar{g}\phi$. But $\bar{f}^{\#-1}A^{\#}\bar{g}^{\#} = \bar{f}^{\#-1}\bar{\alpha}^{\#}\bar{g}^{\#} = \bar{\alpha}^{\#}$. Thus $\bar{f}^{-1}A\bar{g}\phi = \bar{\alpha}^{\#}(\phi)\bar{f}^{-1}A\bar{g}$ and so $\bar{f}^{-1}A\bar{g} \in V_{\bar{\alpha}}$.

(b) T is a contraction mapping: As in Lemma 3 there exist c, λ; $0 < c$ and $1 < \lambda$ such that $d(\bar{f}^{-n}(x), \bar{f}^{-n}(y)) \leqq \dfrac{1}{c\lambda^n} d(x,y)$ for all $x, y \in \bar{M}$ and all $n > 0$. Since $D(A\bar{g}^n, B\bar{g}^n) \leqq D(A,B)$ for all $A, B \in V_{\bar{\alpha}}$, $D(\bar{f}^{-n}A\bar{g}^n, \bar{f}^{-n}B\bar{g}^n) \leqq \dfrac{1}{c\lambda^n} D(A,B)$ for all $n > 0$ and T is a contraction mapping.

THEOREM 3 (*Theorem α of the Introduction*). *Let M be a compact manifold and let $f: M \to M$, $g: M \to M$ be two homotopic expanding endomorphisms. Then f and g are topologically conjugate.*

Proof. Let $m \in M$ then there exists a path β from $g(m)$ to $f(m)$ such that the map $\beta_{\#}: \pi_1(M, f(m)) \to \pi_1(M, g(m)): \alpha \to \beta^{-1}\alpha\beta$ satisfies $\beta_{\#}f_{\#} = g_{\#}$ see [23, p. 52]. Let $\bar{f}$ be any lifting of f; $\bar{f}: \bar{M} \to \bar{M}$, and let $e_0 \in P^{-1}(m)$ and $e_1 = \bar{f}(e_0)$. Let $\bar{g}: \bar{M} \to \bar{M}$ be the lifting of g which takes e_0 into the endpoint of the lifting of the path β^{-1} which starts at e_1. Then it is easy to check that $\bar{f}^{\#} = \bar{g}^{\#}$. Thus Theorem 2 may be applied with α the identity of M, and $\bar{\alpha}$ the identity of $\bar{M}$. Consequently there exist unique $B_1, B_2 \in V_{\mathrm{id}}$ such that (1) $\bar{f}B_1 = B_1\bar{g}$ and $\bar{g}B_2 = B_2\bar{f}$. Thus $B_2\bar{f}B_1 = B_2B_1\bar{g}$ and $B_1\bar{g}B_2 = B_1B_2\bar{f}$. By applying (1) $\bar{f}B_1B_2 = B_1B_2\bar{f}$ and $\bar{g}B_2B_1 = B_2B_1\bar{g}$. But B_1B_2 and B_2B_1 are elements of V_{id} and by the uniqueness part of Theorem 2, $B_1B_2 = B_2B_1 =$ the identity of $\bar{M}$. B_i is the lifting of a continuous function $h_i: M \to M$, $i = 1, 2$ which satisfy $fh_1 = h_1g$ and $gh_2 = h_2$ and $h_1h_2 = h_2h_1$ $=$ identity of M.

THEOREM 4. *Let $g: N \to N$ be a continuous function and have a fixed point n_0, and let m_0 be a fixed point for the expanding endomorphism $f: M \to M$. There is a one to one correspondence between the continuous functions $h: N \to M$, such that $h(n_0) = m_0$ and $fh = hg$, and the group homomorphisms $A: \pi_1(N, n_0) \to \pi_1(M, m_0)$ which make the following diagram commute:*

$$\begin{array}{ccc} \pi_1(N, n_0) & \xrightarrow{A} & \pi_1(M, m_0). \\ \downarrow g_\# & & \downarrow f_\# \\ \pi_1(N, n_0) & \xrightarrow{A} & \pi_1(M, m_0) \end{array}$$

The correspondence is given by $h \to h_\#$.

Proof. Since M is a $K(\pi_1(M, m_0)$ there exists a continuous function $\alpha: N \to M$ such that $\alpha(n_0) = m_0$ and $\alpha_\#: \pi_1(N, n_0) \to \pi_1(M, m_0)$ is equal to A; see [23, p. 427]. Let $\bar{m}_0 \in P_M^{-1}(m_0)$ and $\bar{n}_0 \in P_N^{-1}(n_0)$. Lift f to $\bar{f}: \bar{M} \to \bar{M}$ such that $\bar{f}(\bar{m}_0) = \bar{m}_0$; lift g to $\bar{g}: \bar{N} \to \bar{N}$ such that $\bar{g}(\bar{n}_0) = \bar{n}_0$; and lift α to $\bar{\alpha}: \bar{N} \to \bar{M}$ such that $\bar{\alpha}(\bar{n}_0) = \bar{m}_0$. Then $\bar{f}^\# \bar{\alpha}^\# = \bar{\alpha}^\# \bar{g}^\#$. Let $T: V_{\bar{\alpha}} \to V_{\bar{\alpha}}$ be as in Theorem 2 and let $V_{\bar{\alpha}}{}^* = (B \in V_{\bar{\alpha}} \mid B(\bar{n}_0) = \bar{m}_0)$. Then $V_{\bar{\alpha}}{}^*$ is a closed subset of $V_{\bar{\alpha}}$ and $T(V_{\bar{\alpha}}{}^*) \subset V_{\bar{\alpha}}{}^*$ and the unique fixed point B of Theorem 2 actually lies in $V_{\bar{\alpha}}{}^*$. B lifts a continuous function $h: N \to M$ such that $h(n_0) = m_0$ $h_\# = A$ and $fh = hg$. If h_1 is another continuous function such that $fh_1 = h_1 g$, $h_1(n_0) = m_0$ and $h_{1\#} = A$, then lift h_1 to $\bar{h}_1: \bar{N} \to \bar{M}$ such that $\bar{h}_1(\bar{n}_0) = m_0$; then $\bar{h}_1{}^\# = \bar{\alpha}^\#$. Since $f_1 h_1 = h_1 g$, $\bar{f}\bar{h}_1$ and $\bar{h}_1 \bar{g}$ are liftings of the same continuous function so there exists a $\phi \in D_{\bar{M}}$ such that $\phi \bar{f} \bar{h}_1 = \bar{h}_1 \bar{g}$. Thus $\phi \bar{f} \bar{h}_1(\bar{n}_0) = \bar{h}_1 \bar{g}(\bar{n}_0)$; $\phi \bar{f}(\bar{m}_0) = \bar{m}_0$ and $\phi(\bar{m}_0) = \bar{m}_0$ so ϕ is the identity map of $\bar{M}$. Consequently $\bar{f}\bar{h}_1 = \bar{h}_1 \bar{g}$ and $\bar{h}_1 = B$ so $h_1 = h$.

