EXPANDING ENDOMORPHISMS OF FLAT MANIFOLDS†

DAVID EPSTEIN and MICHAEL SHUB

(Received 28 November 1967)

§1

Let $M$ be a compact differentiable manifold without boundary. A $C^1$-endomorphism $f: M \to M$ is expanding if for some (and hence any) Riemannian metric on $M$ there exist $c > 0$, $\lambda > 1$ such that $\|T^m_f v\| \geq c \lambda^m \|v\|$ for all $v \in TM$ and all integers $m > 0$. In this paper we show that any compact manifold with a flat Riemannian metric admits an expanding endomorphism. The classification of expanding endomorphisms, up to topological conjugacy, was studied in [3]. It is of interest not only abstractly but also because the inverse limit of an expanding endomorphism can be considered as an indecomposable piece of the non-wandering set of diffeomorphism: see [4] and [5].

Preliminaries

We require some standard facts from differential geometry which may all be found in [6]. Let $E(n)$ denote the group of isometries of $\mathbb{R}^n$. So $E(n)$ is the semi-direct product $O(n) \rtimes \mathbb{R}^n$, where $O(n)$ is the orthogonal group. We may consider a compact flat manifold as the orbit space $\mathbb{R}^n/\Gamma$ where $\Gamma$ is a discrete uniform subgroup of $E(n)$. Such a group $\Gamma$ is called a crystallographic or Bieberbach group. Two of the Bieberbach theorems on these groups are:

**Theorem 1.** (Bieberbach). If $\Gamma \subset E(n)$ is a crystallographic group then $\Gamma \cap \mathbb{R}^n$ is a normal subgroup of finite index in $\Gamma$, and any minimal set of generators of $\Gamma \cap \mathbb{R}^n$ is a vector space basis of $\mathbb{R}^n$ relative to which the $O(n)$-components of the elements of $\Gamma$ have all entries integral.

**Theorem 2.** (Bieberbach). Any isomorphism $f: \Gamma \to \Sigma$ of crystallographic subgroups of $E(n)$ is of the form $\gamma \to B\gamma B^{-1}$ for some affine transformation $B: \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 1 is as stated in [6; 3.2.1], and Theorem 2 is as stated in the proof of [6; 3.2.2]. Moreover $\Gamma/\Gamma \cap \mathbb{R}^n$ is isomorphic to the holonomy group of $M$, [6; 3.4.6]. Henceforth, we will write $A$ for $\Gamma \cap \mathbb{R}^n$ and $F$ for $\Gamma/\Gamma \cap \mathbb{R}^n$. The corresponding exact sequence is $0 \to A \to \Gamma \to F \to 0$; it will be called the exact sequence associated to $M$. Recall that an invertible affine map $B: \mathbb{R}^n \to \mathbb{R}^n$ projects to an endomorphism of $M = \mathbb{R}^n/\Gamma$ if the map $\gamma \to B\gamma B^{-1}$ maps $\Gamma$ into itself that is $B\Gamma B^{-1} \subset \Gamma$. The induced map on $M$ is an expanding

† This work was partially supported by National Science Foundation Grants GP-5603 and GP-6868.
endomorphism if the eigenvalues of the linear part of \( B \) are all greater than one in absolute value; in which case the induced map on \( M \) is called an affine expanding endomorphism.

§2. CONSTRUCTION OF AFFINE EXPANDING ENDOMORPHISMS

We begin with examples of affine expanding endomorphisms of the \( n \)-torus, \( T^n \). Consider \( T^n \) as \( \mathbb{R}^n / \mathbb{Z}^n \) where \( \mathbb{Z}^n \) is the integral lattice. Let \( B_1 \) be an \( n \) by \( n \) matrix such that all the entries of \( B_1 \) are integers and all the eigenvalues of \( B_1 \) are greater than one in absolute value. \( B_1 \) may be thought of as a linear map \( B : \mathbb{R}^n \to \mathbb{R}^n \) such that \( B(\mathbb{Z}^n) \subseteq \mathbb{Z}^n \). Thus considering \( \mathbb{Z}^n \) as a group of translations operating on \( \mathbb{R}^n \), \( B(\mathbb{Z}^n) \subseteq \mathbb{Z}^n \) and \( B \) defines an affine expanding endomorphism of \( T^n \). Examples of such \( B \)'s are provided by \( kI_n \), where \( k \) is an integer not equal to \(-1\), \( 0 \), or \( 1 \) and \( I_n \) is the identity map of \( \mathbb{R}^n \).

The torus, \( T^n \), corresponds to \( \Gamma = \mathbb{Z} \cap \mathbb{R}^n = A \). We now consider the case where \( F \) has more than one element. The symbol \( |F| \) denotes the order of \( F \).

Notations

Let \( B : \mathbb{R}^n \to \mathbb{R}^n \) be an affine map. Then \( B = L_B + v_B \) where \( L_B \) is a linear map and \( v_B \) denotes translation by the vector \( v_B \).

We will prove the following theorem:

**Theorem.** Let \( M \) be a compact flat Riemannian manifold with associated exact sequence: \( 0 \to A \to \Gamma \to F \to 0 \). Let \( |F| > 1 \) and let \( k \) be an integer greater than 0. Then there is an affine map \( B : \mathbb{R}^n \to \mathbb{R}^n \) such that \( L_B = (k |F| + 1)I_n \). \( I_n \) and \( B \) projects to an affine expanding endomorphism of \( M \).

