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EXPANDING ENDOMORPHISMS OF FLAT MANIFOLDS†

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§1

LET M be a compact differentiable manifold without boundary. A C^1-endomorphism f: $M \to M$ is expanding if for some (and hence any) Riemannian metric on M there exist $c > 0$, $\lambda > 1$ such that $\|Tf^m v\| \geq c\lambda^m \|v\|$ for all $v \in TM$ and all integers $m > 0$. In this paper we show that any compact manifold with a flat Riemannian metric admits an expanding endomorphism. The classification of expanding endomorphisms, up to topological conjugacy, was studied in [3]. It is of interest not only abstractly but also because the inverse limit of an expanding endomorphism can be considered as an indecomposable piece of the non-wandering set of diffeomorphism: see [4] and [5].

Preliminaries

We require some standard facts from differential geometry which may all be found in [6]. Let $E(n)$ denote the group of isometries of R^n. So $E(n)$ is the semi-direct product $O(n) . R^n$, where $O(n)$ is the orthogonal group. We may consider a compact flat manifold as the orbit space R^n/Γ where Γ is a discrete uniform subgroup of $E(n)$. Such a group Γ is called a crystallographic or Bieberbach group. Two of the Bieberbach theorems on these groups are:

THEOREM 1. (Bieberbach). *If* $\Gamma \subset E(n)$ *is a crystallographic group then* $\Gamma \cap R^n$ *is a normal subgroup of finite index in* Γ, *and any minimal set of generators of* $\Gamma \cap R^n$ *is a vector space basis of* R^n *relative to which the* $O(n)$-*components of the elements of* Γ *have all entries integral.*

THEOREM 2. (Bieberbach). *Any isomorphism* f: $\Gamma \to \Sigma$ *of crystallographic subgroups of* $E(n)$ *is of the form* $\gamma \to B\gamma B^{-1}$ *for some affine transformation* B: $R^n \to R^n$.

Theorem 1 is as stated in [6; 3.2.1], and Theorem 2 is as stated in the proof of [6; 3.2.2]. Moreover $\Gamma/\Gamma \cap R^n$ is isomorphic to the holonomy group of M, [6; 3.4.6]. Henceforth, we will write A for $\Gamma \cap R^n$ and F for $\Gamma/\Gamma \cap R^n$. The corresponding exact sequence is $0 \to A \to \Gamma \to F \to 0$; it will be called the exact sequence associated to M. Recall that an invertible affine map B: $R^n \to R^n$ projects to an endomorphism of $M = R^n/\Gamma$ if the map $\gamma \to B\gamma B^{-1}$ maps Γ into itself that is $B\Gamma B^{-1} \subset \Gamma$. The induced map on M is an expanding

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endomorphism if the eigenvalues of the linear part of B are all greater than one in absolute value; in which case the induced map on M is called an affine expanding endomorphism.

§2. CONSTRUCTION OF AFFINE EXPANDING ENDOMORPHISMS

We begin with examples of affine expanding endomorphisms of the n-torus, T^n. Consider T^n as R^n/Z^n where Z^n is the integral lattice. Let B_1 be an n by n matrix such that all the entries of B_1 are integers and all the eigenvalues of B_1 are greater than one in absolute value. B_1 may be thought of as a linear map $B: R^n \to R^n$ such that $B(Z^n) \subset Z^n$. Thus considering Z^n as a group of translations operating on R^n, $B(Z^n) \subset Z^n$ and B defines an affine expanding endomorphism of T^n. Examples of such B's are provided by $k \,.\, I_{Rn}$ where k is an integer not equal to -1, 0, or 1 and I_{Rn} is the identity map of R^n.

The torus, T^n, corresponds to $\Gamma = \Gamma \cap R^n = A$. We now consider the case where F has more than one element. The symbol $|F|$ denotes the order of F.

Notations

Let $B: R^n \to R^n$ be an affine map. Then $B = L_B + v_B$ where L_B is a linear map and v_B denotes translation by the vector v_B.

We will prove the following theorem:

THEOREM. *Let M be a compact flat Riemannian manifold with associated exact sequence: $0 \to A \to \Gamma \to F \to 0$. Let $|F| > 1$ and let k be an integer greater than 0. Then there is an affine map $B: R^n \to R^n$ such that $L_B = (k\,|F| + 1)\,.\,I_{Rn}$ and B projects to an affine expanding endomorphism of M.*

As an immediate and obvious corollary we have:

COROLLARY. *Any compact flat Riemannian manifold is a non-trivial covering space of itself.*

We proceed as follows: We look for a commutative diagram

$$(*) \qquad \begin{array}{ccccccccc} 0 & \to & A & \to & \Gamma & \to & F & \to & 0 \\ & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I_F} & & \\ 0 & \to & A & \to & \Gamma & \to & F & \to & 0 \end{array}$$

such that L is $(k\,|F| + 1)\,.\,I_A$. For then, since L is injective, $f: \Gamma \to \Gamma$ is a monomorphism. Thus, by Theorem 2 (Bieberbach), there is an affine transformation $B: R^n \to R^n$ such that $f(\gamma) = B\gamma B^{-1}$ for $\gamma \in \Gamma$. Thus B projects to an endomorphism of M and $L_B\,|\,A = L = (k\,|F| + 1)I_A$. But by Theorem 1 (Bieberbach) A contains a vector space basis of R^n so $L_B = (k\,|F| + 1)\,.\,I_{Rn}$.