THEOREM 5. *Let M and N be compact and let $g: N \to N$ and $f: M \to M$ be expanding with fixed points n_0 of g and m_0 of f. There is a one to one correspondence between the homeomorphisms $h: N \to M$, such that $h(n_0) = m_0$ and $fh = hg$, and the group isomorphisms $A: \pi_1(N, n_0) \to \pi_1(M, m_0)$ which make the following diagram commute:*

$$\begin{array}{ccc} \pi_1(N, n_0) & \xrightarrow{A} & \pi_1(M, m_0) \\ \downarrow g_\# & & \downarrow f_\# \\ \pi_1(N, n_0) & \xrightarrow{A} & \pi_1(M, m_0). \end{array}$$

The correspondence is given by $h \to h_\#$.

Proof. By Theorem 4 there exist unique $h_1: N \to M$ and $h_2: M \to N$ which are continuous and which satisfy $h_1(n_0) = m_0$, $h_2(m_0) = n_0$,

$$h_{1\#}: \pi_1(N, n_0) \to \pi_1(M, m_0)$$

is equal to A,

$$h_{2\#}: \pi_1(M, m_0) \to \pi_1(N, n_0)$$

is equal to A^{-1}, $fh_1 = h_1 g$, and $gh_2 = h_2 f$. Consequently $fh_1h_2 = h_1h_2f$, $(h_1h_2)(m_0) = m_0$, and $(h_1h_2)_\#: \pi_1(M, m_0) \to \pi_1(M, m_0)$ is the identity map. By applying Theorem 4 again $h_1h_2 =$ the identity of M. Similarly $h_2h_1 =$ the identity of N. Thus h_1 and h_2 are homeomorphisms.

COROLLARY 2. *Let M be a compact manifold and let $f: M \to M$ be expanding. Then the centralizer of f in the monoid of continuous functions from M to M is a countable discrete subset.*

Proof. f has a finite number of fixed points, and for any $m_0 \in M$, $\pi_1(M, m_0)$ is finitely generated. Thus $\operatorname{Hom}(\pi_1(M, m_0), \pi_1(M, m_1))$ is countable. As any continuous map $h: M \to M$ such that $fh = hf$ must take a fixed point of f into a fixed point of f, Theorem 4 may be applied to the case $N = M$, $g = f$.

In a similar way Theorem 5 may be used to determine the centralizer of f in the group of homeomorphisms of M. It is possible by using Theorem 2 to determine the algebraic structure of the centralizers for some cases.

Remark. Theorem 2 asserts in particular that given a continuous map $g: S^1 \to S^1$ of degree n, $|n| > 1$, there exists a continuous map $\alpha: S^1 \to S^1$ of degree one such that $f_n \alpha = \alpha g$, where $f_n(z) = z^n$. α is, of course, not a homeomorphism in general.

Examples. The examples given here are motivated by the examples of Anosov diffeomorphisms given in [20, sec. I-3]. It would be very helpful to the reader to be familiar with them. Recall that if H is a connected Lie group and $E: H \to H$ is an expanding group automorphism then by Lemma 5, H is diffeomorphic to R^n. In particular, H is simply connected. And as in [20, I-3.6] H is nilpotent. Up to topological conjugacy all examples of expanding maps on compact manifolds, that I know of, are given by the following construction:

Let N be a connected, simply connected nilpotent Lie group, with a left invariant metric. Let C be a compact group of automorphisms of N, and let $G = N \cdot C$, be the Lie group obtained by considering N as acting on itself by left translation and taking the semi-direct product of N and C in Diff(N). Let Γ be a discrete uniform subgroup of G, such that N/Γ (the orbit space of N under the action of Γ) is a compact manifold. Then by a theorem of L. Auslander [5] $\Gamma \cap N$ is a uniform discrete subgroup of N and $\Gamma/\Gamma \cap N$ is finite. Now let $E: N \to N$ be an expanding group automorphism of N such that by conjugating Γ by E in Diff(N) $E\Gamma E^{-1} \subset \Gamma$. Then E projects to an expanding map of N/Γ; $E_0: N/\Gamma \to N/\Gamma$.

Remarks. (1) As $\Gamma/\Gamma \cap N$ is finite, C could have been assumed to be finite to begin with, and as $\Gamma \cap N$ is a uniform discrete subgroup of N, N/Γ is finitely covered by the nilmanifold $N/\Gamma \cap N$. Since $E(N \cap \Gamma)E^{-1} \subset N \cap \Gamma$, E_0 lifts to an expanding map $E_1: N/\Gamma \cap N \to N/\Gamma \cap N$. (Conjugating an element n of N by E in Diff(N) gives $E(N)$).

(2) N/Γ may be considered as the double coset space $\Gamma \backslash G/C$.

(3) Let $0 \to H \to \Gamma \to F \to 0$ be an exact sequence of groups where H is finitely generated, torsion free, discrete nilpotent group, and F is a finite group such that in the induced action F acts effectively on H by isomorphisms. Then by the results of Malcev [14] which are summarized in [6] or [20], H may be embedded as a uniform discrete subgroup of a connected, simply connected nilpotent Lie group N, and the action of F on H extends uniquely to an effective action of F on N by Lie group automorphisms. Thus Γ may be considered as a subgroup of $N \circ F$ and if Γ acts freely on N, N/Γ is a compact manifold.

Problem. Does the general construction above give all examples of expanding endomorphisms of compact manifolds up to topological conjugacy?

I have tried to use Proposition 4 and other facts about the fundamental group of a compact manifold with an expanding endomorphism in conjunction with Theorem 5 to show that it does. Starting with an entirely different approach to the problem, M. Hirsch has shown me some very promising ideas and partial results; but at this writing it remains unsolved.