As an immediate and obvious corollary we have:

**Corollary.** Any compact flat Riemannian manifold is a non-trivial covering space of itself.

We proceed as follows: We look for a commutative diagram

\[
\begin{array}{ccc}
0 & \to & A \\
\downarrow L & & \downarrow f \\
0 & \to & \Gamma \\
\downarrow I_F & & \downarrow I_F \\
0 & \to & F \\
\end{array}
\]

such that \( L \) is \((k |F| + 1)I_A \). \( I_A \). For then, since \( L \) is injective, \( f : \Gamma \to \Gamma \) is a monomorphism. Thus, by Theorem 2 (Bieberbach), there is an affine transformation \( B : \mathbb{R}^n \to \mathbb{R}^n \) such that \( f(\gamma) = ByB^{-1} \) for \( \gamma \in \Gamma \). Thus \( B \) projects to an endomorphism of \( M \) and \( L_B | A = L = (k |F| + 1)I_A \). But by Theorem 1 (Bieberbach) \( A \) contains a vector space basis of \( \mathbb{R}^n \) so \( L_B = (k |F| + 1)I_n \).

**Lemma 1.** Given (\( \ast \)) with \( L \) injective then \( L_B | A = L \).

**Proof.** \( A = \Gamma \cap \mathbb{R}^n \), so if \( a \in A \) we consider \( a \) as the translation \( x \to x + a \). Now \( B^{-1} = L_B^{-1} - L_B^{-1}(v_B) \). So \( BaB^{-1}(x) = x + L_B(a) \) and \( f(a) = BaB^{-1} = L_B(a) \).

We now show the existence of a diagram (\( \ast \)) with the required \( L \)'s. \( A \) is considered as a left \( \Gamma \) module under conjugation. Since \( A \) is abelian the action of \( A \) on itself is trivial
and thus the action of $\Gamma$ on $A$ induces an action of $F$ on $A$. Under these conditions $A^4$, the elements of $A$ left fixed under the action of $A$, equals $A$. $H^1(A, A)^\Gamma$, the $\Gamma$ invariant elements of $H^1(A, A)$, is just $\text{Hom}^\Gamma(A, A)$, the $\Gamma$ module endomorphisms of $A$. (See [2] and [1, p. 190]). Thus the exact sequence in the remark [2, p. 130] becomes for this case:

$$0 \rightarrow H^1(F, A) \rightarrow H^1(\Gamma, A) \rightarrow \text{Hom}^\Gamma(A, A) \rightarrow H^2(F, A) \rightarrow H^2(\Gamma, A).$$

$H^1(\Gamma, A)$ is the group of all crossed homomorphisms $\psi: \Gamma \rightarrow A$ (i.e. all functions satisfying $\psi(xy) = x\psi(y) + \psi(x)$ for $x, y \in \Gamma$) modulo the principal crossed homomorphisms (i.e. functions of the form $\psi(x) = xa$ for a fixed $a \in A$). The map $H^1(\Gamma, A) \rightarrow \text{Hom}^\Gamma(A, A)$ in the sequence is just the restriction map.

**Lemma 2.** There is a correspondence between crossed homomorphisms $\psi: \Gamma \rightarrow A$ and diagrams ($\ast$), defined by $\psi(x) = f(x)x^{-1}$.

**Proof.** If

$$0 \rightarrow A \rightarrow \Gamma \xrightarrow{\rho} F \rightarrow 0$$

is a commutative diagram, then $p(x) = p(f(x))$ for $x \in \Gamma$. So $f(x)x^{-1} \in \ker p$ and there is a unique $a \in A$ such that $f(x)x^{-1} = a$. Now $\psi(xy) = f(xy)(xy)^{-1} = f(x)f(y)y^{-1}x^{-1} = f(x)x^{-1}xf(y)y^{-1}x^{-1}$ which is in additive notation $\psi(x) + x\psi(y)$. On the other hand if $\psi: \Gamma \rightarrow A$ is a crossed homomorphism then $f(x) = \psi(x)x$ defines a homomorphism $f: \Gamma \rightarrow \Gamma$; for $\psi(xy)x = \psi(x)x\psi(y)x^{-1}xy = \psi(x)x\psi(y)y$ and $f(x)x^{-1} = \psi(x)x^{-1} = \psi(x)A$. So $f$ induces the identity map on $F$. That is, the crossed homomorphism $\psi$ corresponds to the diagram

$$0 \rightarrow A \rightarrow \Gamma \xrightarrow{\rho} F \rightarrow 0$$

$$\Downarrow f \Downarrow \text{Id}$$

$$0 \rightarrow A \xrightarrow{L} A \rightarrow 0$$

where $f(x) = \psi(x)x$ and $L(a) = \psi(a) + a$ for $x \in \Gamma$ and $a \in A$.

**Proof of the Theorem.** $I_A$ is obviously a $\Gamma$ module endomorphism of $A$. $|F| \cdot v = 0$ for all $v \in H^2(F, A)$ (see [1; p. 236]). Therefore $k|F| \cdot I_A \in \text{Hom}^\Gamma(A, A)$ is sent to 0 in $H^2(F, A)$ by the map in (1). Thus by the exactness of (1), there is a crossed homomorphism $\psi: \Gamma \rightarrow A$, such that, considered as a crossed homomorphism, $\psi|A = k|F| \cdot I_A$. Thus $f(x) = \psi(x)x$ restricts to $L: A \rightarrow A$, where $L(a) = (k|F| + 1) \cdot I_A$.

**REFERENCES**


Warwick University and University of California, Berkeley
University of California and Brandeis University, Waltham, Mass.