LEMMA 1. *Given $(*)$ with L injective then $L_B|A = L$.*

Proof. $A = \Gamma \cap R^n$, so if $a \in A$ we consider a as the translation $x \to x + a$. Now $B^{-1} = L_B{}^{-1} - L_B{}^{-1}(v_B)$. So $BaB^{-1}(x) = x + L_B(a)$ and $f(a) = BaB^{-1} = L_B(a)$.

We now show the existence of a diagram $(*)$ with the required L's. A is considered as a left Γ module under conjugation. Since A is abelian the action of A on itself is trivial and thus the action of Γ on A induces an action of F on A. Under these conditions A^A, the elements of A left fixed under the action of A, equals A. $H^1(A, A)^\Gamma$, the Γ invariant elements of $H^1(A, A)$, is just $\text{Hom}^\Gamma(A, A)$, the Γ module endomorphisms of A. (See [2] and [1, p. 190]). Thus the exact sequence in the remark [2, p. 130] becomes for this case:

$$\text{(I)} \qquad 0 \to H^1(F, A) \to H^1(\Gamma, A) \to \text{Hom}^\Gamma(A, A) \to H^2(F, A) \to H^2(\Gamma, A).$$

$H^1(\Gamma, A)$ is the group of all crossed homomorphisms $\psi\colon \Gamma \to A$ (i.e. all functions satisfying $\psi(xy) = x\psi(y) + \psi(x)$ for $x, y \in \Gamma$) modulo the principal crossed homomorphisms (i.e. functions of the form $\psi(x) = xa - a$ for a fixed $a \in A$). The map $H^1(\Gamma, A) \to \text{Hom}^\Gamma(A, A)$ in the sequence is just the restriction map.

LEMMA 2. *There is a correspondence between crossed homomorphisms* $\psi\colon \Gamma \to A$ *and diagrams* (*), *defined by* $\psi(x) = f(x)x^{-1}$.

Proof. If

$$\begin{array}{ccccccccc} 0 & \to & A & \to & \Gamma & \xrightarrow{p} & F & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1_F} & & \\ 0 & \to & A & \to & \Gamma & \xrightarrow{p} & F & \to & 0 \end{array}$$

is a commutative diagram, then $p(x) = p(f(x))$ for $x \in \Gamma$. So $f(x)x^{-1} \in \text{Ker}\, p$ and there is a unique $a \in A$ such that $f(x)x^{-1} = a$. Now $\psi(xy) = f(xy)(xy)^{-1} = f(x)f(y)y^{-1}x^{-1} = f(x)x^{-1}xf(y)y^{-1}x^{-1}$ which is in additive notation $\psi(x) + x\psi(y)$. On the other hand if $\psi\colon \Gamma \to A$ is a crossed homomorphism then $f(x) = \psi(x)x$ defines a homomorphism $f\colon \Gamma \to \Gamma$; for $\psi(xy)xy = \psi(x)x\psi(y)x^{-1}xy = \psi(x)x\psi(y)y$ and $f(x)x^{-1} = \psi(x)xx^{-1} = \psi(x) \in A$. So f induces the identity map on F. That is, the crossed homomorphism ψ corresponds to the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & \Gamma & \to & F & \to & 0 \\ & & {\scriptstyle L}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_F} & & \\ 0 & \to & A & \to & \Gamma & \to & F & \to & 0 \end{array}$$

where $f(x) = \psi(x)x$ and $L(a) = \psi(a) + a$ for $x \in \Gamma$ and $a \in A$.

Proof of the Theorem. I_A is obviously a Γ module endomorphism of A. $|F| \,.\, v = 0$ for all $v \in H^2(F, A)$ (see [1; p. 236]). Therefore $k\,|F|\,.\,I_A \in \text{Hom}^\Gamma(A, A)$ is sent to 0 in $H^2(F, A)$ by the map in (I). Thus by the exactness of (I), there is a crossed homomorphism $\psi\colon \Gamma \to A$, such that, considered as a crossed homomorphism, $\psi\,|\,A = k\,|F|\,.\,I_A$. Thus $f(x) = \psi(x)x$ restricts to $L\colon A \to A$, where $L(a) = (k\,|F| + 1)\,.\,I_A$.

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