The toral expanding endomorphisms. The n-Torus, T^n, may be considered as the quotient of the abelian Lie group R^n by the uniform discrete subgroup Z^n, the subgroup of points all of whose entries are integers. Let A be an n by n matrix such that all the entries of A are integers and all the eigenvalues of A are greater than one in absolute value. A may be thought of as an expanding Lie group automorphism of R^n such that $A(Z^n) \subset Z^n$ (Or equivalently considering Z^n as contained in Diff(R^n), $AZ^nA^{-1} \subset Z^n$). Then A projects to an expanding endomorphism of T^n; $A_0: T^n \to T^n$. A_0 is called a linear expanding endomorphism. Conversations with M. Hirsch were very helpful in proving the following:

PROPOSITION 6. *Let $f\colon T^n \to T^n$ be expanding then f is topologically conjugate to a linerar expanding endomorphism.*

Proof. Let $\bar{f}\colon R^n \to R^n$ be a lifting of f. Then $\bar{f}^{\#}\colon D_{R^n} \to D_{R^n}$ may be considered as a group homomorphism of $Z^n \subset R^n$, in which case $\bar{f}^{\#}(z) = \bar{f}(z)$; for $z \in Z^n$. $\bar{f}^{\#}\colon Z^n \to Z^n$ may be written as an n by n matrix A, all the entries of which are integers. A may be considered as a Lie group automorphism (i.e., linear map) of R^n such that $A \mid Z^n = \bar{f} \mid Z^n$. Thus there exist $C > 0$, $\lambda > 1$ such that in the usual metric on R^n, $\| A^n(z) \| \geqq C\lambda^n \| z \|$ for all $z \in Z^n$. By the linearity of A, $\| A^n(y) \| \geqq C\lambda^n \| y \|$ for all $y \in R^n$ and A is an expanding Lie group automorphism of R^n, such that $A(Z^n) \subset Z^n$. Thus A projects to a linear expanding endomorphism $A_0\colon T^n \to T^n$. As A is a lifting of A_0, and $A^{\#} = \bar{f}^{\#}$ the proof of Theorem 3 shows that A_0 and f are topologically conjugate.

Remark. A_0 is homotopic to f, and $A\colon R^n \to R^n$ is $f_*\colon H_1(T^n, R) \to H_1(T^n, R)$. It should be noted, however, that two expanding endomorphisms which are topologically conjugate need not be homotopic.

PROPOSITION 7. *If M is a compact, connected (real analytic) Lie group and if $f\colon M \to M$ is expanding, then M is the n-Torus, T^n, where $n = \dim M$.*

Proof. M is of the form $\dfrac{T \times G}{D}$ where T is a Toroid, G a compact semisimple group and D a finite central subgroup of $T \times G$ such that $D \cap T$ and $D \cap G$ are trivial; see [10, p. 144]. Thus $\bar{M}$ is homeomorphic to $\bar{T} \times \bar{G}$. $\bar{T}$ is homeomorphic to R^m, $m = \dim T$, and $\bar{G}$ is a compact semi-simple group. But since $f\colon M \to M$ is expanding, $\bar{M}$ is diffeomorphic to R^n, $n = \dim N$, thus $\bar{G}$ is trivial and $M = T$.

If M were not compact by using [10, p. 180] the same argument would prove that M is homeomorphic to the direct product of a Torus and a Euclidean space.

If C is a compact group of Lie group automorphisms of R^n and Γ is a uniform discrete subgroup of $C \cdot R^n$, then R^n/Γ is a compact manifold which has a flat Riemannian metric. See [8] and [23] for general references. The following is proven in [9].

THEOREM. *If M is a compact flat Riemannian manifold, then there exists an expanding endomorphism of M.*

The situation when the group N is not abelian seems considerably more complicated. I would like to thank C. Moore for pointing out the following example to me. Let N be the simply connected non-abelian Lie group of dimension three; that is, we may suppose that N is represented by the group of lower triangular matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{bmatrix} \quad x, y, z \in R.$$

Let Γ be the uniform discrete subgroup of matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \gamma & \beta & 1 \end{bmatrix} \quad \alpha, \beta, \gamma \in Z.$$

If a, b, and c are integers which are greater than one in absolute value and $ab = c$, then the map

$$\begin{aligned} x &\to ax \\ y &\to by \\ z &\to cz \end{aligned}$$

defines an expanding Lie group automorphism of N which takes Γ into itself and hence projects to an expanding endomorphism of N/Γ.

* Now let C be the compact group of Lie group automorphisms of N consisting of the identity, and the automorphism A defined by $A(x) = -x$, $A(y) = -y$, and $A(z) = z$. Let $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} A$. Then the group generated by A_1 and Γ is a uniform discrete subgroup of $N \cdot C$; call it Γ_1. It is easy to check that $\Gamma_1 = \Gamma \cup A_1\Gamma$. If $E\colon N \to N$ is the expanding Lie group automorphism defined by

$$\begin{aligned} x &\to ax \\ y &\to by \\ z &\to cz \end{aligned}$$

where a and b are odd integers greater than one in absolute value and $c = ab$, then $E\Gamma_1 E^{-1} \subset \Gamma_1$ and E projects to an expanding endomorphism of N/Γ_1. N/Γ_1 is not a nilmanifold, since Γ_1 is torsion free. But

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \gamma & \beta & 1 \end{bmatrix} A_1 = A_1 \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ \gamma & -\beta & 1 \end{bmatrix} \text{ and in particular } \left[\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_1 \right]^2 = A_1{}^2,$$

and thus Γ_1 is not nilpotent; see [12, p. 247].

Anosov endomorphisms. We begin by recalling some definitions from [20]. A bundle map of a Riemannian vector space bundle $\phi\colon E \to E$ is called contracting if there exists $c > 0$, $0 < \lambda < 1$ such that for all $v \in E$ and all integers $m > 0$, $\| \phi^m(v) \| < c\lambda^m \| v \|$. ϕ will be called expanding if there exists $d > 0$, $u > 1$ such that for all $v \in E$ and all integers $m > 0$, $\| \phi^m(v) \| > du^m \| v \|$. If E is a Riemannian vector bundle over a compact space, then the property of contracting or expanding is independent of the metric. Let M be a Riemannian manifold; $f\colon M \to M$ an endomorphism; and A a subset of M such that $f(A) \subset A$. Then f is said to be hyperbolic on A if $TM \mid A$ is the Whitney sum of two subbundles E^s and E^u such that $Tf(E^s) \subset E^s$; $Tf(E^u) \subset E^u$; Tf is contracting on E^s; and Tf is expanding on E^u.

Definition. Let M be a complete Riemannian manifold without boundary. $f\colon M \to M$ is an Anosov endomorphism if f is an immersion, a covering map, and f is hyperbolic on all of M.

Expanding endomorphisms and Anosov diffeomorphisms are, of course, examples of Anosov endomorphisms. Anosov immersions can be constructed in a similar fashion to the Anosov diffeomorphisms constructed in [20].

THEOREM 6. *The Anosov endomorphisms of a compact manifold M are open in $E^1(M)$ and are structurally stable.*

The proof of this Theorem is the same as the proof of the openness of Anosov diffeomorphisms [20, 1-8] and J. Moser's proof of the stability of Anosov diffeomorphisms, as exposited by J. Mather in the appendix to [20], except for the following modification in the proof of the second statement. Let $C^0(M, M)$ denote the continuous functions from M to M and let $\bar{C}^0(\bar{M}, \bar{M})$ denote the continuous functions from $\bar{M}$ to $\bar{M}$ which are liftings of continuous functions from M to M, both spaces with the compact open topology. Then the natural projection map $p\colon \bar{C}^0(\bar{M}, \bar{M}) \to C^0(M, M)$ is a covering map and defines a Banach manifold structure on $\bar{C}^0(\bar{M}, \bar{M})$. If $\bar{f}$ is any lifting of f, one applies all the arguments of the appendix to the map $h \to \bar{f}^{-1}h\bar{f}$, $h \in \bar{C}^0(\bar{M}, \bar{M})$ which is defined in a neighborhood of $\mathrm{id}_{\bar{M}}$; and then one projects.

I would like to give two examples of Anosov diffeomorphisms of compact manifolds which are not nilmanifolds. The examples were motivated by (*) and are elaborations on [20, 1-3.8]. Once again the general framework is: $\Gamma \subset N \cdot C$ is a uniform discrete subgroup; A is a hyperbolic Lie group automorphism of N such that $A\Gamma A^{-1} = \Gamma$; and A projects to an Anosov diffeomorphism of N/Γ.

Let $Q[\sqrt{3}]$ be the field of rational numbers with the $\sqrt{3}$ adjoined; let $Z[\sqrt{3}]$ be the algebraic integers in $Q[\sqrt{3}]$, and let $\sigma\colon Q[\sqrt{3}] \to Q[\sqrt{3}]$ be the non-trivial Galois automorphism (sending $\sqrt{3}$ into $-\sqrt{3}$). For $\alpha \in Q[\sqrt{3}]$, $\sigma(\alpha)$ will be written $\bar{\alpha}$. Let $\lambda = 2 + \sqrt{3}$, then $\lambda\bar{\lambda} = 1$. Finally, let $w \in Q[\sqrt{3}]$ be chosen such that $w \notin Z[\sqrt{3}]$, $2w \in Z[\sqrt{3}]$, and $\lambda w = \alpha + w$ where $\alpha \in Z[\sqrt{3}]$. It is easy to find such a w.

(1) Let N be the abelian Lie group R^4, and let C be the compact group of Lie group automorphisms of R^4 consisting of the identity and the map B, $B(a, b, c, d) = (-a, b, -c, d)$ for $a, b, c, d \in R$. Let Γ be the uniform discrete subgroup of R^4 consisting of all elements of the form $(\alpha, \beta, \bar{\alpha}, \bar{\beta})$, $\alpha, \beta \in Z[\sqrt{3}]$. Let $A\colon R^4 \to R^4$ be defined by $A(a, b, c, d) = (\lambda a, \lambda b, \bar{\lambda} c, \bar{\lambda} d)$. Let Γ_1 be the uniform discrete subgroup of $N \cdot C$ generated by Γ and B_1, where $B_1 = (0, w, 0, \bar{w})B$. Then $\Gamma_1 = \Gamma \cup B_1\Gamma$ and $A\Gamma_1 A^{-1} = \Gamma_1$, so A projects to an Anosov diffeomorphism of N/Γ_1, which is a Riemannianly flat compact four manifold. N/Γ_1 is not a nilmanifold, however, by the same argument as in (*). $((1, 0, 0, 0)B_1)^2 = B_1{}^2$.

(2) Uet N be the direct product of two copies of the lower triangular 3 by 3 matrices. For convenience we will write the elements of this group as (a, b, c, d, e, f) $a, b, c, d, e, f \in R$ where the multiplication is:

$$\begin{aligned}&(a, b, c, d, e, f)(a_1, b_1, c_1, d_1, e_1, f_1)\\ &\qquad = (a + a_1, b + b_1, c + ba_1 + c_1, d + d_1, e + e_1, f + ed_1 + f_1).\end{aligned}$$

Let Γ be the uniform discrete subgroup of N consisting of all elements of the form $(\alpha, \beta, \gamma, \bar{\alpha}, \bar{\beta}, \gamma)$ $\alpha, \beta, \gamma \in Z[\sqrt{3}]$. Let C be the compact group of Lie group automorphisms of N consisting of the identity and B, where $B(a, b, c, d, e, f) = (-a, -b, c, -d, -e, f)$. Let A be the hyperbolic Lie group automorphism of N defined by $A(a, b, c, d, e, f) = (\bar{\lambda} a, \lambda^2 b, \lambda c, \lambda d, \bar{\lambda}^2 e, \bar{\lambda} f)$. Let Γ_1 be the uniform discrete subgroup of $N \cdot C$ generated by Γ and B_1, where $B_1 = (0, 0, w, 0, 0, \bar{w})B$. $\Gamma_1 = \Gamma \cup B_1\Gamma$ and $A\Gamma_1 A^{-1} = \Gamma_1$, so A projects to an Anosov diffeomorphism of N/Γ_1 which is not a nilmanifold by the argument in (*).

I would like to thank J. Palis for his help in calculating the above examples.

II. Kupka-Smale endomorphisms. The proof of the Kupka-Smale approximation theorem presented here is in its essentials a reworking of the proof for automorphisms given in [22]. The stable and unstable manifolds for endomorphisms are, however, more complicated than those for automorphisms, and the proof of transversality theorems is correspondingly more difficult. The Abraham transversality theorem is used to deal with the new difficulties. In order to apply the theorem, some facts about and examples of Banach manifolds are necessary. The following material may be found in [1], [2], and [3].

Throughout this section N and M will be finite dimensional C^∞ manifolds such that $\partial M = \emptyset$ and N is compact. G will be a Banach Manifold. Let $C^r(N,M)$ denote the space of C^r functions, $r \geqq 1$, from N to M with the topology of uniform convergence of the first r derivatives. (Note that $C^r(M,M) = E^r(M)$). If $f\colon N \to M$ is an element of $C^r(N,M)$ then f^*TM is the vector bundle over N which is the pullback of the tangent bundle of M by f. If E is a vector bundle over a compact manifold then $\Gamma^r(E)$ is the Banach space of C^r sections of E with the topology of uniform convergence of the first r derivatives.

THEOREM (1) *see* [1, 11.1]. *$C^r(N,M)$ is a Banach manifold, and the tangent space at $f \in C^r(N,M)$, $T_f C^r(N,M)$ may be identified with $\Gamma^r(f^*TM)$.*

Definition. $Ev\colon C^r(N,M) \times N \to M$ is the map defined by $Ev(f,n) = f(n)$.

THEOREM (2) *see* [1,11.6 *and* 11.7]. *$Ev\colon C^r(N,M) \times N \to M$ is C^r. Moreover, if $h \in \Gamma^r(f^*TM)$ and $z \in T_nN$, then $TEv_{(f,x)}(h,z) = Tf_n(z) + h(n)$ where $h(n)$ is considered as an element of $T_{f(n)}M$.*

Definition. Let G be a Banach manifold and let $\alpha\colon G \to C^r(N,M)$ be a function. Then $Ev\colon G \times N \to M$ is the function $Ev(\alpha \times \mathrm{id}_M)$. $Ev_\alpha(g,n) = \alpha(g)n$ for all $g \in G$ and all $n \in N$. If Ev is of class C^r, $r \geqq 1$, $\alpha\colon G \to C^r(N,M)$ or briefly, (G,α) is called a manifold of mappings.

Examples. (1) By Theorem 3, if G is a submanifold of $C^r(N,M)$ and $j\colon G \to C^r(N,M)$ is the inclusion map, then (G,j) is a manifold of mappings.

(2) Let M be compact. Let $p\colon E^r(M) \to E^r(M)$ be the map $p(f) = f^p$, and let G be a submanifold of $E^r(M)$. Then $(G, p \mid G)$ is a manifold of mappings.

(3) Let M be compact. Let $\mathrm{graph}_p\colon E^r(M) \to C^r(M, M \times M)$ be defined by $\mathrm{graph}_p(f) = \mathrm{id}_M \times f^p$. Then if G is a submanifold of $E^r(M)$, $(G, \mathrm{graph}_p \mid G)$ is a manifold of mappings.

LEMMA 1. *Let G be a submanifold of $E^r(M)$. Let $f \in G$, and let $m \in M$. If $h \in T_fG$ and $z \in T_mM$ then:*

(a) $TEv_{p_{(f,m)}}(h,z) = \sum_{j=0}^{p-1} Tf_{f^{p-j}(m)}{}^j h(f^{p-j-1}(m)) + Tf_m{}^p(z)$.

(b) $TEv_{\mathrm{graph}_{p_{(f,m)}}}(h,z) = (z, TEv_{p_{(f,m)}}(h,z))$.

Proof. It is sufficient to prove (a). We proceed by induction. For $p = 1$ this is Theorem 3.

$$TEv_{p+1_{(f,m)}}(h,z)$$

$$= TEv_{(f,f^p(x))}(h, \sum_{j=0}^{p-1} Tf_{f^{p-1}(m)}{}^j h(f^{p-j-1}(m)) + Tf_m{}^p(z))$$

$$= Tf_{f^p(m)}(\sum_{j=0}^{p-1} Tf_{f^{p-j}(m)}{}^j h(f^{p-j-1}(m))) + h(f^p(m)) + T_m f^{p+1}(z)$$

which by reindexing (j goes to $j+1$) equals

$$\sum_{j=0}^{p} Tf_{f^{p+1-j}}{}^j h(f^{p+1-j-1}(m)) + T_{fm}^{f}{}^{p+1}(z).$$

The Abraham transversality theorems. Let $K(N)$ be the topological space of compact subsets of N with the weakest topology such that all the compact subsets of an open set of N form an open set. Let $\mathcal{G}_k(TM)$ be the Grassmannian bundle of k planes in TM. Let W_k be the topological space of closed k dimensional sumanifolds of M with the weakest topology which makes the map $\tau\colon W_k \to \mathcal{G}_k(TM)$, $\tau(W) = TW$ for $W \in W_k$ continuous. Recall from [13] that a differentiable map $f\colon G \to M$ is transversal to $W \in W_k$ at a point $g \in G$ if and only if $f(g) \notin W$ or $Tf_g(T_gG) + T_{f(g)}W = T_{f(g)}M$. $f\colon G \to M$ is said to be transversal to $W \in W_k$ on a subset K of G if f is transversal to W at each point of K.

With slight changes, the proofs of the following two theorems are given in [1, 13.4 and 14.2] and [2].

OPENNESS THEOREM. *Let $\alpha\colon G \to C^r(N,M)$ be a manifold of mappings. Let $\beta\colon G \to K(N)$ and $\phi\colon G \to W_k$ be continuous functions. Then $G_{\phi,\beta} = (g \in G \mid \alpha(g)$ is transversal to $\phi(g)$ on $\beta(g))$ is an open subset of G.*

DENSITY THEOREM. *Let $K \in K(N)$, $W \in W_k$, and let $\alpha\colon G \to C^r(N,M)$ be a manifold of mappings. Let $G_{K,W} = (g \in G \mid \alpha(g)$ is transversal to W on $K)$. If Ev_α is transversal to W on $G \times K$ and if Ev_α is C^r where $r > \max(\dim N + k - \dim M, 0)$, then $G_{K,W}$ is an open and dense subset of G.*

Remark. In applying the transversality theorems to the Kupka-Smale approximation theorem for endomorphisms the condition in the density theorem $r > \max(\dim N + k - \dim M, 0)$ will be irrelevant since $E^q(M)$ is dense in $E^p(M)$ for $q > p$ and the openness theorem makes no assumption on r other than $r \geqq 1$. In order to apply the transversality theorems to the approximation theorem some lemmas on the transversality of Ev_α are needed. Instead of writing $f\colon G \to M$ is transversal to W on K we will write $f \pitchfork W$ on K.

LEMMA 1. *If $\alpha\colon G \to C^r(N, M)$ is a manifold of mappings, and if $\alpha(g) \pitchfork W$ at m, then $Ev_\alpha \pitchfork W$ at (g, m). From now on let M be compact.*

LEMMA 2. *Let $f \in E^r(M)$. Let m be a hyperbolic periodic point of f. Then* $\mathrm{graph}_p(f) \pitchfork \Delta_{M\times M}$ *at m, and $Ev_{\mathrm{graph}_p} \pitchfork \Delta_{M\times M}$ at (f, m) for all positive integers p.*

LEMMA 3. *Let $f \in E^r(M)$, and let $m \in M$ be a periodic point of f of period p. Then $Ev_{\mathrm{graph}_p} \pitchfork \Delta_{M\times M}$ at (f, m).*

Proof. Since p is the minimal positive integer n such that $f^n(m) = m$, if $0 \leqq i$, $j \leqq p$, then $f^i(m) \neq f^j(m)$ *when* $i \neq j$. By Lemma 1(b),

$$TEv_{\mathrm{graph}_{p_{(f,m)}}}(h, z) = (z, \sum_{j=0}^{p-1} Tf_{f^{p-j}(m)}{}^j h(f^{p-j-1}(m))) + Tf_m{}^p(z)).$$

We may construct sections $h \in \Gamma^r(f^*TM)$ such that $h(f^{p-j-1}(m)) = 0$ for $j \neq 0$, and $h(f^{p-1}(m))$ considered as an element of $T_{(m)}M$ is any element we wish. Letting $z = 0$, this argument shows that $TEv_{\mathrm{graph}_{p_{(f,m)}}} T_{(f,m)}E^r(M) \times M$ contains $0 \times T_mM$. Thus $Ev_{\mathrm{graph}_p} \pitchfork \Delta_{M\times M}$ *at* (f, m).

Definition. Let $\partial N = \emptyset$, and let $V \subset N$ be a compact submanifold, perhaps with boundary. Then $R\colon C^r(N, M) \to C^r(V, M)$ is the restriction map, $R(f) = f \mid V$ for all $f \in C^r(N, M)$.

LEMMA 4. *$(E^r(M), R \circ p)$ is a manifold of mappings. Moreover, let V and W be closed submanifolds of N such that $V \pitchfork W$, and let $g \in E^r(M)$, then:*

(a) *Let $m \in V$ such that $g^j(m) \neq g^k(m)$ for $0 \leqq j < k \leqq p - 1$ and such that $g^{p-1}(m) \notin W$. Then $Ev_{R\circ p} \pitchfork W$ at (g, m).*

(b) *Let g have maximal rank on W and let $g(W) \subset W$. Let $m \in V$. Then, if $g^j(m) = g^k(m)$ for $0 \leqq j < k \leqq p - 1$ implies that $g^i(m) \in W$ for all $i \geqq j$, $Ev_{R\circ p} \pitchfork W$ at (g, m).*

Proof. *That* $(E^r(M), R\circ p)$ is a manifold of mappings follows from Theorem 2 and the fact that $Ev_{R\circ p} = Ev_p \mid E^r(M)\times V$. The proof of (a) is the same as the proof of Lemma 3. To prove (b) notice that by the hypotheses we may assume that there exists an $n<p$ such that $g^n(m)\notin W$. Let s be the maximal such n. Then by (a) $Ev_{R\circ s+1}\pitchfork W$ at $g(m)$. By Lemma 1 $TEv_{R\circ p} = Tg^{p-s-1}(TEv_{R\circ s+1})$ + terms which can be made equal to zero by the proper choice of h's and z's, as in Lemma 3. Since g has maximal rank on W, $Ev_{R\circ p}\pitchfork W$ at (g, m).

Definition. Let $T_p = (f\in E^r(M) \mid \operatorname{graph} f^p \pitchfork \Delta_{M\times M}$ on all of $M)$. Let $H_p = (f\in T_p \mid$ all periodic points of f of period $\leqq p$ are hyperbolic$)$.

The following proposition is well known.

PROPOSITION 1. *Let* $f\in\bigcap_{k=1}^{n} T_k$, $n<\infty$. *Then f has a finite number of periodic points of period* $\leqq n$; $\beta_1,\cdots,\beta_i$. *Moreover, there exist neighborhoods* U_j *of the* β_j *and* W *of* f *such that* $U_j\cap U_k=\emptyset$ *for* $j\neq k$, *which satisfy the following condition. If* $g\in W$, *then* $g\in\bigcap_{k=1}^{n} T_k$ *and* g *has precisely* i *periodic points of period* $\leqq n$, $\alpha_1,\cdots,\alpha_i$ *such that* $\alpha_j\in U_j$ *and period* $\alpha_j =$ *period* β_j.

THEOREM 1. T_p *and* H_p *are open and dense subsets of* $E^r(M)$.

Proof. By Lemma 3 and the transversality theorems, T_1 is open and dense in $E^r(M)$. By the proof of [22, 5.1] H_1 is open and dense in T_1. We proceed by induction, assuming that H_p is open and dense in $E^r(M)$. We will show that $T_{p+1}\cap H_p$ is open and dense in $E^r(M)$. Then, by the openness of transversality, T_{p+1} is open and dense in $E^r(M)$ and once again by the proof of [22, 5.1] H_{p+1} is open and dense in T_{p+1}. Let $f\in H_p$. If m is a periodic point of f of period $p+1$, then by Lemma 3 $Ev_{\mathrm{graph}_{p+1}}\pitchfork\Delta_{M\times M}$ at (f, m). If $f^{p+1}(m)=m$, and m is not of period $p+1$ then m is of period $k<p+1$ and m is a hyperbolic periodic point of f. By Lemma 2, $Ev_{\mathrm{graph}_{p+1}}\pitchfork\Delta_{M\times M}$. Thus if $f\in H_p$, $Ev_{\mathrm{graph}_{p+1}}\pitchfork\Delta_{M\times M}$ on all of M, and by the transversality theorems $T_{p+1}\cap H_p$ is open and dense in H_p.

Definition. Let $f\in E^r(M)$, and let m be a hyperbolic periodic point of f. Then if $\dim W_{\mathrm{loc}}{}^u(m)=\dim M$, m is called a source. If $0<\dim W_{\mathrm{loc}}{}^u(m) <\dim M$, m is called a saddle. And if $\dim W_{\mathrm{loc}}{}^u(m)=0$, m is called a sink.

Remark. It is well known that the local stable and unstable manifolds vary continuously in the sense of the transversality theorems.

PROPOSITION 2. *There exist open and dense subsets* $G_p \subset H_p$ *and continuous functions* $E_j^p\colon G_p \to R_+$ *for* $0 < j \leqq p$ *such that:*

(a) $G_{p+1} \subset G_p$.

(b) $E_j^p = E_j^{p+1}$ $0 < j \leqq p$.

(c) *If* $f \in G_p$ *and* m *is a saddle point of period* $j \leqq p$ *then* $W_{\mathrm{loc}}{}^s(m)$, $W_{\mathrm{loc}}{}^u(m)$ *contain closed* $E_j^p(f)$ *disks centered at* m, *called* $L_f{}^s(m)$ *and* $L_f{}^u(m)$ *respectively, such that* $W_{\mathrm{loc}}{}^u(m) \supset f^{2j}(L_f{}^u(m))$.

(d) *If* $f \in G_{p+1}$, *no periodic point of* f *of period less than or equal to* $p+1$ *is contained in* $f^j(L_f{}^u(m))$ *where period* $m = j \leqq p$ *and* m *is a saddle, except* m *itself.*

(e) *Let* $f \in G_p$, *and let* n *and* m *be saddle points of* f *of period* j *and* k *respectively. Then* $(L_f{}^s(n) \cup f^j(L_f{}^u(n))) \cap (L_f{}^s(m) \cup f^k(L_f{}^u(m))) = \emptyset$ *unless* $n = m$, *in which case* $f^j(L_f{}^u(n)) \cap L_f{}^s(n) = n$.

Proof. By the proof of the corresponding proposition [22, 6.1a] it is sufficient to verify inductively that the subset of $G_p \cap H_{p+1}$ which satisfies (d) is open and dense. It is clearly open. So we must prove density. It is clear that G_1 may be taken equal to H_1. Let n be a periodic point of period $p+1$ of $f \in G_p \cap H_{p+1}$. Let m be a saddle point of f of period $j \leqq p$ such that $n \in f^j(L_f{}^u(m))$. Let k be the largest integer $0 \leqq k \leqq p$ such that $f^k(n)$ is not an element of any $L_f{}^u(w)$ where period $w \leqq p$. Such a k exists since if $n \in L_f{}^u(m)$ then $f^{ij}(n) \in f^j(L_f{}^u(m)) - L_f{}^u(m)$ for some positive integer i. Now by a standard argument there is an arbitrarily small perturbation of f defined in a compact neighborhood V of $f^k(n)$ such that V does not intersect any $L_f{}^u(w)$ where period $w \leqq p$ and such that the orbit of the new periodic point in V does not intersect any $L_f{}^u(w)$. The continuity of E_j^p finishes the argument.

LEMMA 5. *Let* $f \in G_p$, and let m be a periodic point of f of period j. Let V be a closed submanifold of M such that $V \pitchfork \bigcup_{i=0}^{j-1} f^i(L_f{}^s(m))$. Then $Ev_{R\circ k_j} \pitchfork L_f{}^s(m)$ at (f, q) for any positive integer k and any $q \in V$.

Proof. By definition of $L_f{}^s(m)$, $f^j(L_f{}^s(m)) \subset L_f{}^s(m)$ f has maximal rank on $L_f{}^s(m)$, and m is the only periodic point in $L_f{}^s(m)$. If $f^{jk}(q) = f^{jn}(q)$ for $k > n$, then $f^{jk-jn}(f^{jn}(q)) = f^{jn}(q)$. Hence either $f^{jk}(q) \notin L_f{}^s(m)$ for any k, in which case we are finished, or $f^{jk}(q) = m$ for all $k \geqq n$. Then we may apply Lemma 4(b) to $\bigcup_{j-1}^{i=0} f^i(L_f{}^s(m))$, and $Ev_{R\circ k_j} \pitchfork L_f{}^s(m)$ at (f, q).

PROPOSITION 3. *Let $D_p{}^1 \subset G_p$ be the set of $f \in G_p$ such that if $m \in M$ is a periodic point of f of period $j \leqq p$ then $f^{jk} \pitchfork L_f{}^s(m)$ on M for all $jk \leqq p$. Then $D_p{}^1$ is open and dense in G_p.*

Proof. Let W be the neighborhood of f given by Proposition 1. By Lemma 5 for any fixed k $Ev_{jk} \pitchfork L_f{}^s(m)$ at (f,q) for any $q \in M$. Thus the $h \in W$ such that $h^{jk} \pitchfork L_f{}^s(m)$ on M are dense in W and by the openness of transversality theorem, if h is close enough to f, then $h^{jk} \pitchfork L_h{}^s(\alpha)$ on M. Where α is the point corresponding to m, given in Proposition 1. Since f has a finite number of periodic points of period $\leqq p$ and since p is finite, $D_p{}^1$ is dense in G_p. .By the openness of transversality theorem $D_p{}^1$ is also open in G_p.

PROPOSITION 4. *Let $D_p{}^2 \subset G_p$ be the set of $f \in G_p$ which satisfy the following condition. If m and q are saddle points of f of period $\leqq p$, and period $q = j$, then $f^{jk} \mid L_f{}^u(q) \pitchfork L_f{}^s(m)$ on $L_f{}^u(q)$ for all $jk \leqq p$. Then $D_p{}^2$ is open and dense in G_p.*

Proof. Proposition 2 allows us to use Lemma 5 again, and the proof proceeds as the proof of Proposition 3.

COROLLARY. *$D_p = D_p{}^1 \cap D_p{}^2$ is open and dense in G_p and consequently, $D_\infty = \bigcap_{p=1}^{\infty} D_p$ is a Baire set in $E^r(M)$.*

THEOREM 3. (*Theorem β of the Introduction*). *$KS^r(M) \supset D_\infty$.*

Proof. (1) Since $D_\infty \subset \bigcap_{p=1}^{\infty} H_p$, if $f \in D_\infty$ all the periodic points of f are hyperbolic.

(2) Let m be a periodic point of f of period j. Let $L^s(m)$ be the open $E_f{}^p$ disk contained in $L_f{}^s(m)$ for any $p > j$. Then any point $q \in W^s(m)$ is contained in an inverse image $(f^{jk})^{-1}L^s(m)$ which by Proposition 3 is a submanifold of M. If $q \in (f^{jk})^{-1}L^s(m)$ and $q \in (f^{jn})^{-1}L^s(m)$ then $f^{j(k+n)}(q) \in L^s(m)$. $f^{j(k+n)} \pitchfork L^s(m)$ at q; and thus $(f^{jk})^{-1}L^s(m)$ and $(f^{jn})^{-1}L^s(m)$ coincide on a neighborhood of q. So $W^s(m)$ is a 1-1 immersed submanifold of constant dimension equal to dimension $L^s(m)$.

(3) Let $x \in W^s(m) \cap W^u(q)$.

(a) If q is a sink then $W^u(q) = q$ and $W^s(m)$ contains a periodic point. Thus $m = q$ and $W^u(q) \pitchfork W^s(m)$.

(b) Let q be a source. Let period $q = j$ and let period $m = i$. Then

$x = f^{jk}(z)$ for some positive integer k and some $z \in L_f^u(q)$. There exists an n such that $f^{in}(x) \in L_f^s(m)$. Since $f \in D_{in}^1$, $f^{in} \pitchfork L_f^s(m)$ at x; and since $f \in D_{injk}^1$, $f^{injk} \pitchfork L_f^s(m)$ at z. As

$$Tf\colon TW^s(m) \to T_{f^{im}(x)}W^s(m)\,;$$

$f^{jk} \pitchfork W^s(m)$ at z. Thus $W^u(q) \pitchfork W^s(m)$.

(c) If m and q are both saddle points then the same argument applies, since $f \in D_{injk}^2$. Thus $W^u(q) \pitchfork W^s(m)$.

(d) If m is a sink, then $W^s(m)$ has top dimension, so $W^u(q) \pitchfork W^s(m)$.

(e) If m is a source, then q must be a sink since $f \in D_p^2$ for any p. Thus by (a) $W^u(q) \pitchfork W^s(m)$.

Examples. Expanding endomorphisms are examples of K-S endomorphisms for which the unstable manifold of the periodic points is the entire manifold M. A contracting endomorphism need not be K-S; for instance let $p \in M$ and $f\colon M \to M$ be the constant map $f(m) = p$ for all $m \in M$. As any sufficiently small K-S approximation of f is contracting it has a unique periodic point the stable manifold of which is all of M. More generally any compact submanifold of M with trivial normal bundle can be an arc component of the stable manifold of a periodic point of a K-S endomorphism as follows: Let V be the submanifold and $N(V)$ its embedded normal bundle. By the triviality of $N(V)$ there exists a differentiable map $g\colon N(V) \to R^K$, $K = \dim M - \dim V$, such that $g(V) = 0$ and 0 is a regular value of g. Now let U be the domain of a chart of M, $\psi\colon U \to R^m$, such that $\bar{U} \cap \overline{N(V)} = \emptyset$, $\psi(p) = 0$ for $p \in U$ and ψ is onto. Consider R^m as $R^K \times R^{m-K}$ and define $L\colon R^m \to R^m$ by $L(x,y) = (2x, y/2)$. Let $i\colon R^K \to R^K \times R^{m-K}$ be the map $i(x) = (x,1)$. Now

$$f_1(x) = \begin{cases} \psi^{-1}L\psi(x)\,; x \in U \\ \psi^{-1}ig(x)\,; x \in N(V) \end{cases}$$

defines a differentiable map of $N(V) \cup U$ into U with p as the unique periodic point and V on the stable manifold of p. In fact, $f_1^m \pitchfork W_{\mathrm{loc}}^S(p)$ on V. f_1, perhaps restricted to a slightly smaller set, can be extended to an endomorphism f_2 of M. Let f_3 be any sufficiently small K-S approximation of f_2. Then there is an embedding $j\colon V \to M$ isotopic to the inclusion of V and a periodic point q of f_3 close to p such that $J(V)$ is contained in W_q^S and $\dim W^S(q) = \dim V$. Let $h\colon M \to M$ be a diffeomorphism of M such that $h \mid V = j$. Then $f = h^{-1}f_3h$ is a K-S endomorphism with V in the stable manifold of $h^{-1}(q)$. These examples show, in particular, that while $W^S(h^{-1}(q))$ is an immersed submanifold $f \mid W^S(h^{-1}(q))$ is not, in general, an immersion.

III. Non-wandering sets of endomorphisms. Recall that a property of endomorphisms is called generic, if it is true for a Baire set of $E^1(M)$. As above, M will denote a compact manifold without boundary.

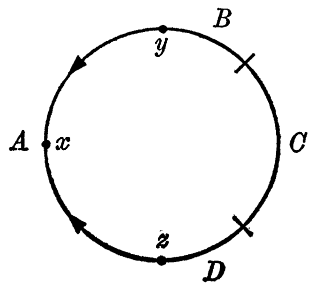
If $f \in E^1(M)$, $x \in M$ is called a wandering point for f if there is a neighborhood U of x such that $\bigcup_{m>0} f^m(U) \cap U = \emptyset$. A point is called non-wandering if it is not wandering. The non-wandering points form a closed subset of M which will be denoted by $\Omega(f)$ or simply by Ω if no confusion is possible. It is clear that $f(\Omega) \subset \Omega$, but as f of a wandering point need not be wandering it is not immediate (as it is for automorphisms) that $f(\Omega) = \Omega$.

Problem. Does $f(\Omega) = \Omega$, if not always at least generically?

The second part of this problem would follow from the next problem, which is a generalization of [29] to endomorphisms.

Problem. Let $P = P(f)$ be the set of periodic points of f. Is $\bar{P} = \Omega$ a generic property of endomorphisms?

Example. Let $f\colon S^1 \to S^1$ be described by the following diagram:



A is the western hemisphere and B, C, and D are each one-third of the eastern hemisphere. f is an immersion of degree two with one fixed sink x, and two fixed soucres y and z. $f(A) = A$ and the interior of A is on the stable manifold of x. B is taken linearly by f onto BCD. C is taken linearly onto A and D is taken linearly onto BCD. It is very easy to see that any point other than x in $W = \bigcup_{n \geqq 0} f^{-n}$ (Interior A) is wandering, and that the complement of $W = W^c$ is non-wandering. W^c is the Cantor middle third set of BCD. A better description of W^c and $f \mid W^c$ is given by the description of Group 0 in [20; I. 9. 4] Map W^c into the Cantor set. $\cdot a_0 a_1 a_2 \cdots, a_j = 0$ or 1, by $a_i(x) = 0$ if $f^i(x) \in B$ and $a_i(x) = 1$ if $f^i(x) \in D$. This correspondence is a homeomorphism and transforms f equivariantly into the map

$$\cdot a_0 a_1 a_2 a_3 \cdot \cdot \cdot \to \cdot a_1 a_2 a_3 \cdot \cdot \cdot .$$

It follows that $\overline{P(f)} = \Omega(f)$. Moreover, f is hyperbolic on W^c. In fact, Tf is expanding on $TS^1 \mid W^c$.

It is possible to extend more of the results about automorphisms to endomorphisms. For example, the rationality of the zeta-function studied in [4] for the case of a sink [25] applies equally well to endomorphisms. There is, however, as yet no satisfactory version of the Ω-stability or spectral decomposition theorems of [20].

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